Hedging Calls in the Black Model with Default

by Peter Carr
Bloomberg/NYU Courant Institute

Initial version: February 9, 2005

Current version: February 16, 2005

File reference: blackdef.tex

We develop the theory of hedging a European call option in the Black model augmented by the possibility of default in the underlying asset. We model default as a drop to zero and we assume that the credit default swap rate is constant. In this setting, we show that the call's payoff can be perfect replicated by dynamic trading in the underlying futures and in credit default swaps.

I thank the members of Bloomberg's QFR and QFD groups for their comments. I am solely responsible for any errors.

I Assumptions

We assume zero interest rates for simplicity. Let F_t be the futures price at time $t \in [0, T]$ for maturity $T' \geq T$. We assume that F is a continuous time stochastic process and that the futures contract enjoys continuous marking to market. Let $\mathbb{P}$ denote statistical probability measure.

Let W be a standard Brownian motion under $\mathbb{P}$. In the Black model, one assumes that the futures price F of the underlying asset is the unique solution of the following stochastic differential equation (SDE):

$$\frac{dF_t}{F_t} = \alpha dt + \sigma dW_t, \qquad t \in [0, T], \tag{1}$$

where F_0 and σ are known positive constants.

Let N be a standard Poisson process under $\mathbb{P}$, which starts at zero and jumps by one at independent exponentially distributed times. In the Cox Ross single jump Poisson model, one alternatively assumes that F uniquely solves the following SDE:

$$\frac{dF_t}{F_{t-}} = \mu dt + (e^j - 1) dN_t, \qquad t \in [0, T], \tag{2}$$

where F_0 is a known positive constant and μ and j are known real numbers of opposite sign.

In (1), one does not need to know the risk premium α, and in (2), one does not need to know the arrival rate λ_p of jumps under $\mathbb{P}$. In both models, dynamic trading in futures and the riskless asset renders the market complete. As a result, the martingale measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ is unique. When one changes measure from $\mathbb{P}$ to $\mathbb{Q}$ in the Black model, the volatility σ is unchanged because it is a path property. Likewise, when one changes measure from $\mathbb{P}$ to $\mathbb{Q}$ in the Cox Ross single jump model, μ and j are unchanged because they are path properties. In contrast, when one changes measure from $\mathbb{P}$ to Q in the Black model, the risk premium α becomes zero (justifying the description of $\mathbb{Q}$ as the risk-neutral measure). Analogously, when one changes measure from $\mathbb{P}$ to Q in the Cox Ross single jump model, recall that the risk-neutral arrival rate of a jump is identified as $\lambda_q \equiv -\frac{\mu}{e^j - 1}$. Intuitively, one does not need to know α in the Black model or λ_p in the pure jump model because their information content is subsumed in the underlying futures price (which is known). Notice that the drift α in the Black model does change when one changes measures, while the drift μ in the pure jump model does not.

All of these results are consequences of the following two fundamental rules:

1. $\mathbb{Q}$ is defined so that F is a $\mathbb{Q}$ martingale
2. A change of measure cannot change the numerical value of a parameter that can be estimated with certainty by continuous observation of a (segment of) a single path.

With all of this in mind, let us develop a third (more realistic) stochastic process for F which combines the features of the above two processes. First, note that the waiting time τ to the first jump of N is an exponentially distributed random variable with constant parameter $\lambda_p > 0$. In our application, we will let τ be the default time of the limited liability asset underlying the futures. Prior to τ, we assume that F follows geometric Brownian motion with constant drift α and constant volatility σ. At the default time τ, we will assume that F drops to zero and remains there afterwards. All of these features are captured if we let F uniquely solve the following SDE under $\mathbb{P}$:

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dN_t, \qquad t \in [0, T], \tag{3}$$

where F_0 and σ are known positive constants. Comparing (3) with (2), we see that the jump size j has been set to negative infinity and that a Brownian motion has been introduced. Once F hits zero, it absorbs there because increments in t, W, and N are multiplied by $F_{t-} = 0$ to get the change in F. We will not need to know α or λ_p, so we can actually assume that α is some real-valued stochastic process and that λ_p is some positive stochastic process.

It will be convenient to work with a default indicator process D defined by $D_t = 1(N_t > 0)$. The futures price process is then:

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dD_t, \qquad t \in [0, T], \tag{4}$$

One can think of (4) as a continuous time trinomial model up to the time of default. Prior to the default time, increments in the Brownian motion generate equally likely small moves up and down of order $\sqrt{dt}$, while the Poisson process generates an $O(dt)$ probability of a large down move in the price of order one. In order to perfectly replicate the payoff of every derivative security written on the futures price path, one needs to be able to dynamically trade three assets. Dynamic trading in just the money market account and the futures contract may work for some payoffs, but it will not suffice for all payoffs.

To deal with this issue, we think of the futures as written on a stock and we introduce a credit default swap (CDS) written on a bond issued by the stock issuer. We assume zero recovery rate for the bond for simplicity. As a result, the default event causes both the bond price and the stock price to vanish (think Enron). For simplicity, we further assume that the CDS rate is constant and observable at $\lambda_q > 0$. We further assume that the CDS rate is paid continuously, rather than periodically. As a result, prior to default, an investor can access the payoff $dD_t - \lambda_q dt$ at zero cost.

II Analysis

To value European-style path-independent contingent claims, let $V(F,t) : \mathbb{R}\times[0,T] \mapsto \mathbb{R}$ be a $C^{2,1}$ function. Let $V_t = V(F_t, t)$ be a continuous time stochastic process, which will eventually be the value process. From Itô's lemma for semi-martingales:

$$\begin{aligned} V(F_T,T) &= V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t + \int_0^T \left[\frac{\partial V}{\partial t}(F_{t-},t) + \frac{\sigma^2 F_{t-}^2}{2}\frac{\partial^2 V}{\partial F^2}(F_{t-},t)\right] dt \\ &\quad + \int_0^T \left[V(0,t) - V(F_{t-},t) - \frac{\partial V}{\partial F}(F_{t-},t)(0-F_{t-})\right] dD_t. \end{aligned} \tag{5}$$

In order to represent the last term in (5) as the gain from dynamically trading a CDS, suppose that we add and subtract the same term:

$$\begin{aligned} V(F_T,T) &= V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t \\ &+ \int_0^T \left\{ \frac{\partial V}{\partial t}(F_{t-},t) + \frac{\sigma^2 F_{t-}^2}{2}\frac{\partial^2 V}{\partial F^2}(F_{t-},t) + \lambda_q \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-} \right] \right\} dt \\ &+ \int_0^T \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-} \right] (dD_t - \lambda_q dt). \end{aligned} \tag{6}$$

Suppose we now require that the function $V(F,t)$ solves the following partial differential difference equation (PDDE):

$$\frac{\partial V}{\partial t}(F,t) + \frac{\sigma^2 F^2}{2}\frac{\partial^2 V}{\partial F^2}(F,t) + \lambda_q \left[V(0,t) - V(F,t) + \frac{\partial V}{\partial F}(F,t)F \right] = 0, \tag{7}$$

on the domain $F > 0, t \in [0,T]$, subject to the following terminal condition:

$$V(F,T) = f(F), \qquad F > 0. \tag{8}$$

The solution to this Cauchy problem exists and is unique. Then (6) reduces to:

$$\begin{aligned} f(F_T) &= V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t \\ &+ \int_0^T \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-} \right] (dD_t - \lambda_q dt). \end{aligned} \tag{9}$$

Hence, by charging $V(F_0,0)$ dollars initially and holding $\frac{\partial V}{\partial F}(F_{t-},t)$ futures and $V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-}$ credit default swaps at each $t \in [0,T]$, the final payoff $f(F_T)$ is achieved at T. Note that the positions in the two risky hedge instruments will vanish after the default time.

Also note that replication is achieved without knowledge of the drift or the arrival rate of jumps (under $\mathbb{P}$). This is fortunate since it is not trivial to determine these parameters from the historical sample path of the futures price, even if it is constant. Also note that if the futures price behaved as in (4), then it would be trivial to estimate σ. As a result, this model has all of the econometric advantages of the simpler Black model in that the parameters which one needs to price are easily determined from the sample path, while the quantities which are difficult to estimate from the sample path are not needed for pricing.

To find the unique solution of the terminal value problem (7) and (8), we specialize the problem to valuing a European call. Let $C(F,t) = V(F,t)$ be the value function when the terminal payoff $f(F) = (F-K)^+$. Since $f(0) = 0$ in this case, we have $C(0,t) = 0$ as well. As a result, the PDDE (7) simplifies to the following PDE:

$$\frac{\partial C}{\partial t}(F,t) + \frac{\sigma^2 F^2}{2}\frac{\partial^2 C}{\partial F^2}(F,t) - \lambda_q C(F,t) + \lambda_q F \frac{\partial C}{\partial F}(F,t) = 0, \tag{10}$$

on the domain $F > 0, t \in [0,T]$, subject to the following terminal condition:

$$C(F,T) = (F-K)^+, \qquad F > 0. \tag{11}$$

We recognize this problem as the same one that Black Scholes solved, but with F replacing S and λ_q replacing r. As a consequence, the defaultable call value is given by:

$$C(F,t) = FN(d_1) - Ke^{-\lambda_q(T-t)}N(d_2), \tag{12}$$

where:

$$d_1 \equiv \frac{\ln(F/K) + (\lambda_q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 \equiv d_1 - \sigma\sqrt{T-t}. \tag{13}$$

III Incomplete Market

We now suppose that there is no credit default swap. Furthermore, we assume quite unrealistically that investors cannot trade any other instrument other than the futures and the money market account. In such a setting, the market is incomplete and the parameter λ_q is not known. Furthermore, the call payoff cannot be perfectly replicated. There exists an infinite number of martingale measures $\mathbb{Q}$ all consistent with the initial observed futures price F_0. For the purpose of pricing calls, there is a one to one correspondence between martingale measures $\mathbb{Q}$ and the parameter λ_q appearing in (12). Each martingale measure produces an arbitrage-free call value $C(F,t;\lambda_q)$ obtained by evaluating (12) at the associated λ_q. Since the martingale measure is not unique. the arbitrage-free price of the call is not unique.

To determine the range of possible call values, recall that the Black Scholes call value is increasing in r and hence that C is increasing in λ_q. As λ_q approaches zero, we know that the call value approaches the

Black model value with volatility σ. As λ_q approaches infinity, we know that the call value approaches F. As a result, the range of arbitrage-free call values is between the Black model value and F. This range reduces to a single point, once the market price of the CDS or the market price of another option becomes known.