LECTURE 2: BLACK-SCHOLES

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Abstract. Here we derive the Black-Scholes equation, find the option prices for common European options, examine the risks, and look at a trader's profit and loss..

1. Modeling asset prices. The future price of most assets can be expected to have both determinstic and random components. A popular model for the deterministic piece is exponential growth,

$$dS = \mu(t,S)Sdt. \tag{1.1a}$$

In a small time interval, $S(t+dt) = S\{1+\mu(t,S)dt\}$. I.e., the asset's return is $\mu(t,S)dt$ for a short time interval dt.If $\mu(t,S)$ is constant, $\mu(t,S)=\mu_0$, then $S(t) = S(0)e^{\mu_0 t}$.

The random part of the price process is commonly modeled in terms of Brownian motion,

$$dS = \sigma(t,S)SdW. \tag{1.1b}$$

Over a short time interval, $S(t+dt) = S(t) + \sigma(t,S)S\sqrt{dt}\xi$, where ξ is Gaussian with mean 0 and variance 1. The Brownian motion increases the variance of the price by $\sigma^2(t,S)S^2dt$ in a short time interval dt. This means that the ratio of the standard deviation to the asset price itself is proportional to the *volatility* $\sigma(t,S)$:

$$\frac{\text{StdDev}\{S(t+dt)\}}{S(t)} = \sigma(t,S)\sqrt{dt} \tag{1.1c}$$

The standard Black-Schole's model uses constant μ and constant σ,

$$dS = \mu Sdt + \sigma SdW. \tag{1.2}$$

We use Ito's formula on $f(S) = \log S$:

$$\begin{aligned} d(\log S) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \\ &= \frac{1}{S}dS - \frac{1}{2S^2}(dS)^2 \\ &= (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW, \end{aligned} \tag{1.3}$$

so

$$\log S = \log S(0) + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma\left[W(t) - W(0)\right]. \tag{1.4}$$

Note that $W(t) - W(0)$ is Gaussian with mean zero and variance t. Thus

$$S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}\xi}, \tag{1.5}$$

where ξ is $G(0,1)$. See the figure below.

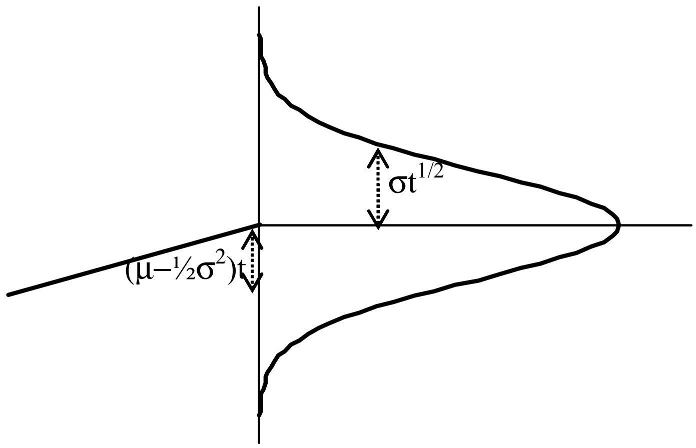


FIG. 1.1. *log S(t) drifts amount* $\left(\mu - \frac{1}{2}\sigma^2\right) t$*, and has a standard deviation of* $\sigma\sqrt{t}$.

1.1. Means. Convexity. In general, whenever $f(X)$ is nonlinear, one expects the average value of $f(X)$ to be different from f at the average value of X :

$$\langle f(X) \rangle \neq f(\langle X \rangle) \tag{1.6}$$

We can illustrate this with $\log S(t)$ and $S(t)$. If we take the expected value of eq. 1.2, we obtain

$$d\langle S \rangle = \mu \langle S \rangle dt + \sigma \langle SdW \rangle = \mu \langle S \rangle dt, \tag{1.7}$$

so the expected value of $S(t)$ is

$$\langle S(t) \rangle = S(0)e^{\mu t}. \tag{1.8}$$

But taking the expected value from eq. 1.4 shows that

$$\langle \log S(t) \rangle = \log S(0) + (\mu - \tfrac{1}{2}\sigma^2)t. \tag{1.9}$$

Thus

$$e^{\langle \log S(t) \rangle} = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t} \neq S(0)e^{\mu t} = \langle S(t) \rangle . \tag{1.10}$$

In general, if we exand $f(X)$ around the average value $\bar{X}$ of X,

$$\begin{aligned} \langle f(X) \rangle = \left\langle f(\bar{X} + (X - \bar{X})) \right\rangle &= f(\bar{X}) + \left\langle X - \bar{X} \right\rangle f'(\bar{X}) + \tfrac{1}{2} \left\langle (X - \bar{X})^2 \right\rangle f"(\bar{X}) + \cdots \\ &= f(\bar{X}) + \left\langle X - \bar{X} \right\rangle f'(\bar{X}) + \tfrac{1}{2} \left\langle (X - \bar{X})^2 \right\rangle f"(\bar{X}) + \cdots \\ &= f(\bar{X}) + \tfrac{1}{2} \operatorname{Var}\{X\} f"(\bar{X}) + \cdots \end{aligned} \tag{1.11}$$

The nonlinear terms $\frac{1}{2} \operatorname{Var}\{X\} f"(\bar{X}) + \cdots$ are generally referred to as "convexity corrections."

2. Black-Scholes.

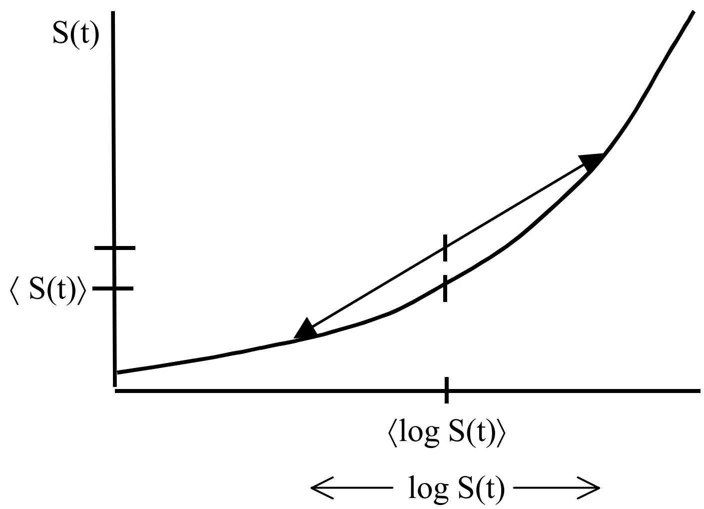


FIG. 1.2.

2.1. The parable of the grandmother. To understand the arbitrage free pricing, pretend that a trader does not work for a vast powerful bank. Pretend instead that he works for a small bookie. Each day this bookie will take, say, \$10,000 worth of bets and from these bets he will try to make \$100 profit each day. It is clear that to survive, the bookie *must* hedge his bets, because if he had an exposure of, say \$2000 and lost, then he would be out of business.

Suppose there is a heavy weight boxing match between, say, my grandmother and Mohammed Ali. A big exciting match with television coverage and lots of action. Suppose that \$10,000,000 is bet on Ali and \$10,000,000 is bet on Granny. In that case the bookie will price his bets (deals) assuming a 50% chance for either Ali or Granny winning, because at this level he can sell all his risk to Vegas. I.e., he can place a bet in Vegas to offset his net exposure. Then regardless of who wins the match, he will make his tiny commision from taking the bets, with any gains/losses against his customers being offset by the bet made to Vegas. Now, the bookie may do his homework and realize that M. Ali doesn't have a chance against Granny. In that case, he may bet, say, \$10 of his own money on Granny.

In finance, using market-implied probabilities to price deals is known as *arbitrage free pricing*, buying and selling deals in the market to eliminate one's risk is *hedging*, and betting the firm's money on a particular outcome is known as *taking a view* or *taking a (proprietary) position*. The main differences between finance and a bookie shop are
(a) the size of financial deals: a swaps desk may average \$5 billion of business a day. This is larger than most bookies;
(b) the margins are much, much tighter. The bid-ask spread of many deals is less than $\frac{1}{2}$bp, or \$50 per \$1,000,000 of notional.
Because these deals have the potential to sink even the largest institution, eliminating risk through hedging is of paramount importance.

2.2. Derivation of Black-Scholes equation. Consider a European call (or put) option on an asset with price $S(t)$. A European option specifies a single exercise date be t_{ex} and a strike price K. If the holder of the option exercises the call (put) on the exercise date, then the holder pays (receives) the strike price K and receives (gives) the asset.

To derive the Black-Scholes equation, we assume

(a) that the asset price can be modeled as

$$dS = \mu S dt + \sigma S dW, \tag{2.1}$$

where the growth rate μ and the volatility σ are constant;
(b) that one can hold both long and short positions in S;
(c) that trading costs are negligible; and
(d) the risk free investment rate r is constant
(e) that there is no arbitrage in the marketplace

Let $V(t, s)$ be the value of the call option at time t, where s is the asset price: $S(t) = s$. One naturally wonders how the value $V(t, S(t))$ evolves due to the fluctuations in the price of the underlying asset. Using Ito's formula, we find that

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} \cdot dt + \frac{\partial V}{\partial s} \cdot dS + \tfrac{1}{2} \frac{\partial^2 V}{\partial s^2} \cdot dS^2 \\ &= \left\{ \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial s} + \tfrac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} \right\} dt + \sigma S \frac{\partial V}{\partial s} dW. \end{aligned} \tag{2.2}$$

The Brownian motion term dW represents risk: it may move in our favor or against us.

Suppose at time t we form a portfolio by going long 1 call option, and shorting δ shares of the underlying asset. Let $s = S(t)$ be the asset price at time t. Then the value of this portfolio is

$$\Pi(t, s) = V(t, s) - \delta s. \tag{2.3}$$

A short time Δt later, the value of the portfolio has changed by an amount

$$\begin{aligned} \Delta \Pi &= V(t + \Delta t, s + \Delta S) - V(t, s) - \delta \Delta S \\ &= \left\{ \frac{\partial V}{\partial t} + \mu s \frac{\partial V}{\partial s} + \tfrac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \right\} \Delta t + \sigma s \frac{\partial V}{\partial s} \Delta W + O(\Delta t)^{3/2} \\ &\quad - \delta \mu s \Delta t - \delta \sigma s \Delta W - O(\Delta t^{3/2}) \\ &= \left\{ \frac{\partial V}{\partial t} + \mu s \left(\frac{\partial V}{\partial s} - \delta \right) + \tfrac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \right\} \Delta t + \sigma s \left(\frac{\partial V}{\partial s} - \delta \right) \Delta W + O(\Delta t)^{3/2} \end{aligned} \tag{2.4}$$

To eliminate as much risk as possible, we choose

$$\delta = \frac{\partial V}{\partial s}. \tag{2.5}$$

This eliminates the random term ΔW, leaving us with

$$\Delta \Pi = \left\{ \frac{\partial V}{\partial t} + \tfrac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \right\} \Delta t + O(\Delta t)^{3/2}. \tag{2.6}$$

This portfolio is riskless, at least over short enough time periods Δt. Therefore, to prevent arbitrage, the value of this portfoliio must grow at the risk-free rate r :

$$\Delta \Pi = r \Pi \Delta t. \tag{2.7}$$

For if this portfolio grew *faster* than the risk free rate, one could start at 0, borrow at the risk free rate, use the money to buy the portfolio, and thus obtain sure profits for free. This would be an arbitrage which we assume does not exist. (In practice, small arbitrages do occur, but when they occur traders snap it up until the prices move back into line and the arbitrage either disappears or becomes too small to bother about.) Substituting **??** for Π and 2.6 for $\Delta\Pi$ yields

$$\left\{\frac{\partial V}{\partial t} + \delta r s + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rV\right\}\Delta t = O(\Delta t)^{3/2}. \tag{2.8}$$

Replacing δ by $\dfrac{\partial V}{\partial s}$, dividing by Δt, and letting Δt go to zero yields

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rV \qquad \text{for } t < t_{ex}. \tag{2.9a}$$

This is the Black-Scholes equation. If the value of the asset is less than K at the exercise date, the call option is worthless; if the value of the asset exceeds K, the value of the option is $s - K$. So,

$$V(t,s) = [s-K]^+ \qquad \text{at } t = t_{ex} \qquad \text{(European call)} \tag{2.9b}$$

The value of the call option can be found by solving the PDE (partial differential equation) 2.9a subject to the boundary condition 2.9b. This will be the topic of the next section, where we will use the Feynman-Kac formula to solve it.

2.3. Other options. There are many other European options with different payoffs at the exercise date. Repeating the above arguments show that these options obey the same PDE, the Black-Scholes equation with different boundary conditions for the different payoffs. A European put is the option to sell the asset for the price K at the exercise date, so the appropriate boundary condition is

$$V(t_{ex},s) = [K-s]^+ \qquad \text{at } t = t_{ex} \qquad \text{(European put).} \tag{2.10a}$$

A digit call pays 1 if the asset price is above the strike K at the exercise date, and a digital put pays 1 if the asset price is below the strike. For these options

$$V(t_{ex},s) = \begin{cases} 0 & \text{if } s < K \\ 1 & \text{if } s \geq K \end{cases} \qquad \text{at } t = t_{ex} \qquad \text{(digital call),} \tag{2.10b}$$

$$V(t_{ex},s) = \begin{cases} 1 & \text{if } s < K \\ 0 & \text{if } s \geq K \end{cases} \qquad \text{at } t = t_{ex} \qquad \text{(digital put).} \tag{2.10c}$$

Similarly, power calls and power puts pay quadratically if one is above or below the strike:

$$V(t_{ex},s) = \left\{[s-K]^+\right\}^2 \qquad \text{at } t = t_{ex} \qquad \text{(power call),} \tag{2.10d}$$

$$V(t_{ex},s) = \left\{[K-s]^+\right\}^2 \qquad \text{at } t = t_{ex} \qquad \text{(power put),} \tag{2.10e}$$

and convexity options have a quadratic payout regardless:

$$V(t_{ex},s) = (s-K)^2 \qquad \text{at } t = t_{ex} \qquad \text{(convexity option).} \tag{2.10f}$$

In general, a European option that has a payoff of $P(S(t_{ex}))$ at t_{ex} solves the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rV \qquad \text{for } t < t_{ex}. \tag{2.11a}$$

with the boundary condition

$$V(t_{ex},s) = P(s) \qquad \text{at } t = t_{ex} \tag{2.11b}$$

3. Solving the Black-Scholes equation. If one is conversant with PDEs, one can solve the Black-Scholes equation analytically (changing variables to $x = \log s$ and solving for the Green's funciton is an effective approach) or numerically (a Crank-Nicolson scheme suffices). Here we use a different approach based on the Feynman-Kac equation.

Suppose we need to find the expected value

$$U(t,x) = \mathrm{E}\left\{ e^{-\int_t^{t_{ex}} r(t',X(t'))dt'} Q(X(t_{ex})) \Big| X(t) = x \right\}, \tag{3.1a}$$

where $S(t)$ follows the Ito process:

$$dX = a(t,X)dt + b(t,X)dW. \tag{3.1b}$$

Recall that the Feynman-Kac formula states that $U(t,x)$ is the solution of the PDE

$$\frac{\partial U}{\partial t} + a(t,x)\frac{\partial U}{\partial x} + \tfrac{1}{2}b^2(t,x)\frac{\partial^2 U}{\partial x^2} = r(t,x)U \qquad \text{for } t < t_{ex}. \tag{3.2a}$$

with the boundary condition

$$U(t_{ex},x) = Q(x) \qquad \text{at } t = t_{ex}. \tag{3.2b}$$

Comparing this to the Black-Scholes formula 2.11a and 2.11b shows another way to find the value $V(t,s)$ of a European option with payoff $P(S(t_{ex}))$. Consider the expected value

$$V(t,s) = \mathrm{E}\left\{ e^{-r(t_{ex}-t)} P(S(t_{ex})) \mid S(t) = s \right\}, \tag{3.3a}$$

where we pretend that $S(t)$ evolves according to

$$dS = rSdt + \sigma SdW. \tag{3.3b}$$

According to the Feynman-Kac formula, $V(t,s)$ then satisfies the Black-Scholes equation with the correct boundary condition. Therefore, this expected value is the option price according to Black-Scholes. Even though the assets growth rate is assumed to be μ in the real world, to calculate the option price correctly, one "takes the expected value in a world in which the growth rate is the risk free rate r." This world is called the risk-neutral world.

Previously, we showed that if

$$dS = rSdt + \sigma SdW \tag{3.4a}$$

with $S(t) = s$, then

$$S(t_{ex}) = s e^{(r-\frac{1}{2}\sigma^2)(t_{ex}-t)+\sigma\sqrt{t_{ex}-t}\xi}, \tag{3.4b}$$

where ξ is a Gaussian random variable with mean 0 and variance 1. Therefore the option price is

$$V(t,s) = \frac{e^{-r(t_{ex}-t)}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2/2} P(s e^{(r-\frac{1}{2}\sigma^2)(t_{ex}-t)+\sigma\sqrt{t_{ex}-t}\xi}) d\xi. \tag{3.5}$$

To find the value of a European call, we substitute $P(s) = [s-K]^+$ into the integral, obtaining

$$V(t,s) = \frac{e^{-r(t_{ex}-t)}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} s e^{-\frac{1}{2}\xi^2+\sigma\sqrt{t_{ex}-t}\xi-\frac{1}{2}\sigma^2(t_{ex}-t)} - K e^{-r(t_{ex}-t)} e^{-\frac{1}{2}\xi^2} d\xi, \tag{3.6}$$

where the lower limit of integration is

$$-d_2 = \frac{\log K/s - \left(r - \frac{1}{2}\sigma^2\right)(t_{ex} - t)}{\sigma\sqrt{t_{ex} - t}} \tag{3.7}$$

Integrating then yields

$$V(t,s) = s\mathcal{N}(d_1) - Ke^{-r(t_{ex}-t)}\mathcal{N}(d_2) \qquad \text{(European call)} \tag{3.8a}$$

with

$$d_{1,2} = \frac{\log s/K + \left(r \pm \frac{1}{2}\sigma^2\right)(t_{ex} - t)}{\sigma\sqrt{t_{ex} - t}}. \tag{3.8b}$$

Substituting the appropriate payoffs for $P(s)$ yields the value of the other European options. For example, one obtains

$$V(t,s) = Ke^{-r(t_{ex}-t)}\mathcal{N}(-d_2) - s\mathcal{N}(-d_1) \qquad \text{(European put)} \tag{3.9a}$$

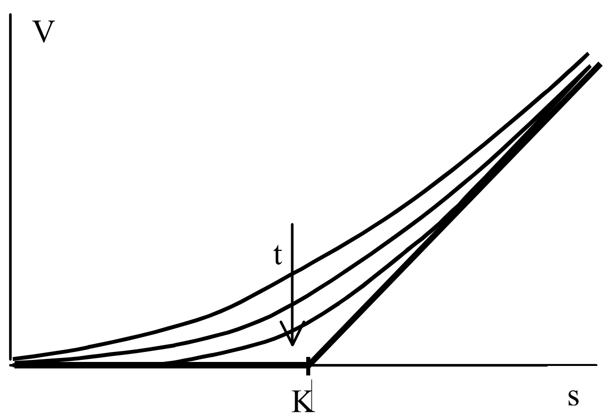
$$V(t,s) = e^{-r(t_{ex}-t)}\mathcal{N}(d_2) \qquad \text{(digital call)} \tag{3.9b}$$

$$V(t,s) = e^{-r(t_{ex}-t)}\mathcal{N}(-d_2) \qquad \text{(digital put)} \tag{3.9c}$$

$$V(t,s) = s^2 e^{(r+\sigma^2)(t_{ex}-t)} - 2sK + K^2 e^{-r(t_{ex}-t)} \qquad \text{(convexity option)} \tag{3.9d}$$

The power call and put are left as an exercise for the TA.

4. Intrinsic value, time value, and risks. Below we graphed the Black-Scholes value for a call option (eqs. 3.8a, 3.8b) as a function of the asset prices s for different values of the time to exercise t. As t approaches the exercise date t_{ex}, the call's value decreases closer and closer to the payoff. An option's value is composed of two pieces: the *intrinsic value* and the *time value.* The intrinsic value is the value the option would have if the volatility σ were zero; it is essentially the payoff. The *time value* is the value the option has as insurance: if you buy the option instead of the asset, one has fixed the price of the asset to be K, and then one gets to see if the asset goes up or down in price over the next few months, before one has to commit oneself to purchasing the option at t_{ex}. Clearly this "look ahead" has some value, the *time value* of the option. Note that this time value decreases steadily (at a fixed s), until it disappears at the exercise date t_{ex}.



Value of a call option with strike K as a function of the asset price s, at different times t.

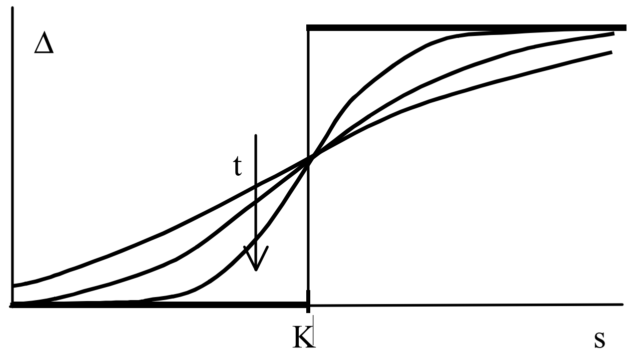


FIG. 4.1. *The hedging ratio* $\Delta = \dfrac{\partial V}{\partial s}$ *as a function of s at different times t. As* $t \to t_{ex}$*, the* Δ *hedge approaches the "all-or-nothing" strategy.*

Recall that the option could be hedged by being short δ shares of the asset, where

$$\delta \equiv \frac{\partial V}{\partial s} = \mathcal{N}(d_1) \tag{4.1a}$$

this hedge ratio is graphed as a function of s for different times t. When the asset price s is far below the strike K (the call is far out of the money), the hedge is to be short a small fraction δ; as the price s of the underlying increases, the hedge δ increases smoothly, reaching 50% when s is at the money; and as s continues to increase, the hedge approaches 100%. As time t increases towards t_{ex}, the hedge δ goes to 0 or 1, depending on whether s is out-of the money or in-the-money.

The gamma risk is the second derivative of the price with respect to the underlying::

$$\Gamma \equiv \frac{\partial^2 V}{\partial s^2} = \frac{e^{-d_1^2/2}}{\sigma s\sqrt{2\pi(t_{ex}-t)}} \tag{4.1b}$$

It is also plotted below as a function of s for various times t. Γ determines how fast one's delta hedges may change. A large gamma means that one may have trouble keeping the hedges up to date. As time t approaches t_{ex}, Γ approaches a delta-function. Short-dated options are notoriously difficult to manage near the money due to the large gamma values there.

According to the Black-Scholes theory, the volatility σ is constant. In actuality, volatilities do vary, with the markets having relatively quiet and relatively noisy periods. Because of this, many trader's hedge their vega risk, the change in an option's value due to small changes in the volatility

$$\text{vega} = \frac{\partial V}{\partial \sigma} = \sqrt{\frac{t_{ex}-t}{2\pi}}\, s e^{-d_1^2/2} \tag{4.1c}$$

As shown below, the vega risk looks like the Γ risk, except that vega goes to zero as $t \to t_{ex}$instead of blowing up at the center.

Finally, the last risk isn't a risk at all, its the change in value with respect to time

$$\frac{\partial V}{\partial t} = -\frac{\sigma s}{2\sqrt{2\pi(t_{ex}-t)}} e^{-d_1^2/2} - rKe^{-r(t_{ex}-t)}\mathcal{N}(d_2). \tag{4.1d}$$

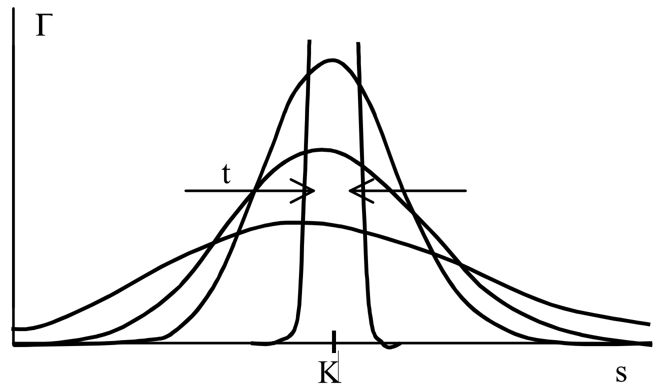


FIG. 4.2.

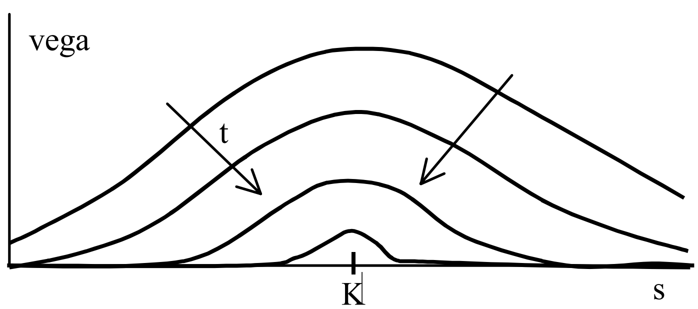


FIG. 4.3. *The vega risk,* $\dfrac{\partial V}{\partial \sigma}$ *of a call option as function of* s*, for different times* t*.*

Every day the option's time value decreases, reaching zero at the exercise date. The *time decay* is the rate at which time value decreases, and is given by the first term. The second term is the *carry*. Note that

$$\text{carry} = -rKe^{-r(t_{ex}-t)}\mathcal{N}(d_2) = r\left\{V - s\frac{\partial V}{\partial s}\right\} = r\Pi. \tag{4.2}$$

Clearly the carry is the rate at which the hedged portfolio Π earns interest at the risk free rate.

5. Trader's P&L. It is insightful to look at a trader's profit and loss calculations. At the beginning of the day, the trader starts with nothing, so his cash position is 0. Suppose a trader decides to buy a call option on some asset. Of course the trader will have δ-hedged his position, so his portfolio will be +1 call option; short δ shares of the asset, and worth

$$\Pi = (V - \delta S) = -Ke^{-r(t_{ex}-t)}\mathcal{N}(d_2) \tag{5.1a}$$

Buying the option cost the trader V, and selling δ shares short gave him cash of δs, so the trader's cash balance will be

$$\text{cash} = -(V - \delta S) = . + Ke^{-r(t_{ex}-t)}\mathcal{N}(d_2) \tag{5.1b}$$

Now consider the trader's position a short time (say, one day) later. At $t \to t + \Delta t$, let the asset price

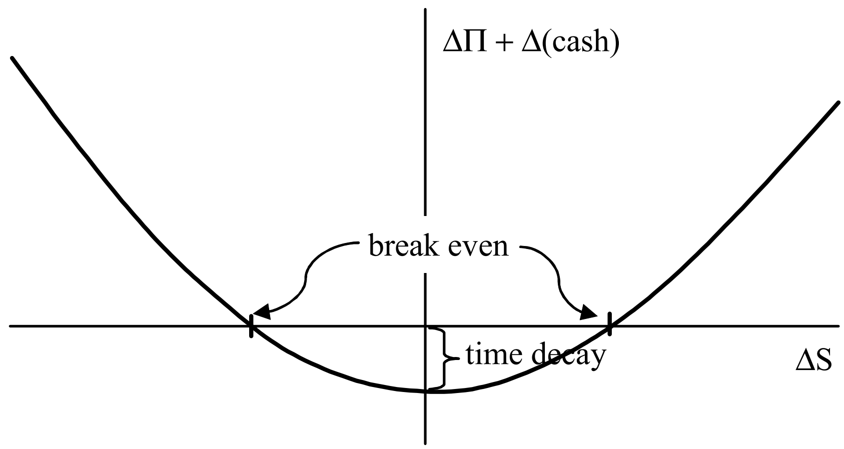


FIG. 5.1.

be $s \to s + \Delta S$.. The change in the trader's portfolio is

$$\begin{aligned} \Delta\Pi &= V(t+\Delta t, s+\Delta S) - V(t,s) - \delta \Delta S \\ &= \frac{\partial V}{\partial t}\Delta t + \left(\frac{\partial V}{\partial t} - \delta\right)\Delta S + \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}(\Delta S)^2 + \cdots . \end{aligned} \tag{5.2a}$$

Over this time interval, the trader's cash balance changes by the *carry*, the interest accrued or owed on the cash balance:

$$\Delta(\text{cash}) = -(V - \delta S) r \Delta t. \tag{5.2b}$$

From Black's equation,

$$\begin{aligned} \frac{\partial V}{\partial t} &= rV - rs\frac{\partial V}{\partial s} - \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \\ &= r(V - s\frac{\partial V}{\partial s}) - \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}. \\ &= \text{carry} + \text{time decay} \end{aligned} \tag{5.3}$$

Adding $\Delta\Pi$ and Δ(cash) together, and substituting this into the result yields

$$\Delta\Pi + \Delta\,(\text{cash}) = \tfrac{1}{2}\frac{\partial^2 V}{\partial s^2}\left\{(\Delta S)^2 - \sigma^2 s^2 \Delta t\right\}. \tag{5.4}$$

This is illustrated below.

The variance of ΔS is

$$\mathrm{E}\left\{(\Delta S)^2\right\} = \sigma^2 s^2 \Delta t \tag{5.5}$$

so on average the trader neither makes nor loses money. If $|\Delta S| > \sigma s\sqrt{\Delta t}$, the trader makes a net gain, with the positive gamma of his portfolio more than offsetting the time decau. If $|\Delta S| \le \sigma s\sqrt{\Delta t}$, then the trader loses more on time decay than he gains from the gamma term.,and he takes a net loss.The points $\Delta S = \pm\sigma s\sqrt{\Delta t}$ are called break even points of the option.

In real life a sell side desk may have hundreds of options writteni on the underlying asset, and the desk hedges the net δ exposure of all the options on that underlying.