Exact Analytical Solution for the Normal SABR Model

Ralf Korn
Fachbereich Mathematik, TU Kaiserslautern, Germany,
e-mail: korn@mathematik.uni-kl.de
Songyin Tang
Fachbereich Mathematik, TU Kaiserslautern, Germany

Abstract
The Stochastic Alpha Beta Rho (SABR) model by Hagan et al. (2002) is currently one of the most popular models in the interest rate derivative markets. In this paper, based on Henry-Labordere's (2008) previous work, we derive a new exact analytical solution for the normal SABR model and verify its correctness by numerical examples.

Keywords
SABR model, Normal SABR, Hyperbolic geometry, Stochastic volatility, Volatility smile.

Introduction
Due to the closed formula of Hagan et al. (2002), building and calculating option prices for the SABR model is valid and comprehensive. But this formula is only approximately and its quality deteriorates for long maturity and high volatility of volatility. In the general case of the SABR model, there is no exact analytical solution. However in the special case of the normal SABR model, Henry-Labordere (2005) & (2008) finds an exact solution for European options. Unfortunately, our numerical experiments show this formula is wrong. After checking each step within the hyperbolic geometry framework proposed by Henry-Labordere, we have been able to localize the mistake. After correcting it, we succeed in obtaining a new formula.

SABR model and Hagan et al.'s formula
The SABR model is a stochastic volatility model for a forward LIBOR rate, a forward swap rate or any other forward rate. The SABR model attempts to capture the dynamics of the volatility smile in the interest rate derivative markets which are dominated by caps, floors and swaptions. It is described by the following equations under the T-forward measure:

$$df_t = \alpha_t f_t^{\beta} dW_t^1 \tag{1}$$

$$d\alpha_t = \nu \alpha_t dW_t^2 \tag{2}$$

$$\mathbb{E}\left[dW_t^1 dW_t^2\right] = \rho dt \tag{3}$$

with initial values: current forward rate f_0 and current volatility α_0. In these equations, f_t is the forward rate process, α_t is the volatility process, W_t^1 and W_t^2 are two correlated Brownian motions with correlation ρ. The model has three constant parameters: the exponent of the forward rate $0 \le \beta \le 1$, the correlation parameter $-1 < \rho < 1$ and the volatility of volatility (vol of vol) $\nu \le 0$.

Consider a European call option on the forward rate f_t with the strike price K and the maturity T years. The value of this option is equal to the discounted expected value of the payoff $\max(f_T - K, 0)$. It is convenient to express the solution in terms of the implied Black volatility of the option. Namely, we force the SABR model price of the option into the valuation formula of the Black (lognormal) model. Then the implied Black volatility, which is the value of the volatility parameter in the Black model such that the Black price matches the SABR price, is approximately given by Hagan et al.'s formula (2002):

$$\sigma_{\text{imp,Black}}^{\text{SABR,Hagan}}(K, f_0) = A \cdot \left(\frac{z}{\chi(z)}\right) \cdot B \tag{4}$$

where z, $\chi(z)$, A and B are given as follows:

$$z = \frac{\nu}{\alpha_0}\left(f_0 K\right)^{\frac{1-\beta}{2}} \ln\frac{f_0}{K}$$

$$\chi(z) = \ln\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right)$$

$$A = \frac{\alpha_0}{\left(f_0 K\right)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\ln^2\frac{f_0}{K} + \frac{(1-\beta)^4}{1920}\ln^4\frac{f_0}{K}\right]}$$

$$B = 1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{(f_0 K)^{1-\beta}} + \frac{1}{4}\frac{\alpha_0 \beta \nu \rho}{(f_0 K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right] T$$

For the special case of at-the-money options, this formula reduces to:

$$\sigma_{\text{imp,Black}}^{\text{SABR,Hagan}}(\text{ATM}) = \sigma_{\text{imp,Black}}^{\text{SABR,Hagan}}(f_0, f_0)$$

$$= \frac{\alpha_0\left[1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{f_0^{2-2\beta}} + \frac{1}{4}\frac{\alpha_0 \beta \rho \nu}{f_0^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right) T\right]}{f_0^{1-\beta}}$$

Normal SABR model and Henry-Labordere's formula

If $\beta = 0$, the SABR model (1) – (3) specializes to the normal SABR case:

$$df_t = \alpha_t dW_t^1 \quad (5)$$

$$d\alpha_t = \nu \alpha_t dW_t^2 \quad (6)$$

$$\mathbb{E}\left[dW_t^1 dW_t^2\right] = \rho dt \quad (7)$$

We should always remember that the analytical solution for the European option price we derive in the following, is based on the assumption that the boundary at zero is neither an absorbing boundary, nor a reflecting boundary. In contrast, in each simulated path of the forward rate, $f_t < 0$ is allowed, which is also the assumption of the Bachelier (normal) model.

In fact, many traders are in the habit of using the normal SABR model, as it seems to capture the rates dynamcics better than other variations of the SABR model.

The value of a European call option for the normal SABR model as claimed by Henry-Labordere (2008) is:

$$V^{\text{European Call}}_{\text{Labordere}} = [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_0^{\frac{\nu^2 T}{2}} \int_{b_{\min}}^{\infty} \quad (8)$$

$$\times \frac{e^{-\frac{t'}{4}}}{(4\pi t')^{\frac{3}{2}}} \frac{b e^{-\frac{b^2}{4t'}} (\alpha_{\max}(b) - \alpha_{\min}(b))}{\sqrt{\cosh b - \cosh b_{\min}}} db\, dt'$$

where $b_{\min}$ is given as follows:

$$b_{\min} = \cosh^{-1}\left(\frac{L}{\alpha_0(1-\rho^2)}\right) \quad (9)$$

$$L = -\rho\left((K - f_0)\nu + \alpha_0 \rho\right) + \sqrt{\alpha_0^2 + 2\alpha_0 \nu \rho (K - f_0) + \nu^2 (K - f_0)^2}$$

where $(\alpha_{\max}(b) - \alpha_{\min}(b))$ is given as follows:

$$(\alpha_{\max}(b) - \alpha_{\min}(b)) = \sqrt{4(1-\rho^2)M} \quad (10)$$

$$M = -(K - f_0)^2 \nu^2 - \alpha_0(\alpha_0 + 2\nu\rho(K - f_0) + \alpha_0\rho^2) + \alpha_0 \cosh b\,(2\rho((K - f_0)\nu + \alpha_0\rho)$$
$$+ \alpha_0(1-\rho^2)\cosh b)$$

where Hyperbolic cosine and its inverse function are respectively defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}); \qquad x \geq 1$$

New formula for the normal SABR model

Below, we derive a new formula of the value of a European call option for the normal SABR model:

$$V^{\text{EuropeanCall}}_{\text{New}} = [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_{b_{\min}}^{\infty} [g(b) \cdot h(b)] db \quad (11)$$

where $b_{\min}$ is given by the formula (9) and $g(b)$ is given as:

$$g(b) = -\frac{e^{-\frac{b}{2}}[-1 - e^b + E + F]}{8\pi}$$

where E and F are given as:

$$E = 2\Phi\left(\frac{b - T'}{\sqrt{2T'}}\right) - 1$$

$$F = e^b\left(2\Phi\left(\frac{b + T'}{\sqrt{2T'}}\right) - 1\right)$$

$$T' = \frac{\nu^2 T}{2}$$

Here $\Phi(x)$ is defined as the cumulative distribution function of the standard normal variable $x \sim N(0,1)$.

In the formula (11), $h(b)$ is given as follows:

$$h(b) = \int_{\alpha_{\min}(b)}^{\alpha_{\max}(b)} \frac{1}{\sqrt{\cosh b - \cosh d(\alpha)}} d\alpha$$

Here $d(\alpha)$ and $\alpha_{\min}(b)$ and $\alpha_{\max}(b)$ are defined as:

$$d(\alpha) = \cosh^{-1}\left(1 + \frac{N}{2(1-\rho^2)\alpha_0 \alpha}\right)$$

$$\alpha_{\min}(b) = \frac{1}{2}(P - \sqrt{Q}) \quad (12)$$

$$\alpha_{\max}(b) = \frac{1}{2}(P + \sqrt{Q}) \quad (13)$$

where N, P, Q are respectively given as follows:

$$N = (-\nu m - \rho\alpha + \rho\alpha_0)^2 + (1-\rho^2)(\alpha - \alpha_0)^2$$

$$P = -2m\nu\rho + 2\alpha_0\rho^2 + 2\alpha_0 \cosh b - 2\alpha_0\rho^2 \cosh b$$

$$Q = 4(-\alpha_0^2 - m^2\nu^2 + 2m\alpha_0\nu\rho) + (-2m\nu\rho + 2\alpha_0\rho^2 + 2\alpha_0 \cosh b - 2\alpha_0\rho^2 \cosh b)^2$$

$$m = f_0 - K$$

Now we give a first example to reveal that our new formula is correct while the one in Henry-Labordere (2008) is flawed. The details about how we derive this formula will be given in the next section. More numerical tests will be given in the last section.

Category	European Call	Implied Black Volatility
Monte Carlo	0.009471	17.48%(Benchmark)
Hagan	0.009516	17.56%(Approximate)
Laborde re	0.005766	10.56%(Wrong)
New Formula	0.009476	17.49%(Right)

Input parameters:
$\beta = 0$, $\alpha_0 = 0.68\%$, $T = 10$, $\nu = 0.3691$, $\rho = -0.0286$
$f_0 = K = 4.35\%$

Monte Carlo Setting:
1.000.000 paths × 1.000 time steps.

Hyperbolic geometry framework

To derive the correct formula, we perform the following steps:

1. Rewrite the option pricing formula by means of the transition probability density.
2. Derive the corresponding Kolmogorov backward Equation (PDE) for this transition probability density.
3. Solve this PDE in the normal SABR case within the framework of hyperbolic geometry.
4. Put the solution of the PDE back into the option pricing formula and obtain the analytical formula.
5. Simplify this formula by reducing a 3-dimensional integral to a 2-dimensional integral.

Our analysis shows that Henry-Labordere only made a mistake in the last step and our main contribution is also in the last step.

1. Step

Let V be the value of a European call option at time 0. Omitting the discount factor, which factors out exactly, the value is:

$$\mathrm{V} = \mathbb{E}\{[f_T - K]^+ | f_0, \alpha_0\} = \int_0^{+\infty} \left[\int_K^{\infty} (f-K) p(T) \mathrm{d}f \right] \mathrm{d}\alpha$$

where $p(t)$ is defined as the transition probability density:

$$p(t) \doteq p(t, f_t, \alpha_t | 0, f_0, \alpha_0) = \mathrm{prob}\{f_t < f < f_t + \mathrm{d}f, \alpha_t < \alpha < \alpha_t + \mathrm{d}\alpha | 0, f_0, \alpha_0\}$$

$p(T)$ is thus the transition probability density at maturity T.

Via a direct integration Hagan et al. (2002) obtain the following formula:

$$\mathrm{V} = [f_0 - K]^+ + \frac{1}{2}(K^\beta)^2 \int_0^T \left[\int_0^{+\infty} \alpha^2 p(t, f_t \equiv K, \alpha | 0, f_0, \alpha_0) \mathrm{d}\alpha \right] \mathrm{d}t$$

Notice that in the formula above f_t is set to be equal to K due to Hagan's integration trick. It remains to calculate the transition probability density function $p(t)$.

2. Step

The Kolmogorov backward equation (KBE) and its adjoint Kolmogorov forward equation (KFE) are partial differential equations characterizing the dynamics of the distribution of the diffusion process (see Bjoerk (2003)). KBE addresses the following question: if we know at a future time s, the state of the system will be given in the target set, what is the probability for each state of the system at time t (t<s) to end up in the target set at time s. Our transition probability p(t) satisfies the KBE:

$$p_t = \frac{1}{2}\alpha^2 (f^\beta)^2 p_{ff} + \rho \nu \alpha^2 (f^\beta) p_{f\alpha} + \frac{1}{2}\nu^2 \alpha^2 p_{\alpha\alpha}$$

where in this equation we write p, f and α instead of $p(t)$, f_t and α_t. And we alo define $p_t \doteq \frac{\partial p(t)}{\partial t}$, $p_{ff} \doteq \frac{\partial^2 p(t)}{\partial f \partial f}$, $p_{f\alpha} \doteq \frac{\partial^2 p(t)}{\partial f \partial \alpha}$, $p_{\alpha\alpha} \doteq \frac{\partial^2 p(t)}{\partial \alpha \partial \alpha}$ for simplicity.

3. Step

Now we solve this PDE for the transition probability density within the Hyperbolic geometry framework. Define a new variable $t' = \frac{\nu^2 t}{2} \in [0, \frac{\nu^2 T}{2}]$, as $t \in [0, T]$. Then we obtain:

$$p_{t'} \doteq \frac{\partial p}{\partial t'} = p_t \frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{2}{\nu^2} p_t = \frac{\alpha^2}{\nu^2}\left((f^\beta)^2 p_{ff} + 2\rho\nu (f^\beta) p_{f\alpha} + \nu^2 p_{\alpha\alpha} \right)$$

In the normal SABR case, $\beta = 0, f^\beta = 1$, the above equation reduces to:

$$p_{t'} = \frac{\alpha^2}{\nu^2}(p_{ff} + 2\nu\rho p_{f\alpha} + \nu^2 p_{\alpha\alpha}) \tag{14}$$

Introduce the new coordinates:

$$x(f, \alpha) = \nu(f - f_0) - \rho\alpha$$
$$y(f, \alpha) = \sqrt{1-\rho^2}\alpha$$

At $t' = 0$, we define a fixed complex point $z_0 = x_0 + iy_0$ with $x_0 = x_0(f_0, \alpha_0)$, $y_0 = y_0(f_0, \alpha_0)$ given by:

$$x_0 = -\rho\alpha_0 \quad y_0 = \sqrt{1-\rho^2}\alpha_0$$

At $t' \in (0, \frac{\nu^2 T}{2}]$, we define another complex point $z_{t'} = x_{t'} + iy_{t'}$ with $x_{t'} = x_{t'}(f_{t'}, \alpha_{t'})$, $y_{t'} = y_{t'}(f_{t'}, \alpha_{t'})$ given by:

$$x_{t'} = \nu(f_{t'} - f_0) - \rho\alpha_{t'} \qquad y_{t'} = \sqrt{1-\rho^2}\alpha_{t'}$$

The geodesic distance $d(z_{t'}, z_0)$ between these two complex points on the Poincare plane is defined as:

$$d(z_{t'}, z_0) = \cosh^{-1}\left(1 + \frac{|z_{t'} - z_0|^2}{2y_{t'}y_0}\right) = \cosh^{-1}\left(1 + \frac{(x_{t'} - x_0)^2 + (y_{t'} - y_0)^2}{2y_{t'}y_0}\right) \tag{15}$$

The first derivatives of p are respectively:

$$p_f \doteq \frac{\partial p}{\partial f} = p_x \frac{\partial x}{\partial f} + p_y \frac{\partial y}{\partial f} = p_x \nu + 0 = \nu p_x$$

$$p_\alpha \doteq \frac{\partial p}{\partial \alpha} = p_x \frac{\partial x}{\partial \alpha} + p_y \frac{\partial y}{\partial \alpha} = -\rho p_x + \sqrt{1-\rho^2} p_y$$

The second derivatives of p are respectively:

$$p_{ff} \doteq \frac{\partial p_f}{\partial f} = \nu \frac{\partial p_x}{\partial f} = \nu\left(\frac{\partial p_x}{\partial x}\frac{\partial x}{\partial f} + \frac{\partial p_x}{\partial y}\frac{\partial y}{\partial f}\right) = \nu^2 p_{xx} \tag{16}$$

$$p_{f\alpha} \doteq \frac{\partial p_f}{\partial \alpha} = \nu \frac{\partial p_x}{\partial \alpha} = \nu\left(\frac{\partial p_x}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial p_x}{\partial y}\frac{\partial y}{\partial \alpha}\right) = -\nu\rho p_{xx} + \nu\sqrt{1-\rho^2} p_{xy} \tag{17}$$

$$\begin{aligned} p_{\alpha\alpha} &\doteq \frac{\partial p_\alpha}{\partial \alpha} = \frac{\partial(-\rho p_x + \sqrt{1-\rho^2} p_y)}{\partial \alpha} = -\rho\frac{\partial p_x}{\partial \alpha} + \sqrt{1-\rho^2}\frac{\partial p_y}{\partial \alpha} \\ &= -\rho(p_{xx}x_\alpha + p_{xy}y_\alpha) + \sqrt{1-\rho^2}(p_{yx}x_\alpha + p_{yy}y_\alpha) \\ &= \rho^2 p_{xx} - 2\rho\sqrt{1-\rho^2} p_{xy} + (1-\rho^2) p_{yy} \end{aligned} \tag{18}$$

We put equations (16) - (18) into the formula (14) and obtain:

$$\begin{aligned} p_{t'} = \frac{\alpha^2}{\nu^2}\Big[&\nu^2 p_{xx} + 2\nu\rho\left(-\nu\rho p_{xx} + \sqrt{1-\rho^2}\nu p_{xy}\right) \\ &+ \nu^2\left(\rho^2 p_{xx} - 2\rho\sqrt{1-\rho^2}p_{xy} + (1-\rho^2)p_{yy}\right)\Big] = y^2(p_{xx} + p_{yy}) \end{aligned} \tag{19}$$

PDE (19) is known as a Heat equation problem on the Poincare plane, for which there is an analytical solution found by McKean (1970). Thus the transition probability density $p(t')$ based on the hyperbolic distance $d(z_{t'}, z_0)$ (15) in terms of the new coordinates $[x,y]$ is solved as:

$$p(t')\mathrm{d}x\mathrm{d}y = \frac{\sqrt{2}e^{-\frac{t'}{4}}\mathrm{d}x\mathrm{d}y}{(4\pi t')^{\frac{3}{2}} y_{t'}^2} \int_{d(z_{t'},z_0)}^{\infty} \frac{be^{\frac{-b^2}{4t'}}}{\sqrt{\cosh b - \cosh d(z_{t'}, z_0)}}\mathrm{d}b$$

Due to

$$\mathrm{d}x\mathrm{d}y = \begin{vmatrix} \frac{\partial x}{\partial f} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial f} & \frac{\partial y}{\partial \alpha} \end{vmatrix} \mathrm{d}f\mathrm{d}\alpha = \begin{vmatrix} \nu & \frac{-\rho}{\sqrt{1-\rho^2}} \\ 0 & \end{vmatrix} \mathrm{d}f\mathrm{d}\alpha = \nu\sqrt{1-\rho^2}\,\mathrm{d}f\mathrm{d}\alpha$$

$$y_{t'}^2 = \left((1-\rho^2)^{\frac{1}{2}}\alpha_{t'}\right)^2 = (1-\rho^2)\alpha_{t'}^2$$

we obtain the transition probability density $p(t')$ based on the hyperbolic distance in terms of the old coordinates $[f,\alpha]$:

$$p(t')\mathrm{d}f\mathrm{d}\alpha = \frac{\nu}{\alpha_{t'}^2\sqrt{1-\rho^2}} \frac{\sqrt{2}e^{-\frac{t'}{4}}\mathrm{d}f\mathrm{d}\alpha}{(4\pi t')^{\frac{3}{2}}} \cdot \int_{d(z_{t'},z_0)}^{\infty} \frac{be^{\frac{-b^2}{4t'}}}{\sqrt{\cosh b - \cosh d(z_{t'}, z_0)}}\mathrm{d}b \qquad (20)$$

At maturity $t = T$, i.e $t' = T' = \frac{\nu^2 T}{2}$, the cumulative distribution function for the forward $f_{T'} \leq F \in (-\infty, +\infty)$ is:

$$\mathrm{prob}(f_{T'} \leq F) = \int_{-\infty}^{F} \int_{0}^{\infty} p(t' = T')\mathrm{d}\alpha\,\mathrm{d}f$$

We check the correctness of this analytical formula in Figure 1 by comparing its values with Monte Carlo simulation with the same input parameters as before.

4. Step

From $t' = \frac{\nu^2}{2}t$ and $T' = \frac{\nu^2}{2}T$, we get $\mathrm{d}t = \frac{2}{\nu^2}\mathrm{d}t'$ and thus $\int_0^T \mathrm{d}t = \frac{2}{\nu^2}\int_0^{T'} \mathrm{d}t'$. Then we put the formula of the transition probability density (20) into the option valuation formula in the first step and set $\beta = 0$ for the normal SABR case. After suitable arrangement we derive the analytical pricing formula containing a 3-dimensional (3D) integral:

$$V = [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_0^{\frac{\nu^2 T}{2}} \int_0^{\infty} \int_{d(\alpha_{t'})}^{\infty} \frac{e^{\frac{-t'}{4}}}{(4\pi t')^{\frac{3}{2}}} \frac{be^{\frac{-b^2}{4t'}}}{\sqrt{\cosh b - \cosh d(\alpha_{t'})}}\mathrm{d}b\,\mathrm{d}\alpha_{t'}\,\mathrm{d}t' \qquad (21)$$

where $z_{t'} = x_{t'} + iy_{t'} = x_{t'}(K,\alpha_{t'}) + iy_{t'}(K,\alpha_{t'})$ and $z_0 = x_0 + iy_0 = x_0(f_0,\alpha_0) + iy_0(f_0,\alpha_0)$. $d(z_{t'},z_0)$ as the geodesic distance between $z_{t'}$ and z_0 is actually a function of $\alpha_{t'}$, since $f_{t'}$ is set to be equal to K. We denote $d(\alpha_{t'}) \doteq d(z_{t'}, z_0)$:

$$\begin{aligned} d(\alpha_{t'}) &= d(z_{t'}, z_0) \overset{(15)}{=} \cosh^{-1}\left(1 + \frac{(x_{t'} - x_0)^2 + (y_{t'} - y_0)^2}{2y_{t'}y_0}\right) \\ &= \cosh^{-1}\left(1 + \frac{N}{2(1-\rho^2)\alpha_0\alpha_{t'}}\right) \end{aligned} \qquad (22)$$

where N is:

$$N = (-\nu m - \rho\alpha_{t'} + \rho\alpha_0)^2 + (1-\rho^2)(\alpha_{t'} - \alpha_0)^2$$
$$m = f_0 - K$$

Figure 1: Exact cumulative distribution function for the forward rate $f_{T'}$ at maturity compared with Monte Carlo simulation.

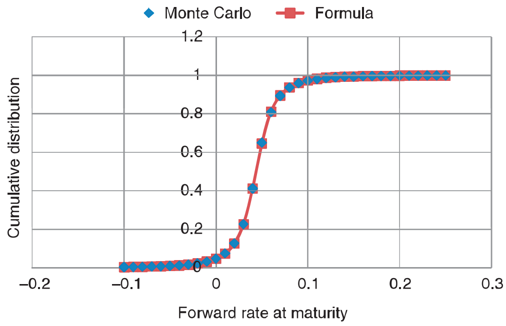
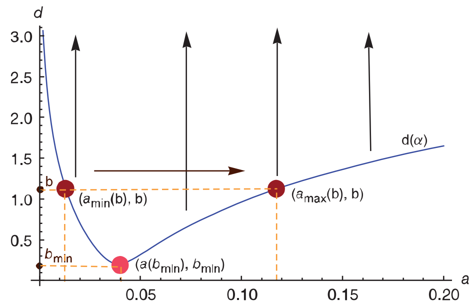


Figure 2: Geodesic distance $d(\alpha)$ between z and $z_{0'}$ (see formula (22)).



We also checked the correctness of this 3D formula (21) in the following with same input parameters as before. Hence, Henry-Labordere must make a mistake in the last step, which we will show later.

3D formula	Monte Carlo	95%-confidence interval
0.009475	0.009471	[0.009432, 0.009509]

5. Step

Notice in the formula (21), at certain time t', in order to integrate over $\alpha_{t'}$ from 0 to ∞, we interchange the order of integration over b and α, where from now on the subscript t' of $\alpha_{t'}$ is omitted for simplicity. We also draw the picture of the geodesic distance $d(\alpha)$ of formula (22) in Figure 2. The half space $b \geq d(\alpha)$ with $\alpha \geq 0$ arbitrary is then mapped to the interval $[\alpha_{min}(b), \alpha_{max}(b)]$, where b is chosen arbitrarily with $b \geq b_{min} \doteq \min d(\alpha)$ (see formula (9)). And $\alpha_{min}(b)$, $\alpha_{max}(b)$ are obtained by solving $d(\alpha) \overset{!}{=} b$ (see formulae (10), (12) and (13)), which are actually functions of b.

Then, Henry-Labordere simplified the 3D formula (21) by changing the integration order over b and α of $d(\alpha)$ and obtained his final 2D formula (8):

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$$V \overset{(21)}{=} [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \times \int_0^{\frac{\nu^2 T}{2}} \int_{b_{\min}}^{\infty} \int_{\alpha_{\min}(b)}^{\alpha_{\max}(b)} \frac{e^{\frac{-t'}{4}}}{(4\pi t')^{\frac{3}{2}}}$$

$$\times \frac{b e^{\frac{-b^2}{4t'}}}{\sqrt{\cosh b - \cosh d(\alpha)}} \mathrm{d}\alpha\, \mathrm{d}b \mathrm{d}t' \tag{23}$$

$$\overset{\text{wrong}}{=} [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \times \int_0^{\frac{\nu^2 T}{2}} \int_{b_{\min}}^{\infty} \frac{e^{\frac{-t'}{4}}}{(4\pi t')^{\frac{3}{2}}}$$

$$\times \frac{b e^{\frac{-b^2}{4t'}} (\alpha_{\max}(b) - \alpha_{\min}(b))}{\sqrt{\cosh b - \cosh b_{\min}}} \mathrm{d}b \mathrm{d}t' \tag{24}$$

We notice that the formula (24) is the same as Henry-Labordere's formula (8). But in the second equation, we denote "wrong" over "=", as that is where the mistake is made. It is not allowed to perform the integration over α, because the function $d(\alpha)$ is a function of α, varies for different α and thus is not equal to the constant $b_{\min}$, which is the minimum of $d(\alpha)$.

In the following, we show how to simplify the 3D formula (23) to get our final 2D formula (11). Note that when we integrate over b from $b_{\min}$ to ∞, $b_{\min}$ does not depend on t', thus we can change the integration order of b and t' and obtain:

$$V \overset{(23)}{=} [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_{b_{\min}}^{\infty} \int_0^{\frac{\nu^2 T}{2}} \int_{\alpha_{\min}(b)}^{\alpha_{\max}(b)} \frac{e^{\frac{-t'}{1}}}{(4\pi t')^{\frac{3}{2}}}$$

$$\times \frac{b e^{\frac{-b^2}{4t'}}}{\sqrt{\cosh b - \cosh d(\alpha)}} \mathrm{d}\alpha\, \mathrm{d}t'\, \mathrm{d}b \tag{25}$$

$$= [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_{b_{\min}}^{\infty} \int_0^{\frac{\nu^2 T}{2}} \frac{b e^{\frac{-t'}{4} - \frac{b^2}{4t'}}}{(4\pi t')^{\frac{3}{2}}} \mathrm{d}t'$$

$$\cdot \int_{\alpha_{\min}(b)}^{\alpha_{\max}(b)} \frac{1}{\sqrt{\cosh b - \cosh d(\alpha)}} \mathrm{d}\alpha\, \mathrm{d}b \tag{26}$$

$$= [f_0 - K]^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_{b_{\min}}^{\infty} [g(b) \cdot h(b)] \mathrm{d}b \tag{27}$$

where $h(b)$ and $g(b)$ are defined as:

$$h(b) \doteq \int_{\alpha_{\min}(b)}^{\alpha_{\max}(b)} \frac{1}{\sqrt{\cosh b - \cosh d(\alpha)}} \mathrm{d}\alpha$$

$$g(b) \doteq \int_0^{\frac{\nu^2 T}{2}} \frac{b e^{-\frac{t'}{4} - \frac{b^2}{4t'}}}{(4\pi t')^{\frac{3}{2}}} \mathrm{d}t' = -\frac{e^{-\frac{b}{2}}[-1 - e^b + E + F]}{8\pi}$$

where E and F are:

$$E = 2\Phi\left(\frac{b - T'}{\sqrt{2T'}}\right) - 1$$

$$F = e^b\left(2\Phi\left(\frac{b + T'}{\sqrt{2T'}}\right) - 1\right)$$

$$T' = \frac{\nu^2 T}{2}$$

The argument that we can simplify formula (25) to formula (26) is that for certain b, $\alpha_{\min}(b)$, $\alpha_{\max}(b)$ and $d(\alpha)$ are all functions of b and do not depend on t', when we integrate over α from $\alpha_{\min}(b)$ to $\alpha_{\max}(b)$.

Note that the last formula (27) is indeed our new formula (11). $h(b)$ and $g(b)$ are also defined as before. In this way, we derive our new formula which contains a 2D integral.

Numerical Results

In this section, we illustrate the correctness of our new formula (11) by a series of numerical experiments. The differences to the formula (4) and (8) are in particular highlighted in test case II.

Test case I: $\rho \neq 0$

Input parameters: $\beta = 0$, $\alpha_0 = 0.68\%$, $T = 10$, $\nu = 0.3691$, $\rho = -0.0286$, $f_0 = 4.35\%$.

Monte Carlo Setting: 1.000.000 paths × 1.000 time steps.

See: Table 1, Table 2 and Figure 3.

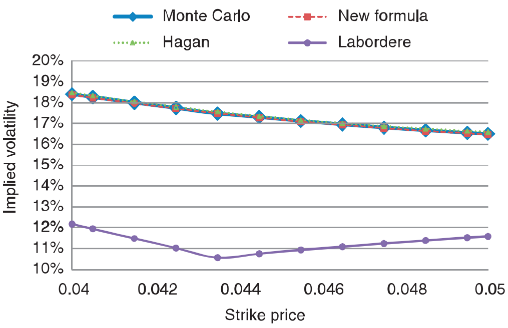
Table 1: Test case I: European Call Option Price

Strike	MonteCarlo	New	Hagan	Labordere
0.0400	0.011408	0.011392	0.011444	0.008267
0.0405	0.011117	0.011100	0.011150	0.007898
0.0415	0.010551	0.010535	0.010580	0.007171
0.0425	0.010009	0.009994	0.010035	0.006460
0.0435	0.009471	0.009476	0.009516	0.005766
0.0445	0.008998	0.008983	0.009023	0.005452
0.0455	0.008529	0.008513	0.008555	0.005156
0.0465	0.008083	0.008068	0.008113	0.004876
0.0475	0.007661	0.007646	0.007696	0.004613
0.0485	0.007262	0.007247	0.007303	0.004365
0.0495	0.006885	0.006870	0.006934	0.004132
0.0500	0.006705	0.006690	0.006759	0.004021

Table 2: Test case I: Implied Black Volatility

Strike	MonteCarlo	New	Hagan	Labordere
0.0400	18.41%	18.38%	18.48%	12.17%
0.0405	18.27%	18.23%	18.33%	11.94%
0.0415	18.00%	17.96%	18.05%	11.48%
0.0425	17.75%	17.72%	17.80%	11.02%
0.0435	17.48%	17.49%	17.56%	10.56%
0.0445	17.31%	17.28%	17.36%	10.75%
0.0455	17.12%	17.09%	17.17%	10.93%
0.0465	16.96%	16.93%	17.01%	11.09%
0.0475	16.80%	16.78%	16.87%	11.24%
0.0485	16.67%	16.64%	16.75%	11.38%
0.0495	16.55%	16.53%	16.64%	11.52%
0.0500	16.50%	16.47%	16.60%	11.58%

Figure 3: Test Case I: Volatility Smile



Test case II: $\rho = 0$

Input parameters: $\beta = 0$, $\alpha_0 = 1\%$, $T = 30$, $\nu = 0.5$, $\rho = 0$, $f_0 = 3.5\%$.
Monte Carlo Setting: 1.000.000 paths × 1.000 time steps.
See: Table 3, Table 4 and Figure 4.

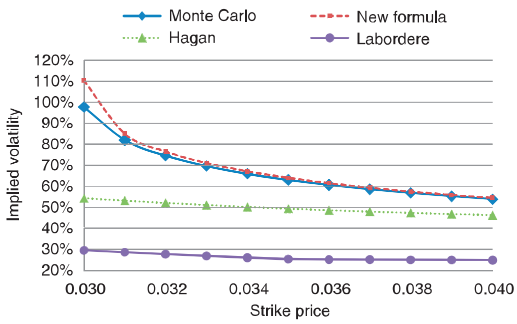
Table 3: Test case II: European Call Option Price

Strike	MonteCarlo	New	Hagan	Labordere
0.030	0.034760	0.034919	0.030569	0.021470
0.031	0.034187	0.034346	0.030209	0.020730
0.032	0.033630	0.033789	0.029851	0.020003
0.033	0.033089	0.033248	0.029499	0.019289
0.034	0.032565	0.032724	0.029155	0.018587
0.035	0.032057	0.032216	0.028819	0.017899
0.036	0.031565	0.031724	0.028495	0.017587
0.037	0.031089	0.031248	0.028182	0.017289
0.038	0.030630	0.030789	0.027883	0.017003
0.039	0.030187	0.030346	0.027597	0.016730
0.040	0.029760	0.029919	0.027325	0.016470

Table 4: Test case II: Implied Black Volatility

Strike	MonteCarlo	New	Hagan	Labordere
0.030	97,78%	110,40%	54,32%	29,55%
0.031	82,02%	85,04%	53,15%	28,58%
0.032	74,64%	76,49%	52,07%	27,68%
0.033	69,71%	71,09%	51,08%	26,83%
0.034	66,02%	67,13%	50,17%	26,04%
0.035	63,08%	64,02%	49,34%	25,29%
0.036	60,64%	61,47%	48,59%	25,17%
0.037	58,58%	59,32%	47,91%	25,07%
0.038	56,80%	57,47%	47,30%	24,98%
0.039	55,24%	55,87%	46,75%	24,91%
0.040	53,87%	54,45%	46,25%	24,85%

Figure 4: Test Case II: Volatility Smile



Conclusion

In this paper we make use of the hyperbolic geometry framework to solve the option pricing problem in the normal SABR model and thereby correct the European call formula (8) by Henry-Labordere (2008). The new formula (11) is very practical when using it for calibration. Derivation of the explicit analytical formula for the log-normal SABR case along similar lines is a challenging topic for future investigation.

Acknowledgments

Both authors thank the State Research Center for Mathematical and Computational Modeling (CM)^2 of TU Kaiserslautern for financial support.

Ralf Korn is Full Professor for Financial Mathematics at the University of Kaiserslautern, Germany. He received his chair in 1999 after his PhD in 1993 and his Habilitation in 1997 at the University of Mainz, Germany. He has been an academic visitor to Imperial College London in 1995/96 and has been a visiting professor at the University of Strasbourg in 2004 and 2006. His main research areas are financial mathematics (in particular portfolio optimization, derivative pricing, computational) and stochastic control (in particular impulse control and worst-case control). He has published five books (the most recent one, Monte Carlo Methods and Models in Finance and Insurance, with his wife Elke Korn and Gerald Kroisandt) and over 60 refereed journal articles. He is the head of the Financial Mathematics Group of Fraunhofer ITWM, deputy head of the German Society for Actuarial and Financial Mathematics, the managing director of the European Institute for Quality Management of Products and Methods in Finance, and editor/associate editor of various bookseries and journals.

Songyin Tang obtained his Diploma in Mathematics at the University of Kaiserslautern in Germany. Now he is a PhD student of Prof. Dr. Ralf Korn in Financial Mathematics and a member of the State Research Center (CM)^2. He is the overall winner of the german university competition "Assenagon Thesis Award in Finance 2012".

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