SABR spreads its wings

Traditional methods for the stochastic alpha beta rho model tend to focus on expansion approximations that are inaccurate in the long maturity ‘wings’. However, if the Brownian motions driving the forward and its volatility are uncorrelated, option prices are analytically tractable. In the correlated case, model parameters can be mapped to a mimicking uncorrelated model for accurate option pricing. Alexander Antonov, Michael Konikov and Michael Spector explain how

The stochastic alpha beta rho (SABR) model introduced in Hagan, Lesniewski & Woodward (2001) and Hagan et al (2002) is widely used by practitioners to capture the volatility skew and smile effects of interest rate options. The underlying forward rate process \( F_t \) and its volatility \( \nu_t \) are assumed to evolve according to the system of stochastic differential equations (SDEs):

\[
dF_t = F_t^\beta \nu_t dW_t \\
\nu_t = \gamma F_t^\beta dW_t
\]

where \( F_0 \) is the local volatility function with \( 0 \leq \beta \leq 1 \), \( \gamma \) is the volatility of the volatility process, and \( W_t \) are two standard Brownian motions under the risk-neutral measure with correlation \( \rho \). We assume an absorbing boundary condition for \( F_t \) at zero to guarantee that it is a martingale.

The primary use of the SABR model is in volatility surface interpolation and extrapolation. Another important application is pricing constant maturity swap (CMS) products. The CMS price is calculated via integrals of European swaption prices using a static replication formula (Hagan, 2003). The integration is done over swaption strikes from zero to infinity. This means that a SABR approximation of European swaption prices must be robust and coherent for a wide range of strikes.

In the original articles (Hagan, Lesniewski & Woodward, 2001, and Hagan et al, 2002), the authors came up with an approximation formula for forward values of European options \( C(t, K) = \mathbb{E}(F_t - K)^+ \) using Riemannian geometry and the heat kernel approach. Later on, the logic was refined by many other authors, including Berestycki, Busca & Florent (2004), Henry-Labordère (2008) and Paulot (2009).

However, the approximation quality rapidly deteriorates with time. For maturities larger than 10 years, for example, the error in implied volatility can be 1% or more even for at-the-money values. One can easily observe bad approximation behaviour for extreme strikes as well, which can prevent a valid probability density function being obtained. These undesirable properties in the distribution’s tails are especially dangerous for CMS calculations by static replication.

The initial approximation formula in Hagan et al (2002) is used as a standard tool for volatility surface interpolation, which has led somehow to the approximation rather than the model itself becoming an industry standard. However, the model price is more coherent.

A different approach to SABR option pricing was undertaken in Islah (2009), with an exact formula in terms of a multi-dimensional integration for the zero correlation case and a conditional Bessel process approximation for non-zero correlation. Nevertheless, a practical implementation of this exact result for calibration is hardly possible – the final formula consists of a three-dimensional integration of special functions and is computationally costly.

Finally, Andreasen & Huge (2011) proposed an approximation-based one-step partial differential equation (PDE) solver. The procedure was proven to be arbitrage-free, but still only delivers a rough approximation of the theoretical SABR model.

In this article, we improve the approximations for SABR option pricing. We first give an exact formula for the zero correlation case in terms of a simple two-dimensional integration of elementary functions. The corresponding integrands have plausible asymptotics, which permit an efficient numerical implementation. Moreover, we have found a very efficient approximation in terms of one-dimensional quasi-Gaussian integration. Although an order of magnitude slower than the almost instantaneous Hagan formula, it is significantly more accurate, especially in the wings (see table A). This makes the swaption volatility cube calculation speed suitable for practical applications.

The second technical result covers a general correlation case where we propose a very accurate approximation based on a model mapping procedure. We calculate effective coefficients of a zero-correlation SABR model, the so-called mimicking model, such that its small-time asymptotics coincide with the initial non-zero correlation case. The coefficient expressions involve simple algebra without numerical integration. Then we calculate the option price using the effective zero-correlation SABR model.

Our new results provide reduced approximation error and correct behaviour in the tails of the distribution for most model parameters. When \( |\rho| \) is close to one and \( \beta \) is close to zero, the option price can occasionally not be convex for small strikes, but this undesirable effect is much less pronounced than in previous approximations. Moreover, due to its small amplitude and localisation, it does not affect CMS pricing by static replication.

The high accuracy of our approximation is very important for dynamic SABR models, where analytic approximation used in calibration should provide results close to those used in pricing, for example, SABR Libor market models, as in Mercurio & Morini (2009) and Rebonato, McKay & White (2009).

The classical SABR model (1) and (2) uses a widely known set of five parameters \( \{F_0, \nu_0, \beta, \gamma, \rho\} \). Usually, the initial rate \( F_0 \) is known and the other parameters \( \{\nu_0, \beta, \gamma, \rho\} \) are subject to cali-
bution. In some papers, \(v_t\) is denoted as \(\alpha\). It is natural to impose an absorbing boundary condition on the rate behaviour at zero, which guarantees the martingale property of the rate. This also implies a non-zero probability of the rate being zero. We can transform the SABR process \(F_t\) into a stochastic volatility Bessel process \(Q_t\) via:

\[
Q_t = \frac{F_t^{1-\beta}}{1-\beta}
\]

which, with the volatility \(v_t\), satisfies the following SDE system:

\[
dQ_t = \frac{\beta}{2(\beta-1)}Q_t^{-1}\gamma v_t dt + \rho v_t dW_t
\]

\[
dv_t = \gamma v_t dW_2
\]

Denote marginal probability density functions (PDFs) of the processes \(F_t\) and \(Q_t\) as \(p(t, f) = \mathbb{E}[\delta(f_t - f)]\) and \(\delta(t, q) = \mathbb{E}[\delta(Q_t - q)]\) respectively, where \(\delta(x)\) denotes the Dirac delta-function. These PDFs are related by:

\[
p(t, f) = p(t, q) f^{-\beta} \tag{6}
\]

We consider the classical SABR model with stochastic volatility without mean-reversion. Analytical results cannot be easily adapted to mean-reverting volatility models.

**Zero correlation case**

Here, we assume \(\rho = 0\). The SABR rate (1) is distributed as a time-changed elasticity of variance (CEV) process, that is:

\[
F_t \sim X_t
\]

where \(X_t\) is a CEV process with an absorbing boundary, \(dX_t = \lambda v_t dW_t\), and the stochastic time \(\tau\) is independent of the Brownian motion \(W_t\) and defined as the cumulative variance:

\[
\tau = \int_0^t v_s^2 ds \tag{7}
\]

A distribution of the CEV process for a given time involves a modified Bessel function and can be found, for example, in Jeanblanc, Yor & Chesney (2009). The stochastic time density \(p(t, \tau) = \mathbb{E}[\delta(\tau - \tau)]\) was found by Yor (1992) as an integral over \(v\) of the joint density \(p(t, v, \tau) = \mathbb{E}[\delta(v - v(\tau, \tau))\), which, in turn, was expressed in terms of a one-dimensional integral. The forward value of a European call option can be written as:

\[
C(t, K) = \mathbb{E}[\left(F_t - K\right)^+] = \mathbb{E}[\left(X_t - K\right)^+] = \int_0^\tau dt \int_0^\tau d\tau p(t, v, \tau) C_{vev}(t, K) \tag{8}
\]

where the CEV call option forward value \(C_{vev}(t, K) = \mathbb{E}[\left(X_t - K\right)^+]\) can be found in Jeanblanc, Yor & Chesney (2009) and references therein. Substituting Yor’s expression for the joint density \(p(t, v, \tau)\), we obtain:

\[
C(t, K) = 2e^{-r\tau/\beta} \int_0^\tau dv \int_0^\tau \frac{v^{-1/2}}{\sqrt{2\pi}} \int_0^\tau d\tau C_{vev}(\tau, K) \frac{e^{-r\tau^2/(2\tau)}}{2\tau} \sinh(\frac{v\tau}{\tau}) (8)
\]

The result is expressed via the kernel function:

\[
G(t, s) = \frac{2\sqrt{2} \left( -e^{-t/s} \right)^{s/2}}{\sqrt{2\pi} t^2} \int_0^t ds \sqrt{\cosh u - \cosh s} \tag{9}
\]

which is closely related to the McKean (1970) heat kernel \(G_{MK}(t, s)\). Both kernels are associated with Brownian motion on the Poincare hyperbolic plane \(H^2\). The kernel \(G(t, s)\) describes the cumulative probability \(P(s(x, y) > s)\) for the hyperbolic distance \(s(x, y)\) on \(H^2\) with the probability density given by the McKean kernel \(G_{MK}(t, s)\), \(G(t, s) = 2\pi e^{2\xi}G_{MK}(t, s)\). \(G_{MK}\) is norm preserving, so \(G(t, 0) = 1\), as can be shown directly.

The final option price formula for zero correlation is reduced to integration over the distance \(s\):

\[
C(t, K) = (F_0 - K)^+ = \frac{2\pi \sqrt{KF_0}}{\sqrt{2\pi}} \sinh(\frac{\eta \phi(s)}{s}) G(t^2/\tau, s)
\]

\[
+ \sin(\eta \phi(s)) \int_0^\tau ds \frac{\sinh(\frac{v\tau}{\tau})}{\sinh(s)} G(t^2/\tau, s)
\]

where:

\[
\eta = \left[ \frac{1}{2(1-\beta)} \right]
\]

The underlying functions \(\phi(s)\) and \(\psi(s)\) are defined as:

\[
\phi(s) = 2\arctan\left( \frac{\sinh^2 s - \sinh^2 s_0}{\sinh^2 s - \sinh^2 s_0} \right)
\]

\[
\psi(s) = 2\arctanh\left( \frac{\sinh^2 s - \sinh^2 s_0}{\sinh^2 s - \sinh^2 s_0} \right)
\]

with the integration limits \(s_0\) and \(s_0\) given by:

\[
s_0 = \arcsinh\left( \frac{\sqrt{v_0}}{v_0} \right) \quad \text{and} \quad s_0 = \arcsinh\left( \frac{\sqrt{\gamma q + q_0}}{v_0} \right)
\]

Here \(q\) and \(q_0\) are the transformed values of the spot and strike:
Note that the option price depends on the parameters \( q, g \) and \( v \) through the dimensionless quantities \( s \) and \( s_0 \). The two-dimensional integration in formula (10) can be performed numerically in an efficient manner; the integrands are smooth functions of the parameters. Moreover, it can be shown that the function \( G(t,s) \) can be closely approximated as:

\[
G(t,s) = \frac{\sinh s}{s} e^{-\frac{1}{2} \left( R(t,s) + \delta R(t,s) \right)}
\]

where:

\[
R(t,s) = 1 + \frac{3g(s)}{8s^2} \left( 5r^2 \left( -8s^2 + 3g^2(s) + 24g(s) \right) \right) + \frac{35r^3 \left( -40s^4 + 3g^4(s) + 24g^2(s) + 120g(s) \right)}{1024s^8}
\]

and:

\[
g(s) = s \coth s - 1
\]

and the correction \( \delta R(t,s) \) is defined as:

\[
\delta R(t,s) = e^t \left( \frac{3072 + 384r + 24r^2 + r^3}{3072} \right)
\]

to guarantee that \( G(t,0) = 1 \). In computation, \( R(t,s) \) is replaced by its fourth-order expansion for small \( s \), as is the square root expression in (11). The effective small-time expansion (12) can be derived following Section 4.3 of Antonov & Spector (2012) and taking few other expansion terms. The technique is based on the Hagan et al (2002) result for the McKean kernel. Substituting the kernel \( G(t,s) \) (11) in the equation (10) leads to a one-dimensional integration formula. This considerably speeds up the calculation without sacrificing precision (see results below). As mentioned in the introduction, we consider \( \beta \in [0,1) \). The limiting case \( \beta = 1 \) requires taking careful limits in the expression (10) and will be addressed in future articles.

The general case: non-zero correlation

Heat kernel expansion. The heat kernel expansion (DeWitt, 1969) is a small-time asymptotic approximation for parabolic PDEs. This is a regular recipe for general stochastic systems to obtain PDF expansions as a fundamental solution to the Kolmogorov equation. The density \( p(t,f,v) \) expansion for the SABR model was calculated in Henry-Labordère (2008) and Paulot (2009).

The marginal PDF \( p(t,f) \) is obtained by integration over volatility \( v \), which is performed with the help of the saddle-point method, implying that the main contribution is due to the 'optimal' volatility, given by:

\[
v_{min}^2 = \gamma^2 \delta q^2 + 2\gamma \delta q v_0 + v_0^2 \quad \text{with} \quad \delta q = \frac{K^{1-\beta} - F_0^{1-\beta}}{1-\beta}
\]

This gives the following small-time expansion for a call option time-value with strike \( K \) and maturity \( T \):

\[
C(T,K) - (F_0 - K)^+ = \frac{T^2}{2\gamma} \exp \left\{ -\frac{1}{2} \ln \frac{v_{min}}{T} \right\} \quad \text{for} \quad L < 1
\]

\[
I = \left\{ \frac{1}{\sqrt{1-L^2}} \left( \arctan \frac{u_{0,L}}{\sqrt{1-L^2}} - \arctan u_{0,L} \right) \right\} \quad \text{for} \quad L > 1
\]

where:

\[
u_0 = \frac{\delta q + v_0 - v_{min}}{v_{min}} \quad \text{and} \quad L = \frac{\ln \left( \frac{1-\beta}{K^{1-\beta}} \right)}{\sqrt{1-r^2}}
\]

Expression (14) was derived for strikes away from the forward, that is, \( |K - F_0| \gg \sqrt{T} \), and a formal substitution of \( K = F_0 \) leads to a divergence. Paulot (2009) came up with a correct limit expansion and explained how to find the at-the-money option time-value.


Mapping to the zero-correlation SABR model. The expansion works well for small times, but for moderate and large ones it is ineffective. We use the mapping technique of Antonov & Misirpeashyev (2009), which works as follows. We produce a so-called mimicking model that has the same small-time expansion for the option, and calculate the option value based on this. For example, Hagan used the Black-Scholes or normal model. Paulot has proposed the CEV process as the mimicking model. The most popular case of the Black-Scholes mimicking model is the SABR model with zero correlation (SABR ZC), which is given by:

\[
s_{min}(q,v_{min}) = \ln \left( v_{min} + q_0 + \sqrt{q_0^2 + v_{min}^2} \right)
\]

The optimal parallel transport, which also depends on the strike \( K \), is given by:

\[
A_{min}(q,v_{min}) = \frac{\beta}{2} \ln \left( \frac{K}{F_0} \right) + B_{min}
\]

with:

\[
q_0 = \text{arccos} \left( \frac{\delta q + v_0 - v_{min}}{v_{min}} \right)
\]

and:

\[
B_{min} = \frac{1}{2} \beta \ln \left( \sqrt{1-r^2} \right) \quad \text{for} \quad L < 1
\]

\[
B_{min} = \left\{ \frac{1}{\sqrt{1-L^2}} \right\} \quad \text{for} \quad L > 1
\]

Expression (14) was derived for strikes away from the forward, that is, \( |K - F_0| \gg \sqrt{T} \), and a formal substitution of \( K = F_0 \) leads to a divergence. Paulot (2009) came up with a correct limit expansion and explained how to find the at-the-money option time-value.

one does. The mimicking model parameters can be strike-dependent. We denote them as in SABR ZC but with a tilde. Then, using option value (14), we should match:

\[
\frac{1}{2} \frac{\tilde{\gamma}^2}{2 \gamma^2} + \ln \frac{\tilde{\gamma}_{\min}}{2 \gamma} - \ln \left( K^\tilde{\beta} \sqrt{\tilde{\nu}_{\min}} \right) + \tilde{A}_{\min} = \frac{1}{2} \frac{\gamma^2}{2 \gamma^2} + \ln \frac{\gamma_{\min}}{2 \gamma} - \ln \left( K^\beta \sqrt{\nu_{\min}} \right) + A_{\min}
\]

\[ (17) \]

We fix \( \tilde{\gamma} \) and \( \tilde{\beta} \) in the mimicking model and look for time-expansion of \( \tilde{v}_0 \):

\[
\tilde{v}_0 = v_0(0) + T v_0^{(1)} + \cdots
\]

\[ (18) \]

such that the fit (17) is satisfied for both \( O(T^{-1}) \) and \( O(T^0) \) orders. After some algebra, we get:

\[
\tilde{v}_0^{(0)} = \frac{2 \Phi \delta \tilde{\gamma}^2}{\Phi^2 - 1}
\]

\[ (19) \]

where:

\[
\Phi = \left( \frac{v_{\min} + \rho \tilde{v}_0 + \gamma \tilde{\nu}_{\min}}{(1 + \rho) v_0} \right)^{\frac{1}{\gamma}} \text{ and } \delta \tilde{\gamma} = \frac{K^{1-\beta} - F_0^{1-\beta}}{1-\beta}
\]

The next correction term is slightly more complicated:

\[
\frac{\tilde{v}_0^{(1)}}{v_0^{(0)}} = \tilde{\gamma}^2 + \frac{1}{2} \left( \frac{\beta - \beta}{\beta - \beta} \right) \ln (K F_0) + \frac{1}{2} \ln (v_0 v_{\min}) \left( \frac{\Phi^{-1} \ln \Phi}{\Phi^{-1} \ln \Phi} \right) - \frac{1}{2} \ln \left( \frac{v_0^{(0)}}{\Phi^{-1} \ln \Phi} \right)
\]

\[ (20) \]

Let us stress that the effective volatility expansion \( \tilde{v}_0 = v_0^{(0)} + T v_0^{(1)} \), as well as the optimal quantities, volatility \( v_{\min} \) and parallel transport term \( B_{\min} \) explicitly depend on the strike \( K \).

Now let us come back to fixed skew \( \beta \) and volatility-of-volatility \( \gamma \). The approximation accuracy is quite sensitive to these parameters. A good choice based primarily on our numerical experiments is:

\[
\tilde{\beta} = \beta
\]

\[ (21) \]

\[
\tilde{\gamma}^2 = \gamma^2 - \frac{3}{2} \left( \gamma^2 p^2 + v_0 \gamma (1-\beta) F_0^{\beta-1} \right)
\]

\[ (22) \]

The intuition behind this is the following. The same power \( \beta \) helps with asymptotics for small strikes (see below for a detailed discussion on asymptotics). The volatility-of-volatility \( \gamma \) choice is inspired by a fit of the at-the-money implied volatility short-time curvature, obtained as the second derivative over the at-the-money strike \( K = F_0 \) of the leading term of the implied volatility expansion \( \sigma_0(K) \). We fixed \( \gamma \) and \( \tilde{\beta} \), and calculated \( \tilde{v}_0 \) because the resulting option price is most sensitive to \( \tilde{v}_0 \) and because the main fit of the \( O(T^{-1}) \) terms could be explicitly solved for \( \tilde{v}_0 \), but not for \( \gamma \) and \( \tilde{\beta} \). The at-the-money case is just the limit \( K \to F_0 \). The leading-order term is:

\[
\frac{\tilde{v}_0^{(0)}}{v_0} \bigg|_{K=F_0} = \tilde{v}_0
\]

in this case, and the next correction is given by:

\[ (23) \]
responding SABR rate probability density has the following leading asymptotics:

\[ p(t, f) \sim e^{-\frac{\ln^2 f}{2t}} \text{ as } f \to \infty \] (24)

This asymptotic behaviour coincides with the result of Benaim, Friz & Lee (2008) and Piterbarg (2004). The authors also derived the limit implied Black-Scholes volatility for call options \( C(t, K) \) for large strikes:

\[ \lim_{K \to \infty} \sigma_{BS}(t, K) = \frac{\gamma}{1-\beta} \] (25)

which appeared to be strike-independent. We notice also that it coincides with the large strike limit of the leading volatility term \( \sigma_1 \) (15). Lee (2004) related minimum (maximum) finite moments to left (respectively, right) wings asymptotics of implied volatilities. The SABR model does not satisfy the right Lee condition: it has all positive finite moments, that is, \( \mathbb{E}[F_t^p] < \infty \) for \( p > 0 \) and \( \beta < 1 \) due to its PDF strong decay (24). On the other hand, as shown in Benaim, Friz & Lee (2008), the left Lee condition is satisfied, leading to the left-wing implied volatility asymptotics:

\[ \lim_{K \to -\infty} \sigma_{BS}(t, K) = 2 \] (26)

The approximation gives a close fit for the distribution for a wide range of strikes. Nevertheless, the approximate PDF can have small negative values for small strikes, for small \( \beta \) and \( |p| \) close to one. Of course, these negative values are tiny with respect to huge negative probabilities for existing approximations based on the effective implied volatility. For large strikes, our approximation appears to be close numerically to the heat kernel small-time expansion (23). We will address it rigorously elsewhere.

**Numerical experiments**

Here, we demonstrate the efficiency of our approach using the following data: \( F_0 = 1, \nu_0 = 0.25, \gamma = 0.3, p = -0.5, \beta = 0.6 \) and \( T = 20 \) years. We present the Black implied volatility for European call options \( C(T, K) = \mathbb{E}[(F_T - K)^+] \) for a range of strikes \( K \) and second-moment underlying CMS calculations.

CMS convexity adjustments depend on the second moment of the rate process, which can be evaluated by the usual static replication formula (Hagan, 2003):

\[ \mathbb{E}[F_T^2] = 2 \int_0^\infty dK \mathbb{E}[(F_T - K)^+] \] (26)

For the SABR ZC map option approximation, one can use this formula directly for the second-moment calculations without any heuristic tricks, such as strike domain limitations or tail replacements. The tiny negativity of certain density approximations for the SABR ZC map does not influence the quality of the CMS calculations. For close-to-zero correlations and large skews, the big-strike tail is very fat, which produces a very slow convergence of the static replication integral.

To optimise the numerical integration, one can adapt different variance reduction techniques (for example, using the CEV model). The large strike option price can be approximated with the help of the implied volatility limit (25) with strike-independent efficient skew and volatility-of-volatility of the zero correlation SABR model.

In our numerical experiments, we compare the following methods:

- Monte Carlo simulation (MC).
- Map to the zero-correlation SABR model (ZC map).
For the Monte Carlo simulations, we have used 100 time-steps a year, 50,000 paths of good low-discrepancy numbers and an Euler scheme with an absorbing condition for a zero rate. The simulation results are presented as the mean over 50 independent runs after a careful convergence study in both time-steps and paths.

The computer time for the ZC map approximation is almost entirely spent in the numerical two-dimensional integration (9) and (10). Using efficient high-order integration schemes allows us to obtain the ZC map approximation 100 times slower than the almost instantaneous classical Hagan one. However, one-dimensional integration with the kernel $G(t,s)$ approximation (10) and (11) slows calculation by a factor of 10 compared with the Hagan formula, due to quasi-Gaussian nature of the former. This makes the swaption volatility cube calibration speed acceptable for practical applications. Note that the error between two-dimensional integration and one-dimensional one is tiny, at most 0.3 basis points in the implied volatility. We present it in our numerical experiments under the heading 1D–2D.

When the new formula’s slowness presents a bottleneck, one can use hardware to accelerate, for example, graphics processing units. Another way to speed up calibration is to find a solution with the original Hagan formula where it is known to be accurate – for example, for tiny maturities and close to at-the-money strikes – or use it as an initial guess for final calibration with a more precise new formula.

In figure 1 and table A, we present the implied volatility and its error for different methods for a large maturity of 20 years.

We observe an excellent approximation quality around the at-the-money region for the SABR ZC map, with only slight degeneration on the edges and insufficient approximation accuracy for the other methods.

Table B demonstrates an excellent approximation quality for the SABR ZC map and insufficient accuracy for the other methods. This means that our approximation works correctly even for extreme strikes. Indeed, for a 20-year maturity the second-moment integration (26) goes quite far in strikes: the option price reaches $10^{-6}$ for strikes around 35.

Conclusion

The commonly used Hagan expansion for the SABR model is well known to be imprecise in the distribution’s tails, and in pricing longer expiry options, implying negative densities and arbitrage. Most known alternatives either exhibit similar behaviour, nor consistent with the theoretical SABR model, or have an extremely slow numerical implementation. The approach presented here is quite precise and near arbitrage-free for all practical purposes, consistent with the theoretical SABR, and still reasonably fast. There is a new exact option pricing formula for the zero-correlation case, and the general case is handled by mapping the model parameters into an uncorrelated version without much loss of precision. Although there is a reduction in computation speed of an order of magnitude, the accuracy gained is significant.

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