Numerical Valuation of Cross-Currency Swaps and Swaptions

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October 24, 1996

Abstract

We investigate numerical valuation of cross-currency interest rate-based derivatives under Babbs' extended Vasicek-style model by numerical solution of the associated partial differential equation (PDE) — in particular, we consider the terminable differential (diff) swap.

Firstly we precisely formulate, in terms of their cash flows, various types of single and cross-currency swaps and swaptions. We describe Babbs' model for the domestic and foreign term structures and the exchange rate, its formulation in terms of three correlated driftless Gaussian processes and the associated three state variable parabolic PDE. We then formulate finite difference approximations to the PDE, and discuss explicit and implicit methods. With this discrete approximation to the valuation problem in a period, we proceed to value the terminable diff swap and other deals numerically by backwards recursion through the payment dates, and investigate the solutions found graphically.

We conclude that it is certainly practical, on a fast workstation, to solve for the value function of a wide range of cross-currency derivative securities by solution of explicit finite difference approximations of the PDE.

1 Introduction

In this paper we consider the numerical valuation of interest rate-based derivatives, in particular, valuation of cross-currency swap agreements. The motivation behind this choice is that cross-currency interest rate derivatives form a topic of enormous current practical importance, but such derivatives are under-represented in the literature on numerical valuation. These derivatives, assuming single stochastic factors driving the term structures and exchange rate, are dependent on three stochastic state variables, and so the PDE which their value functions must satisfy has three state variables plus time. Swap deals have the added complexity of multiple cash flow dates. The question arises as to whether such deals can be valued to any reasonable accuracy in a reasonable time on a standard workstation, since the size of the matrix representing the discretised PDE is third order in the number of grid points per axis.

In §2, we introduce various swaps, concentrating on the cross-currency case. We consider models where the value function solves a (model-dependent) PDE between cash flow dates, and give precise specifications of boundary and recursive terminal conditions for the value function of these different swap deals. In particular we look at an extension to the diff swap — new to the literature and suggested to us by Simon Babbs — which involves exchange of domestic and foreign LIBOR rates, both paid on the same domestic principle, with the additional feature that the counterparty may terminate the deal at any of the LIBOR payment dates for a fixed cost in his native currency. At the end of a period, the counterparty terminates if the termination cost is less than the continuation cost, which gives a terminal condition for a period that depends on the value at the start of the next period, and so we may solve for the value of the deal by backwards recursion, the solution in the last period giving a terminal condition for the penultimate period, and so on.

In §3 we give a general cross-currency model due to Babbs, with Ito process models of domestic and foreign bond prices, which are consistent with initial term structures, and of the exchange rate. We describe Babbs' specialisation of the general model that produces an 'extended Vasicek' model for the short rate with term structure processes driven by three correlated driftless Gaussian stochastic state variables, and give his PDE with respect to these variables which any European-style derivative must follow.

Then in §4 we discretise a general three state variable backward parabolic PDE, and consider standard finite difference approximations in each period. Preliminary to actually valuing deals numerically using Babbs' model, we describe the data that must be supplied to the model and we derive step function integration formulae for functions in the bond price and exchange rate formulae which facilitate numerical evaluation.

Finally, in §5, we solve numerically for the value function of some of the deals described in §2, including the terminable diff swap, by backwards recursion, and present results on convergence of the solution and timing of the various routines, as well as giving various plots of cross-sections through the (4D) solution surface of a call option on a terminable diff swap.

We conclude that the *explicit method* is the best of the standard methods for this multivariable type of problem, and that with it we may solve for the value function of a wide range of cross-currency derivatives. Note that we immediately obtain, from this value surface, many of the partial derivatives required for hedging (see Carr [8]). There are many possible directions one could take to speed up and increase the accuracy of the solution, and some of these are discussed in §6.

Throughout, we denote random variables by a **bold** typeface.

2 Swaps and Swaptions

The single currency fixed rate-floating rate (*vanilla*) swap is by far the most common among all swaps — Litzenberger [16] claims that, as of 1992, over two-thirds of the current total \$3 trillion outstanding interest rate swaps are vanilla fixed-floating swaps. However, cross-currency swaps are becoming increasingly popular.

In the types of swap which we will consider, a floating rate of interest is swapped for another floating or fixed rate, and this floating rate is usually taken to be some margin above the 1, 3 or 6 month LIBOR rate. The swap period, say $[0, T]$, is then divided into periods of the same length as the LIBOR term, with swapped payments made at the end of each period according to the rates prevailing at the start of the period, i.e. *in arrears.* If the zero-coupon bond price at time t for maturity M is $P(t, M)$, then we define the *LIBOR rate* $L(t, M)$ *for the period* $[t, M]$ by the annualised return on the corresponding zero-coupon bond, specifically

$$L(t, M) := \frac{1}{\delta} \times \frac{1 - P(t, M)}{P(t, M)}, \tag{1}$$

where δ is the *accrual factor* defined by

$$\delta := \frac{\text{number of days in } [t, M]}{\text{basis}}, \tag{2}$$

where basis is typically 365 for pounds sterling and 360 for U.S. dollars.

2.1 Vanilla floating-fixed swaps and swaptions

We define a *vanilla fixed-floating interest rate swap* as an agreement between two parties, the 'bank' and the 'counterparty', whereby the bank pays the counterparty a floating annualised rate of interest on a cash amount (or *principal*) Z, and the counterparty pays the bank a agreed fixed rate of interest r^* on the same principal amount, all for a fixed period $[0, T]$. Typically, the life of such a swap is anything from 2 to 15 years. Of course, equally the roles of bank and counterparty could be reversed, We adopt the convention throughout of valuation in domestic terms and from the bank's point of view, and denote value to the bank at time t by $V(t)$. It may be the case that the counterparty has an option, typically at no cost, to enter into such a swap contract at some point in the future — this type of deal we refer to as a *swaption.* In addition, the counterparty may have the option at various points to terminate the deal at a cost, in which case we call the swap *terminable.*

To use domestic LIBOR L_d as the floating rate, the swap period $[0, T]$ is divided into the corresponding LIBOR periods, and we denote period j by $[t_{j-1}, t_j)$ for $j = 1, \ldots, N$, where $t_0 = 0$. Swap payments[1] $p_1, \ldots, p_N$ are made (to the bank) at $t_1-, \ldots, t_N-$, where

[1]Note that periods are defined as closed below and open above and payments are made at t_j- so that the value function is RCLL everywhere.

$t- := \lim_{s \searrow 0}(t-s)$, and are given by

$$p_j := Z\delta_j \left[r^* - L_d(t_{j-1}, t_j) \right], \tag{3}$$

where δ_j is the swap rate accrual factor for period j, defined as in (2). Note that the LIBOR rate L is determined at the start of the period, but payment is made at the end — this is a path dependence in the payoff, which we may eliminate by reformulating the deal, so that instead present values of p_j are paid at the end of the preceding period $t_{j-1}-$. So in fact, we have a payment of $P_d(t_{j-1}, t_j)p_j$ at each $t_{j-1}-$, for $j = 2, \ldots, N$, which is determined completely by values of appropriate state variables at time $t_{j-1}-$, and now the final period is redundant. We will use this trick repeatedly in the cross-currency swaps of §2.2.

The value at $t < T$ of the swap is simply the sum of the present values $V_j(t)$ of all remaining swap payments after t. In a particular period we have a PDE (depending on a term structure model) in V_j with the terminal condition

$$V_j(t_{j-1}-) = P(t_{j-1}, t_j)p_j \qquad j = 2, \ldots, N. \tag{4}$$

In fact, we may calculate the current plain vanilla swap value, given the current term structure, since receiving $L(t_{j-1}, t_j)Z$ at t_j is equivalent to receiving Z at t_{j-1} and paying Z at time t_j. However, when the cash flows at period dates are more complex, with option structures such as we consider below, we must resort to numerical solution. In general, numerical solution of such deals is rapid and efficient for virtually any single factor model, because of the low dimensionality. Once we extend the idea of a vanilla swap to a swap across currencies, the resulting increase in the number of state variables makes *efficient* numerical solution much more key.

2.2 Cross-currency swaps

We thus turn to swaps where the two interest rates being swapped are in different, *domestic* and *foreign*, currencies. These have the additional complexity of requiring models of the two term structures and the exchange rate between them. Indeed, it might also be appropriate to use a cross-currency model to price single currency swaps, since it incorporates two additional explanatory variables that affect the domestic term structure through correlation.

The most common (*vanilla*) *cross-currency swap* is the exchange of floating or fixed rate interest payments on principals Z_d and Z_f in two currencies, domestic and foreign, which we define as follows. Again, we divide the swap period $[0, T]$ into N periods, and domestic and foreign payments p_{dj} and p_{fj} based on LIBOR are made at the end of each period, given by

$$p_{dj} := \delta_j \left[k_d L_d(t_{j-1}-, t_j-) + m_d \right] Z_d \tag{5}$$

$$p_{fj} := \delta_j \left[k_f L_f(t_{j-1}-, t_j-) + m_f \right] Z_f, \tag{6}$$

where $L_d(t, M)$ and $L_f(t, M)$ are domestic and foreign LIBOR at time t for the period $[t, M]$ respectively, defined as in (1), k_d and k_f are floating rate parameters, m_d and m_f are fixed rate components or margins above domestic and foreign LIBOR respectively and δ_j is the accrual factor (2) — the parameters k_d, k_f, m_d and m_f determine which party receives which rate. Finally, in the case that the swap is with *exchange of principal*, the party paying the domestic rate receives Z_d and pays Z_f at the start of the deal, with the reverse exchange at the end of the deal.

Again the swap value $V(t)$ is the sum of values of individual payments $V_j(t)$, each value function solving a model-determined PDE. We eliminate the path dependence in the payoff by equivalently exchanging the present values, discounted to time $t_{j-1}-$ using domestic and foreign term structures accordingly, and so that the terminal condition for the period $[t_{j-2}, t_{j-1})$ is

$$V_j(t_{j-1}-) \quad = \quad P_d(t_{j-1}, t_j)p_{dj} - S(t_{j-1})P_f(t_{j-1}, t_j)p_{fj}, \tag{7}$$

where $S(t_{j-1})$ denotes the exchange rate in domestic currency prevailing at time t_{j-1}. Since we have no option features, we can again price this deal analytically by equating each LIBOR payment to paying and receiving the principal — we can then see that the vanilla cross-currency swap with exchange of principal has value zero at time zero and we use this as a test of solution accuracy in §5. If in addition we set $Z_f := Z_d/S(0)$, then there is even no initial exchange of principal.

We now consider an increasingly popular variant of the above deal which has the feature that it avoids any explicit exchange rate exposure, and such deals, even without option features, cannot be valued in this simple way.

Differential swap A *vanilla differential (diff)* or *switch LIBOR* swap is an exchange of domestic and foreign LIBOR, but foreign interest rates are paid on the *same* domestic principal amount Z as the domestic rate, so there is no explicit exchange rate exposure. The payment to the bank at the end of period j is given by

$$p_j := Z\delta_j \left[k_d(L_d(t_{j-1}, t_j) + m) - k_f L_f(t_{j-1}, t_j)\right], \tag{8}$$

and then the formulation in a particular period as a PDE problem is the same as for the domestic vanilla swap above, with the terminal condition (4).

The diff swap was introduced to the academic literature by Litzenberger [16], who discusses practical estimation and hedging issues, and was taken up by Babbs [5] as an application of his cross-currency model of §3. Under this model, he derives a simple closed form expression for the diff swap using the risk-adjusted valuation formula (27) and calculating the expectation by exploiting the Gaussian state variables. The expression is couched in terms of current bond prices and integrals of the various volatility and correlation functions, and is relatively straightforward to evaluate numerically — we will use this closed-form formula as a check on our numerical procedure.

Terminable diff swap Consider now the *terminable* diff swap, suggested to the authors by Babbs [4], where the counterparty has the option to terminate the deal at the start of every interest period for a termination cost of X in the counterparty's native currency. This is altogether a more complicated deal than those discussed earlier, and does not have the same simple European-style payoff structure — it is a 'Bermudan' option, which is an American option with only a *finite* number of early exercise dates. We formulate it as follows.

At the end of each period, the counterparty must either terminate the deal, at a cost in foreign currency of X, or continue the deal by making the diff swap payment (8). As usual, we equivalently exchange present values so that the last period $[t_{N-1}, t_N)$ is redundant. In the penultimate period, we have the boundary condition at t_{N-1} for the solution in the penultimate period

$$V(t_{N-1}-) = \min\{XS(t_{N-1}), P_d(t_{N-1}, t_N)p_N\}, \tag{9}$$

since the counterparty terminates if the termination cost is less than the cost of continuing. In an earlier period j, we have the same payment, but we have to take into account the payments still remaining if the counterparty chooses to continue rather than pay to terminate. So we have the same boundary condition as (9), except that the value of the remaining deal periods $V(t_j)$ must be added to the payment p_j as the reward for continuing, thus:

$$V(t_j-) = \min\{XS(t_j), P_d(t_j, t_{j+1})p_{j+1} + V(t_j)\}. \tag{10}$$

We may thus value this terminable diff swap by solving the PDE in the penultimate period $N-1$ with the terminal condition (9), substituting the resulting solution value at t_{N-2} into (10) to give a terminal condition for period $N-2$, and repeating this procedure, stepping backwards in time until we get the solution at $t_0 = 0$. In practice, a terminable diff swap may be sold with the margin m reduced so that the initial value is zero — to find this zero value margin is a root finding problem, albeit simple, on top of numerical valuation, and we do not consider it here.

We may of course allow additional option features. For example, we might consider a call option on a terminable diff swap, with maturity t_1 and exercise price K, so that we have the same terminal condition as the terminable diff swap in each period except in period 1, for which we have the call option payoff

$$V(t_1-) = \min\{K, V(t_1)\}. \tag{11}$$

We solve such a deal (with $K := 0$, as is usually the case) in §5.

3 Babbs' Cross-Currency Term Structure Model

To completely specify the valuation problem for any of the deals discussed above, we need to specify a term structure model. The classical term structure models are concerned with contingent claims in only one, so-called *domestic*, economy. Once we include a second economy, which we call *foreign*, we have different term structure processes and risk preferences in each economy, and a rate of exchange between their currencies. Until recently, models for pricing derivatives in this setting either assumed constant interest rates and a stochastic exchange rate, or modelled stochastic interest rates in the same manner as Merton [17]. Neither of these approaches is satisfactory: the first approach is appealing only for its simplicity and cannot be justified empirically; the second suffers all the flaws of the Merton model — it does not model the full term structure, and as a result cannot support American-style payoffs, which require a continuum of bond price maturities. See Amin and Jarrow [1] for a review and references to empirical work.

Amin and Jarrow [1] extend the Heath, Jarrow and Morton [12] Gaussian model, and Babbs [5, 4, 6] applies his similar model of [2], both in an attempt to extend full term-structure models to the cross-currency case. We consider here the Babbs model, in particular his 'extended Vasicek' specialisation. For more details on the model see Babbs [5], or, in the present context, Hutton [14].

3.1 Model structure

We start by specifying term structure dynamics in terms of the zero-coupon bond prices P_d and P_f in both the domestic and foreign economy, and the exchange rate S between their currencies, in terms of the *objective* probability measure. By convention, we value assets and derivative securities in terms of the domestic currency, and our exchange rate is the domestic price per unit of foreign currency.

We specify our bond price and exchange rate Ito processes as satisfying the stochastic differential system

$$
\begin{aligned}
\frac{d\mathbf{P}_d(t,T)}{\mathbf{P}_d(t,T)} &= \left[\mathbf{r}_d(t) + \boldsymbol{\theta}_d(t)\boldsymbol{\sigma}_d(t,T)\right] dt + \boldsymbol{\sigma}_d(t,T)d\mathbf{Z}_d(t) \\
\frac{d\mathbf{P}_f(t,T)}{\mathbf{P}_f(t,T)} &= \left[\mathbf{r}_f(t) + \boldsymbol{\theta}_f(t)\boldsymbol{\sigma}_f(t,T)\right] dt + \boldsymbol{\sigma}_f(t,T)d\mathbf{Z}_f(t) \\
\frac{d\mathbf{S}(t)}{\mathbf{S}(t)} &= \left[\mathbf{r}_d(t) - \mathbf{r}_f(t) + \boldsymbol{\theta}_S(t)\boldsymbol{\sigma}_S(t)\right] dt + \boldsymbol{\sigma}_S(t)d\mathbf{Z}_S(t),
\end{aligned}
\tag{12}
$$

where σ_d, σ_f and σ_S represent bond price and exchange rate *volatilities*, so that $\sigma_d(t,t) = \sigma_f(t,t) = 0$ for all $t \in [0,T]$ and are strictly positive elsewhere; θ_d, θ_f and θ_S are related to the market prices of risk of domestic and foreign bonds and exchange rate[2]; $\mathbf{Z}_d$, $\mathbf{Z}_f$ and $\mathbf{Z}_S$ are imperfectly correlated Wiener processes with correlation processes

$$
\begin{aligned}
d\mathbf{Z}_d(t)d\mathbf{Z}_f(t) &= \boldsymbol{\rho}_{df}(t)dt \\
d\mathbf{Z}_d(t)d\mathbf{Z}_S(t) &= \boldsymbol{\rho}_{dS}(t)dt \\
d\mathbf{Z}_f(t)f\mathbf{Z}_S(t) &= \boldsymbol{\rho}_{fS}(t)dt.
\end{aligned}
\tag{13}
$$

[2] θ_d is exactly the market price of risk for domestic bonds, but θ_f is the market price of foreign bond risk in foreign currency.

3.1.1 'Separable Extended Vasicek' restriction

From the above general specification of the term structure dynamics we may derive the resulting process for the short rate in either economy. See Hull [13] for this result, but the process for the short rate may be non-Markovian, because of path-dependent integrals involving bond price volatility in the drift, and thus in general the current short rate is not sufficient to determine the current term structure. However, if we restrict the *deterministic* volatility to be independent of the bond price and of the functional form

$$\sigma_k(t,T) := [G_k(T) - G_k(t)]\,\lambda_k(t) \qquad k = d, f \tag{14}$$

for arbitrary functions G_k and λ_k, we eliminate any path-dependency in the short rate process $\mathbf{r}_k$, which then satisfies

$$d\mathbf{r}_k(t) = \left\{\mu_k'(t) - \frac{G_k''(t)}{G_k'(t)}\left[\mu_k(t) - \mathbf{r}_k(t)\right]\right\} dt - G_k'(t)\lambda_k(t)\left[\boldsymbol{\theta}_k(t)dt + d\mathbf{Z}_k(t)\right] \qquad k = d, f, \tag{15}$$

where

$$\mu_k(t) := F_k(0,t) + G_k'(t)\int_0^t \left[G_k(t) - G_k(s)\right]\lambda_k^2(s)ds$$

and $F_k(0,t)$ is the instantaneous forward rate for time t at time zero, which is determined by the initial term structure. Babbs [2] shows that this volatility specification is in fact a necessary and sufficient condition for the existence of a single state variable for the term structure. Furthermore, the resulting short rate process (15) is recognisable as an extended Vasicek-type model, i.e. of the (risk-adjusted) form

$$d\mathbf{r}(t) = (\alpha(t) - \beta(t)\mathbf{r}(t))dt + \sigma(t)d\mathbf{Z}(t), \tag{16}$$

where α/β is the *long run mean level*, β is the *mean reversion rate* and σ is the *variability* of the short rate.

Separable models If we also ask for the more easily specifiable property that the bond price volatility be separable into a product of functions of time to maturity $T-t$ and calendar time t, then it follows (see e.g. Babbs [2]) that this is equivalent to requiring

$$G_k(t) = \frac{1 - e^{-\xi_k t}}{\xi_k} \qquad k = d, f, \tag{17}$$

for some constants ξ_d and ξ_f, and

$$\lambda_k(t) = e^{\xi_k t}\kappa_k(t) \qquad k = d, f, \tag{18}$$

where κ_k is the afore-mentioned function of calendar time. These constants may be interpreted in the context of the extended Vasicek short rate process (15) thus: the *mean reversion rate* of the short rate is $-G_k''(t)/G_k'(t)$, which on substituting for G_k from (17) yields ξ_k; the *variability* of the short rate is $G_k'(t)\lambda_k(t)$, which on substituting from (17) and (18) yields $\kappa_k(t)$.

3.2 Risk-adjusted processes in domestic terms

We now express the three price processes in domestic terms. First, we write each of the correlated Wiener increments $d\mathbf{Z}_d$, $d\mathbf{Z}_f$ and $d\mathbf{Z}_S$ of (12) as a linear combination of increments of three independent (*orthogonal*) Wiener processes $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ thus:

$$d\mathbf{Z}_k(t) = \sum_{j=1}^{3} \alpha_{kj}(t) d\mathbf{W}_j(t) \qquad k = d,\ f,\ S. \tag{19}$$

Substituting this into (13) and using the fact that the $\mathbf{W}_j$ are uncorrelated, we see that the matrix $A := [\alpha_{kj}]_{j=1,2,3}^{k=d,f,S}$ must be the square root of the correlation matrix $[\rho_{kl}]_{k,l=d,f,S}$[3].

In everything that follows, we value securities under the *numeraire* $P_d(0,H)$, i.e. normalised by the initial domestic bond price at some suitably distant horizon date H, since then price processes are martingales under the risk-adjusted probability measure. Under this measure there are no arbitrage opportunities between domestic and foreign bonds of any maturities up to H.

To specify the risk-adjusted measure on continuous paths of the independent coordinate Wiener process $\mathbf{W} := (\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3)'$ in 3-space, we utilize the Radon-Nikodym derivative process

$$\exp\left\{-\int_0^t [\boldsymbol{\theta}_d(s) + \frac{1}{2}\boldsymbol{\theta}_d^2(s) + \boldsymbol{\theta}_f(s) + \frac{1}{2}\boldsymbol{\theta}_f^2(s) + \boldsymbol{\theta}_S(s) + \frac{1}{2}\boldsymbol{\theta}_S^2(s)] ds\right\}$$

on $0 \leq t \leq H$ and apply Girsanov's theorem to obtain the independent coordinate Wiener process $\tilde{\mathbf{W}} := (\tilde{\mathbf{W}}_1, \tilde{\mathbf{W}}_2, \tilde{\mathbf{W}}_3)'$ under the risk-adjusted measure , where $d\tilde{\mathbf{W}}_k = d\mathbf{W}_k + \boldsymbol{\theta}_k dt$, $k = d, f, S$, and remove the market price of risk terms in the risk-adjusted analogue of (12).

It turns out that the two bond prices and the exchange rate may then be captured by three driftless Gaussian state variable processes, a property which leads to simpler numerical procedures, either for computation of the value by integration or for solution of the PDE.

Theorem 1 *The domestic and foreign bond prices and the exchange rate are given in terms of driftless Gaussian state variables X_d, X_f, X_S by*

$$\begin{aligned}
\mathbf{P}_d(t,T) &= \frac{P_d(0,T)}{P_d(0,t)} \exp\left\{[G_d(T) - G_d(t)] \left[\mathbf{X}_d(t) - \int_0^t h_d(t,T,H,s) ds\right]\right\} \\
\mathbf{P}_f(t,T) &= \frac{P_f(0,T)}{P_f(0,t)} \exp\left\{[G_f(T) - G_f(t)] \left[\mathbf{X}_f(t) - \int_0^t h_f(t,T,H,s) ds\right]\right\} \\
\mathbf{S}(t) &= \frac{P_f(0,t)S(0)}{P_d(0,t)} \exp\left\{-G_d(t)\mathbf{X}_d(t) + G_f(t)\mathbf{X}_f(t) + \mathbf{X}_S(t) - \frac{1}{2}\int_0^t h_S(t,H,s) ds\right\},
\end{aligned} \tag{20}$$

where

$$h_d(t,T,H,s) := \left(\frac{G_d(T) + G_d(t)}{2} - G_d(H)\right) \lambda_d^2(s)$$

[3] Since the square root has three free parameters, we follow Babbs [5] in choosing $\alpha_{d1} = \alpha_{d2} = \alpha_{d3} = 0$.

$$
\begin{aligned}
h_f(t,T,H,s) &:= \lambda_f(s)\left[\frac{\sigma_f(s,T)+\sigma_f(s,t)}{2} + \sigma_S(s)\rho_{fS}(s) - \sigma_d(s,H)\rho_{df}(s)\right] \\
h_S(t,H,s) &:= \sigma_f^2(s,t) + \sigma_S^2(s) - \sigma_d^2(s,t) + 2\sigma_d(s,t)\sigma_d(s,H) + 2\sigma_f(s,t)\rho_{fS}(s)\sigma_S(s) \\
&\quad -2\sigma_f(s,t)\rho_{df}(s)\sigma_d(s,H) - 2\sigma_S(s)\rho_{dS}(s)\sigma_d(s,H) \qquad (21)
\end{aligned}
$$

and the state variables X_d, X_f and X_S are defined by

$$
\begin{aligned}
\mathbf{X}_d(t) &:= \sum_{j=1}^{3}\int_0^t \alpha_{dj}(s)\lambda_d(s)d\tilde{\mathbf{W}}_j(s) \\
\mathbf{X}_f(t) &:= \sum_{j=1}^{3}\int_0^t \alpha_{fj}(s)\lambda_f(s)d\tilde{\mathbf{W}}_j(s) \\
\mathbf{X}_S(t) &:= \sum_{j=1}^{3}\int_0^t \left[\alpha_{dj}(s)G_d(s)\lambda_d(s) - \alpha_{fj}(s)G_f(s)\lambda_f(s) + \alpha_{Sj}(s)\sigma_S(s)\right]d\tilde{\mathbf{W}}_j(s), \qquad (22)
\end{aligned}
$$

where $\tilde{\mathbf{W}}_j(t)$ is a Wiener process under the risk-adjusted probability measure.

Proof: See Babbs [5] for the original proof, or Hutton [14]. The proof uses Ito's lemma to derive the log-price processes, which have constant coefficients and so are simple to integrate. ■

3.3 Pricing European derivative securities

We now give a PDE which any derivative security must satisfy between cash flow dates in Babbs' model. The following lemma gives the variances and covariances of the random variables $\mathbf{X}_d(t)$, $\mathbf{X}_f(t)$ and $\mathbf{X}_S(t)$, the integrands of which will essentially form the PDE coefficients and will also enable us to place sensible bounds on the underlying variables of this PDE when we come to considering numerical solution in § 4.1.

Lemma 1 *The driftless Gaussian processes $\mathbf{X}_d$, $\mathbf{X}_f$ and $\mathbf{X}_S$ defined by (22) have the following variances and covariances at time $t \in [0,H]$:*

$$
\begin{aligned}
\mathrm{var}\left[\mathbf{X}_d(t)\right] &= \int_0^t \lambda_d^2(s)ds \\
\mathrm{var}\left[\mathbf{X}_f(t)\right] &= \int_0^t \lambda_f^2(s)ds \\
\mathrm{var}\left[\mathbf{X}_S(t)\right] &= \int_0^t H^{SS}(s)ds \\
\mathrm{cov}\left[\mathbf{X}_d(t),\mathbf{X}_f(t)\right] &= \int_0^t H^{df}(s)ds \\
\mathrm{cov}\left[\mathbf{X}_d(t),\mathbf{X}_S(t)\right] &= \int_0^t H^{dS}(s)ds \\
\mathrm{cov}\left[\mathbf{X}_f(t),\mathbf{X}_S(t)\right] &= \int_0^t H^{fS}(s)ds, \qquad (23)
\end{aligned}
$$

where the functions H^{SS}, H^{df}, H^{dS} and H^{fS} are defined by

$$
\begin{aligned}
H^{SS}(s) &:= G_d^2(s)\lambda_d^2(s) + G_f^2(s)\lambda_f^2(s) + \sigma_S^2(s) - 2\rho_{df}(s)G_d(s)\lambda_d(s)G_f(s)\lambda_f(s) \\
&\quad +2\rho_{dS}(s)G_d(s)\lambda_d(s)\sigma_S(s) - 2\rho_{fS}(s)G_f(s)\lambda_f(s)\sigma_S(s) \\
H^{df}(s) &:= \rho_{df}(s)\lambda_d(s)\lambda_f(s) \\
H^{dS}(s) &:= \lambda_d(s)\left[G_d(s)\lambda_d(s) - \rho_{df}(s)G_f(s)\lambda_f(s) + \rho_{dS}(s)\sigma_S(s)\right] \\
H^{fS}(s) &:= \lambda_f(s)\left[\rho_{df}(s)G_d(s)\lambda_d(s) - G_f(s)\lambda_f(s) + \rho_{fS}(s)\sigma_S(s)\right].
\end{aligned} \tag{24}
$$

Proof: See Babbs [5] for the original proof, or Hutton [14]. However, it is simply an application of Fubini's theorem to take expectations through the integrals $\tilde{\mathbb{E}}\left[\mathbf{X}_k(t)\mathbf{X}_l(t)|\,\mathbf{X}(0)\right]$ for $k, l = d, f, S$, with $\mathbf{X}_k(t)$ defined by (22). ■

We now give a PDE for any European-style derivative security whose payoff is a function of the domestic and foreign bond prices and exchange rate and hence in turn the state variables X_d, X_f and X_S. The closed form expressions of Theorem 1 for the bond prices and exchange rate enable us to express the terminal payoff and boundary conditions, formulated in terms of bond prices and rates, in terms of the state variables X_d, X_f and X_S.

Theorem 2 *Let $V := V(X_d, X_f, X_S, t)$ denote the domestic value function of a security with a terminal payoff measurable with respect to information σ-field at T and no intermediate payments, and assume that $V \in C^{2,1}\left(\mathbb{R}^3 \times [0,T)\right)$. Then the normalised domestic value function, defined by*

$$
V^*(t) := \frac{V(t)}{P_d(t,T)}, \tag{25}
$$

satisfies the PDE

$$
\frac{1}{2}\lambda_d^2\frac{\partial^2 V^*}{\partial X_d^2} + \frac{1}{2}\lambda_f^2\frac{\partial^2 V^*}{\partial X_f^2} + \frac{1}{2}H^{SS}\frac{\partial^2 V^*}{\partial X_S^2} + H^{df}\frac{\partial^2 V^*}{\partial X_d \partial X_f} + H^{dS}\frac{\partial^2 V^*}{\partial X_d \partial X_S} + H^{fS}\frac{\partial^2 V^*}{\partial X_f \partial X_S} + \frac{\partial V^*}{\partial t} = 0 \tag{26}
$$

on $\mathbb{R}^3 \times [0,T)$, where H^{SS}, H^{df}, H^{dS} and H^{fS} are defined by (24).

Proof: See Babbs [6] for the original proof or Hutton [14] for more details. The proof is straightforward though: under the risk-adjusted measure, the normalised price process of a traded European security is a martingale, so that, since it is an Ito process, it must have zero drift. Calculating the drift from Ito's lemma and setting it to zero gives us the PDE (26). ■

Babbs [5] shows that the value $V(t)$ at time t of a derivative security which pays $\psi(X_d, X_f, X_S)$ at time T is the discounted expected payoff

$$
V(X_d, X_f, X_S, t) = P_d(X_d(t), t, T)\tilde{\mathbb{E}}\left[\psi(\mathbf{X}_d(T), \mathbf{X}_f(T), \mathbf{X}_S(T))|X_d(t), X_f(t), X_S(t)\right], \tag{27}
$$

which is, after normalisation, the solution to (26) with the boundary condition $V(T) = \psi$. From a numerical point of view, for a standard European-style derivative security, we may either integrate (27) numerically, exploiting the Gaussian state variables, or solve the PDE with the appropriate boundary conditions.

3.4 Modelling issues

There is of course as much choice for the term structure model as in the single currency case. An important consideration for the cross-currency case is that of dimensionality — any more than one state variable for each term structure process and for the exchange rate would make numerical valuation computationally very demanding. As it is, a single factor model gives us a three state variable (plus time) PDE to solve, which is computationally non-trivial.

The choice of the extended Vasicek form gives us lognormal bond prices, which holds out the possibility of analytic solutions to many European-style derivatives. Parametrisation of the model is also important — a short rate model would require a two step solution procedure: firstly, solving for the zero-coupon bond price as a function of the short rate, and secondly using this to express the terminal condition as a function of the short rate. Furthermore, the PDE with the short rates as state variables has first and zero order derivatives, and so is more difficult to solve numerically.

Under Babbs' model we avoid this first step, since boundary conditions are expressed in terms of the abstract Gaussian state variables (22) by the bond price formulae of Theorem 1 — although inspection of these formulae reveals that this is not altogether trivial from a numerical point of view, many integrals and exponentials must be calculated and, if this is not done efficiently, can represent a significant overhead. One disadvantage of this parametrisation is that the state variables are not observable in the market, making interpretation of the resulting solution more difficult away from $t = 0$.

Probably the most serious fault in extended Vasicek-style models is that they allow negative interest rates. However, Babbs [2] shows, by valuing a contingent claim that pays only when short rates are negative for certain realistic parameter values to find very low values relative to the payoff (of the order of a basis point, i.e. 0.01%), that this model feature has a quite small effect on derivative valuation.

We will not discuss the calibration of Babbs' model to market data in detail here, as it is not our area of expertise. Suffice it to say that interest rate volatilities of the form utilized (as in Figure 3) can be fitted independently from analytic formula for suitably liquid instruments, such as foreign and domestic swaptions, and correlation data must be estimated historically.

4 Discretisation and Solution of the PDE

We next describe the numerical solution procedure — including localisation and discretisation of the PDE in a swap period — used to produce a discrete system on a finite domain, as well as the specification of data and the evaluation of bond prices and exchange rate.

4.1 Localisation of the PDE

We restrict the spatial domain $\mathbb{R}^3$ to a finite region, which we denote[4]

$$[L_x, U_x] \times [L_y, U_y] \times [L_z, U_z]. \tag{28}$$

The lower and upper bounds on the space variables L_x, L_y, L_z, U_x, U_y and U_z should be chosen in each period to be 'large enough' so as not to introduce significant errors at the boundary. To specify this precisely requires lengthy analysis, so we take an intuitive probabilistic approach. At any instant the state variables are correlated Gaussian with mean zero, so that we may find a confidence interval about zero in $\mathbb{R}^3$ for their position at any future time, which we take to be our truncated state variable region. We take as our confidence level three standard deviations[5], where the required variances are given by Lemma 1, and the resulting monotonic increasing time-dependent confidence intervals are plotted for specimen data in Figure 4. For simplicity in computing the bounds in a period, we take as our standard deviation that at the end of the last non-trivial period, t_{N-1}. Thus in every period we choose

$$\begin{aligned} [L_x, U_x] &:= [-3\mathrm{var}[\mathbf{X}_d(t_{N-1})], 3\mathrm{var}[\mathbf{Y}_d(t_{N-1})]] \\ [L_y, U_y] &:= [-3\mathrm{var}[\mathbf{X}_f(t_{N-1})], 3\mathrm{var}[\mathbf{Y}_f(t_{N-1})]] \\ [L_z, U_z] &:= [-3\mathrm{var}[\mathbf{X}_S(t_{N-1})], 3\mathrm{var}[\mathbf{X}_S(t_{N-1})]], \end{aligned} \tag{29}$$

where the variances are given by (23).

A more sophisticated approach to bound setting would be to allow for different bounds in each period, increasing according to the variance $\mathrm{var}(\mathbf{X}(t_j))$. This was attempted in Hutton [14] — the differing grid points between successive periods complicates matters when computing the recursive terminal condition between them, necessitating linear interpolation to compute $V(t_j+)$, and this was found to produce numerical difficulties.

This localisation is justified as long as we impose the growth condition that the payoff is at most exponential, but we do not attempt here to formulate this more precisely. Note that bond price, LIBOR rates and exchange rates are exponential functions of X_d, X_f and X_S, so this is not a problem here.

4.1.1 Boundary conditions

We must also specify values on the boundaries of the spatial variables, i.e. at $X_d(t) = L_x, \ldots, X_S(t) = U_z$ for all t in $[t_{j-1}, t_j)$. The difficulty with choosing these boundary conditions is that, for an arbitrary payoff function, they are not known, and if we are

[4] For notational convenience we associate X_d, X_f and X_S with the canonical space variables x, y and z respectively.

[5] The probability that a zero-mean Gaussian random variable lies outside three standard deviations of zero is, from tables in [7], approximately .0026.

not to perform quite detailed analysis for each different type of deal, we can only posit quite general approximate boundary conditions. Examples which we investigate include simply setting first or second derivatives constant at the boundary and a more complicated 'stopped process' boundary condition, where we stop the processes $\mathbf{X}_d(t)$, $\mathbf{X}_f(t)$ and $\mathbf{X}_S(t)$ when one hits the boundary, hence the value on the boundary is simply the discounted payoff for current values of the state variables. In §5.2 we present results from some different specifications, the variation between which proves to be gratifyingly small.

4.2 Discretisation of a general 3-D quasi-linear parabolic PDE with Dirichlet conditions

We now formulate the finite difference discretisation of a general quasi-linear PDE, of which the PDE (26) is a special case. We allow for specification of the discretisation scheme, be it explicit, implicit or Crank-Nicolson, by means of setting a parameter[6] $\theta \in [0,1]$. θ may be a function of the discretised state variables and time, to allow for *Alternating Direction Implicit* (ADI) methods, step length on an axis may vary along that axis and the coefficients are functions of at most time — although this is simply for notational convenience and all steps follow through for space-dependent coefficients.

So we have a general PDE

$$\alpha(t)u_{xx} + \beta(t)u_{yy} + \gamma(t)u_{zz} + \delta(t)u_{xy} + \epsilon(t)u_{xz} + \zeta(t)u_{yz} - u_t = 0 \tag{30}$$

on the domain $[L_x, U_x] \times [L_y, U_y] \times [L_z, U_z] \times [L_t, U_t)$, where t now represents time until the end of the period (t_j – calendar time); hence the sign on the time partial derivative. We write the value function solution at variable width mesh points

$$u\left(L_x + \sum_{n=1}^{i}(\Delta x)_n,\ L_y + \sum_{n=1}^{j}(\Delta y)_n,\ L_z + \sum_{n=1}^{k}(\Delta z)_n,\ L_t + \sum_{n=1}^{m}(\Delta t)_n\right) \tag{31}$$

as $u^m_{i,j,k}$, adopting a convention that $\sum_{n=1}^{0} := 0$, where

$$\begin{aligned} i &\in \{0,1,\ldots,I\} := \mathcal{I}, \quad j \in \{0,1,\ldots,J\} := \mathcal{J}, \\ k &\in \{0,1,\ldots,K\} := \mathcal{K}, \quad m \in \{0,1,\ldots,M\} := \mathcal{M}. \end{aligned}$$

We write $\alpha(t), \ldots, \zeta(t)$ as $\alpha^m, \ldots, \zeta^m$. Finite difference approximations to the partial derivatives in (30) at the point indexed by (i,j,k,m), in the interior of the index domain $\mathcal{I} \times \mathcal{J} \times \mathcal{K} \times \mathcal{M}$, are given by

$$\begin{aligned} u_{xx} &\approx \theta_1 \left(\frac{u^m_{i+1,j,k} - 2u^m_{i,j,k} + u^m_{i-1,j,k}}{(\Delta x)_i^2} \right) + (1-\theta_1) \left(\frac{u^{m-1}_{i+1,j,k} - 2u^{m-1}_{i,j,k} + u^{m-1}_{i-1,j,k}}{(\Delta x)_i^2} \right) \\ u_{xy} &\approx \theta_4 \left(\frac{u^m_{i+1,j+1,k} - u^m_{i+1,j-1,k} - u^m_{i-1,j+1,k} + u^m_{i-1,j-1,k}}{4(\Delta x)_i (\Delta y)_j} \right) \end{aligned}$$

[6] Note that θ no longer denotes the market price of risk.

$$
\begin{aligned}
&+(1-\theta_4)\left(\frac{u^{m-1}_{i+1,j+1,k}-u^{m-1}_{i+1,j-1,k}-u^{m-1}_{i-1,j+1,k}+u^{m-1}_{i-1,j-1,k}}{4(\Delta x)_i(\Delta y)_j}\right)\\
u_t &\approx \frac{u^m_{i,j,k}-u^{m-1}_{i,j,k}}{(\Delta t)_m}, \qquad (32)
\end{aligned}
$$

with u_{yy}, u_{zz}, u_{xz}, u_{yz} approximated (with parameters θ_2, θ_3, θ_5 and θ_6) in an analogous manner. All of these approximations are accurate to second order in the step length apart from the time derivative, which is first order accurate. The parameter θ_n determines the discretisation scheme: $\theta_n = 0, \frac{1}{2}, 1$ gives the explicit, Crank-Nicolson and implicit finite difference scheme for the corresponding derivative respectively. Substituting the approximations of (32) into (30) gives[7], suppressing arguments of θ_n and $\Delta x,\ldots,\Delta t$,

$$
\begin{aligned}
&\delta^m\frac{(1-\theta_4)}{4\Delta x\Delta y}u^{(m-1)}_{i-1,j-1,k}+\epsilon^m\frac{(1-\theta_5)}{4\Delta x\Delta z}u^{(m-1)}_{i-1,j,k-1}+\alpha^m\frac{(1-\theta_1)}{(\Delta x)^2}u^{(m-1)}_{i-1,j,k}\\
&-\epsilon^m\frac{(1-\theta_5)}{4\Delta x\Delta z}u^{(m-1)}_{i-1,j,k+1}-\delta^m\frac{(1-\theta_4)}{4\Delta x\Delta y}u^{(m-1)}_{i-1,j+1,k}+\zeta^m\frac{(1-\theta_6)}{4\Delta y\Delta z}u^{(m-1)}_{i,j-1,k-1}\\
&+\beta^m\frac{(1-\theta_2)}{(\Delta y)^2}u^{(m-1)}_{i,j-1,k}-\zeta^m\frac{(1-\theta_6)}{4\Delta y\Delta z}u^{(m-1)}_{i,j-1,k+1}+\gamma^m\frac{(1-\theta_3)}{(\Delta z)^2}u^{(m-1)}_{i,j,k-1}\\
&+\left(-2\alpha^m\frac{(1-\theta_1)}{(\Delta x)^2}-2\beta^m\frac{(1-\theta_2)}{(\Delta y)^2}-2\gamma^m\frac{(1-\theta_3)}{(\Delta z)^2}+\frac{1}{\Delta t}\right)u^{(m-1)}_{i,j,k}\\
&+\gamma^m\frac{(1-\theta_3)}{(\Delta z)^2}u^{(m-1)}_{i,j,k+1}-\zeta^m\frac{(1-\theta_6)}{4\Delta y\Delta z}u^{(m-1)}_{i,j+1,k-1}+\beta^m\frac{(1-\theta_2)}{(\Delta y)^2}u^{(m-1)}_{i,j+1,k}\\
&+\zeta^m\frac{(1-\theta_6)}{4\Delta y\Delta z}u^{(m-1)}_{i,j+1,k+1}-\delta^m\frac{(1-\theta_4)}{4\Delta x\Delta y}u^{(m-1)}_{i+1,j-1,k}-\epsilon^m\frac{(1-\theta_5)}{4\Delta x\Delta z}u^{(m-1)}_{i+1,j,k-1}\\
&+\alpha^m\frac{(1-\theta_1)}{(\Delta x)^2}u^{(m-1)}_{i+1,j,k}+\epsilon^m\frac{(1-\theta_5)}{4\Delta x\Delta z}u^{(m-1)}_{i+1,j,k+1}+\delta^m\frac{(1-\theta_4)}{4\Delta x\Delta y}u^{(m-1)}_{i+1,j+1,k}\\
&+\text{ the same again with } (m-1)\to(m) \textit{ and } (1-\theta_n)\to\theta_n,\ \text{ except for the } u^{(m-1)}_{i,j,k} \text{ term}\\
&+\left(-2\alpha^m\frac{\theta_1}{(\Delta x)^2}-2\beta^m\frac{\theta_2}{(\Delta y)^2}-2\gamma^m\frac{\theta_3}{(\Delta z)^2}-\frac{1}{\Delta t}\right)u^m_{i,j,k}=0\\
&\qquad i\in\{1,\ldots,I-1\},\ldots,k\in\{1,\ldots,K-1\},\ m\in\{1,\ldots,M\}. \qquad (33)
\end{aligned}
$$

We now write the unwieldy expression (33) in vector form. This is rather complicated algebraically, since we have four variables, so we omit the details — see Hutton [14] for more information. However, we collapse each index by replacing it with a vector, indexed

[7] We note that a computer algebra system, such as Maple or Mathematica, is invaluable for such work, especially when the output can be translated into C code.

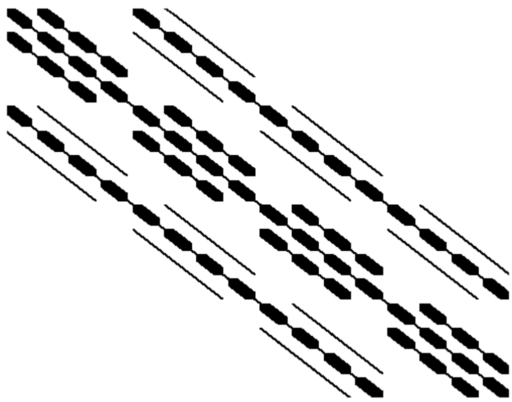

Figure 1: Bitmap of typical F^m or G^m matrix, $I = J = K = 5$

by the remaining indices, each entry of which corresponds to one value of that index, which gives us the following linear equation system.

Putting $u := (u^1 \ldots u^M)'$ and $u^m := (u_{0,0,0}, u_{0,0,1}, \ldots, u_{I,J,K-1}, u_{I,J,K})'$, equations (33) become

$$Hu + \phi = 0, \tag{34}$$

where

$$H := \begin{pmatrix} G^1 & & & \\ F^2 & G^2 & & \\ & \ddots & \ddots & \\ & & F^M & G^M \end{pmatrix} \tag{35}$$

and ϕ is a vector that contains boundary condition information — for details see Hutton [14]. The matrices F^m and G^m are square, symmetric and of size $(I-1)(J-1)(K-1)$, and their general bitmap (pattern of non-zeroes) is shown in Figure 1. The structure consists of diagonal bands of non-zero elements, arranged in a nested tridiagonal structure.

To solve the linear system (34), we do not solve it directly, but by forward substitution starting from the initial condition[8] u^0 thus:

$$\text{solve} \quad G^{m+1}u^{m+1} \quad = \quad \phi^{m+1} - F^{m+1}u^m \qquad m = 0, \ldots, M-1, \tag{36}$$

and so existence of a solution is determined by invertibility of the matrix G^m for each $m = 1, \ldots, M$. A well-known sufficient condition for any square matrix to be invertible is that it be strictly *diagonally dominant*[9], and if, for illustrative purposes, we put $\Delta x := \Delta y := \Delta z := \Delta$, then (see Hutton [14]) this condition reduces to the neat form

$$\frac{\Delta^2}{\Delta t} > \theta_4|\delta^m| + \theta_5|\epsilon^m| + \theta_6|\zeta^m|. \tag{37}$$

[8]Note that we have changed to a backward time variable, so that the initial condition is given by the usual terminal condition in forward time.

[9]A matrix is *diagonally dominant* if the absolute value of the diagonal element is greater than the sum of the absolute values of the off-diagonal elements in each row.

Diagonal dominance is satisfied in virtually all practical cases, and is always satisfied if the PDE has no cross-derivatives.

4.3 Solution of the Discrete Problem

Precisely how best to solve the discretised problem depends on the discretisation scheme used. In all cases, attention must be paid to exploiting the structure and sparsity of the typically very large square matrices F^m and G^m to achieve reasonable computing time and efficient use of computer memory.

4.3.1 The explicit method

Explicit methods are the simplest to implement and are memory-efficient. If we set $\theta_n := 0$ for all n the matrix G^m is simply the diagonal matrix $\text{diag}(\frac{1}{\Delta t})$ and so is trivial to invert — we see from (36) that u^{m+1} then depends *explicitly* on u^m. Putting $n := (I-1) = (J-1) = (K-1)$, at each time step m we have only to do two matrix multiplications, each of which takes $O(n^3)$ floating point operations since both $\text{diag}(\Delta t)$ and the matrix F^m have $O(n^3)$ non-zero elements. There are M time steps, so the total operations count is $O(Mn^3)$.

The main disadvantage of the explicit method is that it is not necessarily stable. For a version of (26) with no cross-derivatives (i.e. $\delta = \epsilon = \zeta = 0$), the criterion that guarantees stability at each time step m is that

$$\alpha^m \frac{\Delta t}{(\Delta x)^2} + \beta^m \frac{\Delta t}{(\Delta y)^2} + \gamma^m \frac{\Delta t}{(\Delta z)^2} \leq \frac{1}{2}. \tag{38}$$

No similar characterisation is known for the case of mixed derivatives, so in §5 we determine the critical time step experimentally — we find that (38) is very nearly sufficient in practice, since the coefficients of the mixed derivatives are relatively small. In any case, we have to take the number of time steps M of the order of the square of the number of space steps, so that the operation count for the explicit method is $O(n^5)$.

The approximation is accurate to second order in space and first order in time, inherited from the finite difference approximations (32). Of course if we take $M = O(n^2)$, as we must for stability, the method is second order accurate in time. Note that computer storage need be allocated only for the current and previous time step solution vectors.

4.3.2 General implicit methods

If $\theta_n(i,j,k,m) > 0$ for some i,j,k,m,n, the matrix G^m is not simply diagonal, and then, from (36), at each time step we have to solve a linear equation system involving the matrix $G^m x = b$. A possible approach is to adapt the general LU decomposition method to take advantage of the band diagonal[10] structure of G^m (see Figure 1) — the resulting L and

[10] A matrix is *band diagonal* if all non-zeroes lie in a diagonal band containing the diagonal.

U factors are both band diagonal lower and upper-triangular, and so computation associated with elements out of the diagonal band may be eliminated and storage requirements reduced. The total operation count is $O(n^7)$, instead of $O(n^9)$ for the standard LU algorithm — note that we have to recompute the LU factors at each time step because the PDE has time-dependent coefficients. The best of the simple implicit schemes (i.e. with θ constant) is the Crank-Nicolson method ($\theta_n = \frac{1}{2}$), which has the advantages of being second order accurate in time and unconditionally stable. However, on comparison with the $O(n^5)$ second order accurate in time explicit scheme, it is clear that implicit methods solved in the manner proposed here are uncompetitive, and storage of dense LU factors is impractical for all but small n, as we discuss in §5.1. We mention some alternatives to LU decomposition in §6.

4.4 Data Functions and Evaluation of Formulae

Before we can proceed with empirical tests of the terminable diff swap problem, we must supply data to the many and various functions involved, and consider how to evaluate the bond price and exchange rate functions.

4.4.1 Data functions

To specify the bond price and exchange rate functions (21), we supply the observed initial exchange rate $S(0)$ and a horizon date $H > t_N$ as positive constants and the observed initial term structures $P_d(0,T)$ and $P_f(0,T)$ as RCLL step functions approximating the observed initial term structures, for which we supply a set of grid points (a *time set*) $0 = \tau_0, \tau_1, \dots, \tau_n := H$ and corresponding positive bond price values constant for each $t \in [\tau_{j-1}, \tau_j)$. The bond price volatility functions $\sigma_d(t,T)$ and $\sigma_f(t,T)$ are defined by (14), so that we need to supply constant mean reversion rates ξ_d and ξ_f, and the time-dependent variabilities of each short rate, $\kappa_d(t)$ and $\kappa_f(t)$, as step functions, as described above. We also specify the exchange rate volatility $\sigma_S(t)$ as a step function. Finally, we specify the three correlation functions $\rho_{df}(t)$, $\rho_{dS}(t)$ and $\rho_{fS}(t)$ as step functions. Specimen bond prices, exchange rates and short rate variabilities are plotted in Figures 2 and 3.

4.4.2 Evaluating bond price and exchange rate formulae

All our possible expressions for terminal conditions are in terms of bond prices and the exchange rate, but the PDE (26) has X_d, X_f and X_S as state variables, so we must consider in detail the efficient evaluation of bond prices and exchange rate, given in Theorem 1, as functions of the state variables.

According to Theorem 1, we need to evaluate the following three integrals:

(i) $\int_0^t \lambda_d^2(u)du$,

(ii) $\int_0^t h_f(t,T,H,u)du$,

(iii) $\int_0^t h_S(t,H,u)du$.

The integrands are all products of functions of time with step functions of time, and we may calculate the integrals as a sum of integrals over the intervals in which the step function remains constant. In each integrand, we have a function $f(t)$ and a step function $g(t)$ which is a product of other RCLL step functions and so has a time set $\tau_1 := 0, \tau_2, \ldots, \tau_n$ given by the ordered union of the time sets of the step functions which comprise it. Putting $\tau_n := t$, we have $\int_0^t f(u)g(u)du = \sum_{j=1}^n g(\tau_{j-1}) \left[\int_{\tau_{j-1}}^{\tau_j} f(u)du\right]$. In this manner we proceed to calculate integrals (i)-(iii), trying in general to make the resulting expressions amenable to numerical evaluation, for example by multiplying out the product of two exponentials to give one exponential, since exponentials are costly to compute. Thus:-

(i)

$$\begin{aligned}\int_0^t \lambda_d^2(u)du &= \sum_{j=1}^n \int_{\tau_{j-1}}^{\tau_j} e^{2\xi_d u}\kappa_d^2(u)du \\ &= \sum_{j=1}^n \frac{\kappa_d^2(u)}{2\xi_d}(e^{2\xi_d\tau_j} - e^{2\xi_d\tau_{j-1}}).\end{aligned} \tag{39}$$

(ii) Similarly, although the expressions involved are lengthy and the reader is referred to Hutton [14] for details,

$$\begin{aligned}&\int_0^t h_f(t,T,H,u)du \\ &\quad= \sum_{j=1}^n \left[\frac{1}{\xi_f^2}(e^{\xi_f\tau_j} - e^{\xi_f\tau_{j-1}}) - \frac{1}{4\xi_f^2}(e^{\xi_f(2\tau_j-t)} - e^{\xi_f(2\tau_{j-1}-t)} + e^{\xi_f(2\tau_j-T)} - e^{\xi_f(2\tau_{j-1}-T)})\right]\kappa_f^2(\tau_{j-1}) \\ &\quad+ \sum_{j=1}^n \left[\frac{1}{\xi_f}(e^{\xi_f\tau_j} - e^{\xi_f\tau_{j-1}})\right]\kappa_f(\tau_{j-1})\sigma_S(\tau_{j-1})\rho_{fS}(\tau_{j-1}) \\ &\quad- \sum_{j=1}^n \left[\frac{1}{\xi_d\xi_f}(e^{\xi_f\tau_j} - e^{\xi_f\tau_{j-1}}) - \frac{1}{\xi_d(\xi_f+\xi_d)}(e^{(\xi_f+\xi_d)\tau_j-\xi_d H} - e^{(\xi_f+\xi_d)\tau_{j-1}-\xi_d H})\right] \\ &\qquad\times\kappa_f(\tau_{j-1})\kappa_d(\tau_{j-1})\rho_{fd}(\tau_{j-1}).\end{aligned} \tag{40}$$

(iii) Again, similar manipulations gives us

$$\begin{aligned}&\int_0^t h_S(t,H,u)du \\ &\quad= \sum_{j=1}^n \left[\frac{1}{\xi_f^2}\left(\tau_j - \tau_{j-1} - \frac{2}{\xi_f}(e^{\xi_f(\tau_j-t)} - e^{\xi_f(\tau_{j-1}-t)}) + \frac{1}{2\xi_f}(e^{2\xi_f(\tau_j-t)} - e^{2\xi_f(\tau_{j-1}-t)})\right)\right]\kappa_f^2(\tau_{j-1}) \\ &\quad+ \sum_{j=1}^n (\tau_j - \tau_{j-1})\sigma_S^2(\tau_{j-1}) \\ &\quad+ \sum_{j=1}^n \left[\frac{1}{\xi_d^2}\left(\tau_j - \tau_{j-1} - \frac{2}{\xi_d}(e^{\xi_d(\tau_j-H)} - e^{\xi_d(\tau_{j-1}-H)}) - \frac{1}{2\xi_d}(e^{2\xi_d(\tau_j-t)} - e^{2\xi_d(\tau_{j-1}-t)})\right.\right. \\ &\qquad\left.\left.+\frac{1}{\xi_d}(e^{\xi_d(2\tau_j-t-H)} - e^{\xi_d(2\tau_{j-1}-t-H)})\right)\right]\kappa_d^2(\tau_{j-1})\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\left[\frac{2}{\xi_f}(\tau_j-\tau_{j-1})-\frac{2}{\xi_f^2}(e^{\xi_f(\tau_j-t)}-e^{\xi_f(\tau_{j-1}-t)})\right]\kappa_f(\tau_{j-1})\rho_{fS}(\tau_{j-1})\sigma_S(\tau_{j-1})\\
&-\sum_{j=1}^{n}\left[\frac{2}{\xi_d\xi_f}(\tau_j-\tau_{j-1}-\frac{1}{\xi_d}(e^{\xi_d(\tau_j-H)}-e^{\xi_d(\tau_{j-1}-H)})-\frac{1}{\xi_f}(e^{\xi_f(\tau_j-t)}-e^{\xi_f(\tau_{j-1}-t)})\right.\\
&\quad\left.+\frac{1}{\xi_d+\xi_f}\left(e^{(\xi_d+\xi_f)\tau_j-\xi_f t-\xi_d H}-e^{(\xi_d+\xi_f)\tau_{j-1}-\xi_f t-\xi_d H}\right)\right]\kappa_f(\tau_{j-1})\rho_{fd}(\tau_{j-1})\kappa_d(\tau_{j-1})\\
&-\sum_{j=1}^{n}\left[\frac{2}{\xi_f}(\tau_j-\tau_{j-1})-\frac{2}{\xi_f^2}(e^{\xi_f(\tau_j-H)}-e^{\xi_f(\tau_{j-1}-H)})\right]\sigma_S(\tau_{j-1})\rho_{Sd}(\tau_{j-1})\kappa_d(\tau_{j-1}).
\end{aligned}
\tag{41}
$$

5 Numerical Results and Visualisation

In this section we present the results of the numerical valuation method proposed in §4, using Babbs' model with specimen financial data, applied to specific deals of the type discussed in §2. We also present various cross-sections through the resulting 4-D solution surface in a period.

5.1 Computational details

All results here were computed on an IBM RS/6000 590 serial computer with 128 MB of RAM running under AIX 3.2.5. The code was written in C with double precision arithmetic, using the IBM MASS library to speed up computation of exponentials required for bond prices and exchange rates at a grid point according to (21) and in turn (39)–(41), with inlining to speed up calls to these nested functions.

Since the explicit method requires many time steps for stability, it is important to do these efficiently. In the code, the solution vector u^{m+1} is computed from u^m simply by evaluating (33) with $\theta_n = 0$ for each (i, j, k), taking basic precautions to preserve efficiency, such as computing coefficients outside the main loop. Apart from the boundary condition experiment in Table 3, boundary conditions are set by simply extrapolating the new solution vector, i.e. the result of one explicit iteration, linearly to the boundary points. To test implicit methods we used the routine `bandec` and `banbks` for LU decomposition and substitution of band-diagonal matrices in Press *et al* [18]. However, the total memory requirement is prohibitive: the number of elements stored is $3n^5 + 3n^4 + n^3$, so that taking $n = 25$, for example, requires 244MB of RAM for double precision storage. For this reason, we do not pursue implicit methods further here, but for numerical results see Hutton [14]. All results that follow in §5.2 are for the explicit method.

5.2 Numerical Results

All deals valued here are based on 3 month pound sterling and U.S. dollar LIBOR (and hence have quarterly swap payments) and are quoted per unit of sterling (domestic) principal, with initial term structures, volatilities and all other data as specified in Hutton

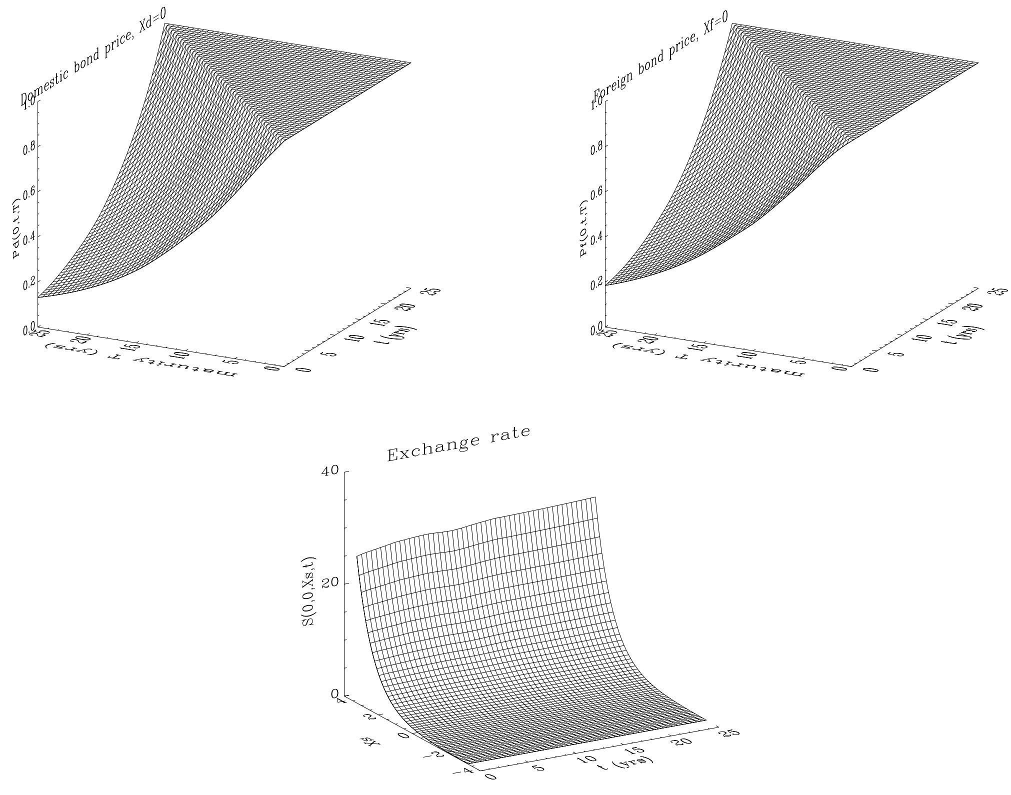

Figure 2: Bond prices $P_d(0,t,T)$ and $P_f(0,t,T)$ and exchange rate $S(0,0,X_S,t)$

[14][11], except that we take the initial exchange rate to be $S(0) := .64516129$. Some of the data supplied to the model are plotted in Figures 2 and 3 and the resulting 3σ-confidence intervals for the state variables, used to truncate the state variables as described in §4.1, are illustrated in Figure 4.

Vanilla cross-currency swap In Table 1, we give numerical results for the vanilla cross-currency swap with exchange of principal, defined by the terminal condition (7) with $Z_d := 1$, $Z_f := Z_d/S(0)$, $m_d := m_f := 0$, $\delta_j := .25$, $k_d := k_f := -1$. The state variable bounds for all three deals are fixed to the 10 year value, to aid comparison, and times quoted are for the 10 year deal. To estimate comparative times for the shorter deals, simply scale the times in the ratio of the deal lengths. As discussed in §2.2, this deal has zero initial value, and we see clearly the accuracy of the numerical solution. Clearly the accuracy deteriorates as the duration of deal lengthens, although all step widths are constant — this is simply accumulation of standard explicit method discretisation error, which is linear in the total number of time steps, but may also reflect the greater variance of the underlying processes in the later periods. In the case of the 1 year deal, we achieve accuracy of 1 bp (.0001), with $n = 40$ in 25 s, but for the 10 year deal we have to take

[11]This data was originally supplied by Simon Babbs, then of Midland Global Markets.

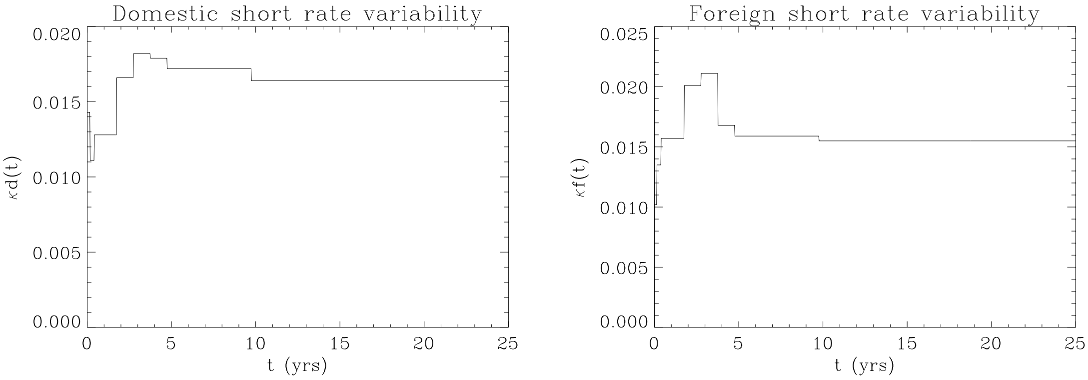

Figure 3: Prospective short rate variabilities $\kappa_d(t)$ and $\kappa_f(t)$

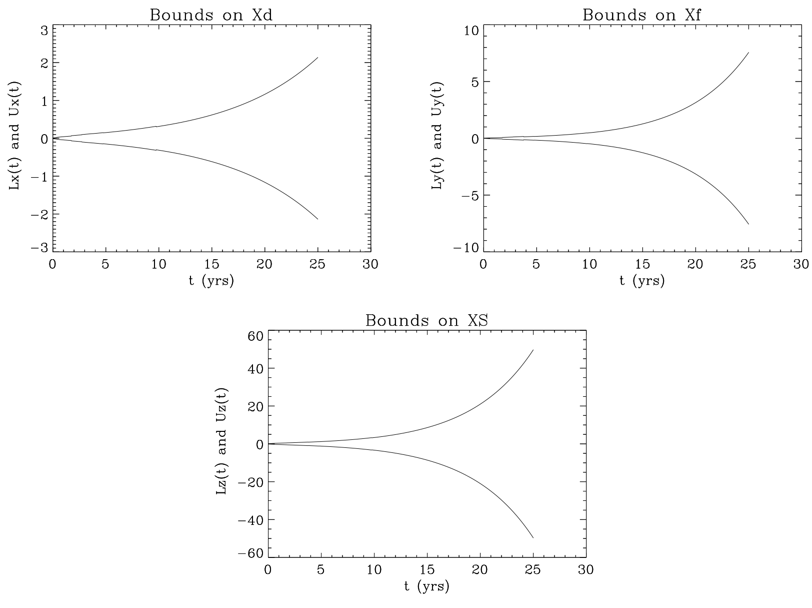

Figure 4: Bounds on Gaussian state variables $\mathbf{X}_d(t)$, $\mathbf{X}_f(t)$ and $\mathbf{X}_S(t)$

discretisation	1 year	5 years	10 years	
$M \times I \times J \times K$	V	V	V	time (s)
20×6^3	.001733	.001733	.154813	0.20
20×10^3	.000564	.009462	.050369	0.63
20×20^3	.000149	.002322	.012407	3.94
20×40^3	.000052	.000629	.003233	31.97
40×80^3	.000014	.000169	.000897	416.64
100×160^3	.000003	.000044	.000294	$\sim$ 7200.00
true value	0	0	0	

Table 1: Vanilla cross-currency swap with exchange of principal, deal value with varying discretisation and deal length.

discretisation $M \times I \times J \times K$	vanilla V	terminable $X = .01$ V	time (s)
20×6^3	-.086798	-.124087	0.21
20×10^3	-.086293	-.129086	0.57
20×20^3	-.085919	-.123529	3.90
20×40^3	-.085815	-.123216	31.29
40×80^3	-.085750	-.123057	411.12
100×160^3	-.085721	-.122993	$\sim$ 7300.00
true value	-.085712		

Table 2: Vanilla and Terminable Diff swap deal values with varying discretisation.

$n = 160$ and hence a solution time of about 2 hours, to approach a similar accuracy. Note that the explicit method stability requirement (38) affects the solution time significantly for higher spatial discretisations, which we need for the 10 year deal.

Diff swap Table 2 gives results for 10 year vanilla and terminable diff swaps, defined by the end-of-period payoff (10), with the known vanilla diff swap solution value computed from the formula in Babbs [5]. Solution times are essentially the same as for the vanilla swap of Table 1, and are given for the sake of completeness. We see that we achieve much better convergence than for the vanilla swap with exchange of principal of Table 1, with basis point accuracy in 31 s for the vanilla diff swap and apparently in 411 s for the terminable version. In both cases this improvement is due to the flatter solution surface than for the vanilla swap — from (10), we see that the vanilla diff swap part of the payoff in each period is flat with respect to the exchange rate, and hence to X_S.

In Table 3 we demonstrate the variation of the numerical solution with the boundary condition type, discussed in §4.1.1, for the 10 year vanilla diff swap. We take six examples,

discretisation $M \times I \times J \times K$	boundary condition 1 V	2 V	3 V	4 V	5 V	6 V
20×6^3	-.086460	-.087388	-.086798	-.087236	-.087360	-.087362
20×10^3	-.086117	-.086360	-.086293	-.086352	-.086360	-.086357
20×20^3	-.085856	-.085926	-.085919	-.085926	-.085926	-.085925
20×40^3	-.085777	-.085817	-.085815	-.085818	-.085817	-.085817
40×80^3	-.085717	-.085751	-.085750	-.085752	-.085752	-.085751
100×160^3	-.085689	-.085722	-.085721	-.085722	-.085722	unstable
true value	-.085712	-.085712	-.085712	-.085712	-.085712	-.085712

boundary condition key:

1. $V = 0$ on boundary.
2. $V(X_d, X_f, X_S, t) = P_d(X_d, t, U_t)V(X_d, X_f, X_S, U_t)$ on boundary ('stopped process' condition).
3. linear extrapolation to boundary points.
4. quadratic extrapolation to boundary points.
5. $\frac{\partial V(t)}{\partial X_k} = \frac{\partial V(U_t)}{\partial X_k}$ on X_k boundary
6. $\frac{\partial^2 V(t)}{\partial X_k^2} = \frac{\partial^2 V(U_t)}{\partial X_k^2}$ on X_k boundary.

Table 3: Diff swap deal value with varying discretisation and varying boundary conditions.

discretisation $M \times I \times J \times K$	1 year forward V	1 year call V	time (s)
20×6^3	-.071080	-.071350	0.10
20×10^3	-.067289	-.067973	0.30
20×20^3	-.067963	-.068175	2.03
20×40^3	-.067872	-.068007	16.21
80×80^3	-.067751	-.067871	377.36
200×160^3	-.067732	-.067850	$\sim$ 3200.00

Table 4: One year call and one year forward on a five year terminable diff swap with varying discretisation.

described in the key to the table[12]. Boundary conditions have some effect on solution time — solution time for type 1 (fastest) for the case $n = 80$ is 400 s, type 2 (i.e. stopped process, the slowest here) is 420 s, with the rest following closely the times in Table 2, so that $n = 80$ takes 410 s. We see from the table that, whilst there is some variation in solution value for coarser grids, variation is well within a basis point for grids finer than $n = 20$ apart from type 1. The type 1 case ($V = 0$ on the boundary) corresponds in practical code terms to not specifying a boundary condition, which makes for easy implementation and fast computation but is not reliable, since it is far from convergent as we increase M — in fact $u^M \to 0$ as $M \to \infty$. This remark illustrates the main problem with many approximate boundary conditions, including all those here — the resulting method is convergent with I but not necessarily with M.

Diff swaption Finally, in Table 4, we give results for a 1 year (zero strike price) call on a 5 year terminable diff swap i.e. a 1 year into 5 year diff swaption, mainly in order to demonstrate the simplicity of the PDE method when option structure is complicated. We also give, so as to determine the additional value to the counterparty of the call option, a 1 year forward 5 year terminable diff swap. Since the deals are shorter than 10 years, the spatial boundaries are tighter and hence we must take more time steps than for previous deals for stability — although we could have set the spatial boundaries to the 10 year case, as we did for the 1 year and 5 year vanilla swaps of Table 1. Basis point accuracy is apparently achieved in both cases within 380s, and the additional option value is about 1bp to the counterparty. That the difference should be small is unsurprising, since the counterparty has many future termination options and so the deal is already weighted in his favour, even without the additional option — this can be further appreciated by noting the limited range of the effect of the option in period 1 in Figures 5d–5f. We discuss the solution surface of this deal further in §5.3.

[12]Note that condition 3 is that used for all other tables, so the column headed '3' in Table 3 is the same as column 2 of Table 2.

5.3 Visualisation of the swaption value surface

In Figure 5 we give various plots of the (4D) value surface[13] as a function of the three state variables X_d, X_f, X_S and time, for the 1 year into 5 year swaption structure of the previous section, results for which are given in Table 4. We choose this deal because it incorporates most features of simpler deals — for example, after period 1 the deal is simply a terminable diff swap. In general, we see good agreement with the theoretical behaviour, which we try to illustrate in the following remarks.

Figures 5a–5c show the value surface for period 20 as a function of X_d, X_f and X_S respectively, with the remaining variables in each case set to their expected value of zero. The termination boundary[14] is clearly visible in Figures 5a and 5b — the payoff, or terminal condition, is 'capped' in the termination region — at $.01S(X_S, t_j)$ at the end of the period. In Figure 5c, variation in X_S cannot take the value into the termination region, but clearly the shape of the surface is influenced by possible termination through variation in X_d and X_f from zero.

Figures 5d–5f show the same plots but for period 1. In Figures 5d and 5e the effect of the option to buy the swap is apparent — the value surface is 'capped' at the end of the period at zero, but in Figure 5f the variables X_d and X_f are set so that the value lies strictly in the buy region[15]. The buy region is simply a section of the terminable diff swap surface of period 2 — it is increasing with X_d, decreasing with X_f and increasing with X_S, since domestic and foreign LIBOR are negative exponential in X_d and X_f respectively, and exchange rate, and hence termination cost, is exponential in X_S.

This is clearly not an exhaustive study of the solution surface and there are many other possible cross-sections we could take, but those presented here are fairly representative.

6 Conclusions and Future Directions

Valuation of cross-currency terminable swaps represents a computational task that would usually only be attempted on parallel supercomputers, and as a result we have been restricted to quite coarse grids by the standards of numerical PDE literature — that we get reasonable convergence is due to the fact that the solutions to practical valuation problems do not in general have high curvature. We have in most cases obtained convergence to within a basis point in reasonable computing time.

We conclude that applying band diagonal LU decomposition routines to solving implicit schemes is infeasible for this problem, and it is not clear that any other numerical solution method for implicit schemes could out-perform the ordinary explicit method used here,

[13] We note here that the use of modern data visualisation computer packages, such as PV-Wave used here, is invaluable for debugging the relatively complex computer code and for understanding the solution produced.

[14] The *termination boundary* is the set of points at the end of a period j at which the counterparty is indifferent between terminating and continuing.

[15] The *buy region* is the set of points at the end of a period j at which the counterparty exercises his option to buy the swap.

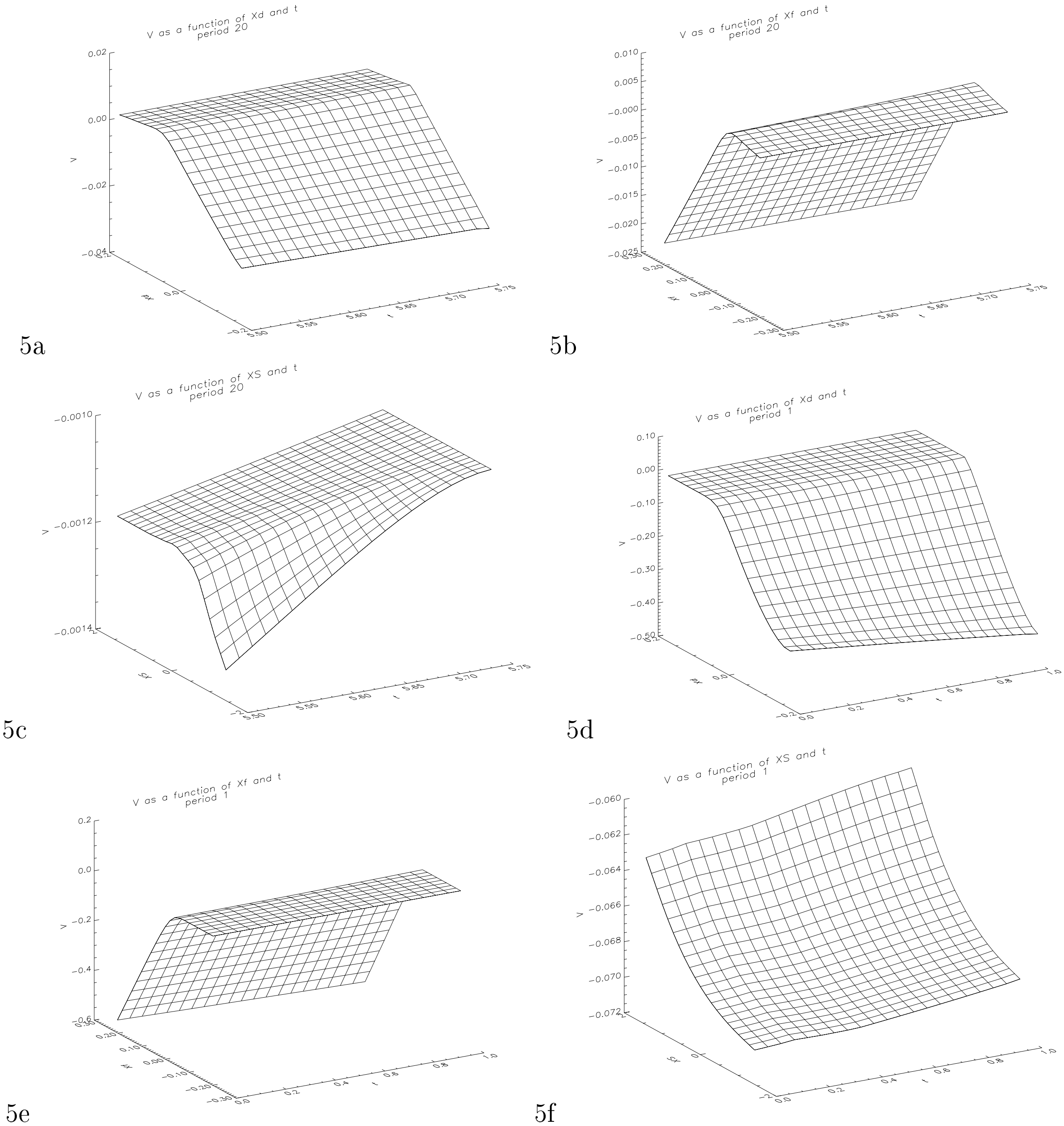

Figure 5: 1 year call on 5 year terminable diff swap: solution surfaces for period 20 (plots a, b, c), $t \in [5.5, 5.75]$ and period 1 (plots d, e, f), $t \in [0, 1]$)

except in cases where stability is restrictive, such as for the vanilla swap. Obvious methods to try include Successive Over-Relaxation (SOR) and recursive tridiagonal or sparse matrix factorisation techniques (such as that advocated in Keast and Muir [15]) applied to the Crank-Nicolson scheme, offering stable second order accuracy in time and efficient use of memory. For PDEs with fewer state variables such methods may well be more efficient, especially for fine spatial discretisations and single state variable problems.

The explicit method approach advocated here could be further improved. It is possible to linearly transform our state variables (essentially to diagonalise the state variables covariance matrix) so as to eliminate the cross-derivative terms and hence transform the PDE into a time-dependent version of the heat equation, which reduces the number of non-zero bands in the matrix F^m from 19 to 7, with a corresponding reduction in time for matrix multiplication. Preliminary experiments with this approach are underway, but so far we have had numerical difficulties when X_S appears in the terminal condition. Since only matrix multiplications are required, it should also be a simple matter to implement the explicit method on a parallel computer, particularly a *fine grain* parallel or *vector* machine. For example, Ekvall [11] investigated parallelised explicit and ADI methods on a 3-D Black-Scholes-type PDE on a Connection Machine CM200 with 4096 processors. The drawbacks of the explicit method are of course its poor stability characteristics and first order time accuracy, and since we do not usually know the critical mesh ratio in advance, some solution time has to be spent determining it — time which we have not added to our results.

One approach which is worth further investigation for this particular problem is the Fourier method, which uses the Fast Fourier Transform to solve the heat equation — its $O(n^3 \log n)$ solution time for a single time point is very appealing, and further work should investigate whether this is realisable. It cannot be used for state variable-dependent coefficients, so whilst it applies here, it is not immediately applicable to many other models. However, a more general and hence more attractive fast method is that of multi-grid, which is the method choice for many physical applications, and could probably be used to good effect in financial problems.

Of particular interest, given the work in Dempster and Hutton [9, 10] (see also Hutton [14]), would be American-style interest rate derivatives, with numerical valuation via linear programming solution of the finite difference approximation. However, it is clear that the difficulties with implicit methods here would carry over to our LP method for an American derivative, and work should be directed towards producing an ADI or multi-grid linear programming solver for American-style derivatives contingent on up to three stochastic variables.

7 Acknowledgements

Much of the material in this paper appears in the Ph.D. thesis [14] of the second author, written under the supervision of the first. We are grateful to the EPSRC (UK), the University of Essex and HSBC Markets for partial support of the research. In particular, we would like to thank Robert Benson, Daniel George and Steven Blyth of the Specialised Derivatives Group at HSBC Markets, London for their support, criticism, encouragement and permission to publish this work. Without Simon Babbs, formerly of HSBC Markets and now at First National Bank of Chicago, London, we would not have undertaken — nor completed — this project; we owe him a special debt of gratitude. Finally, we would like to acknowledge helpful discussions with and comments from Michael Selby, Stewart Hodges, Mark Davis and participants at seminars at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, the Futures and Options Research Centre, University of Warwick and NatWest Capital Markets, London.

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