# The Pricing Models of Cross-currency Equity Swaps and Swaptions

Liao, Szu-Lang
National Chengchi University

Wang, Ming-Jie
Southern Taiwan University of Technology

## Abstract

The paper derives the arbitrage prices of equity swaps and swaptions when extending the transactions into international equity markets. The currency hedging cost is incorporated into the swap values through adding a correction term. To obtain the pricing models using the risk-neutral valuation formula, we adjust the price process of each traded asset such that follows the forward martingale measure. According to the results, we find that the values of cross-currency equity swap and swaption also depend on the dynamics of equity prices and exchange rate, not only the forward interest rates process. Therefore, if the swap payments are included by extra conditions, then the results of pricing interest rate swap cannot be applied fully to equity swap.

*Keywords:* Cross-currency Equity Swap; Equity Swaption; Martingale Measure; Currency Risk

## Introduction

The cross-currency equity swap is an instrument which allows financial managers to capture the existing and expected foreign equity return without incurring foreign exchange exposure. A typical cross-currency equity swap involves a notional principal, a specified tenor and pre-specified payment intervals, where the underlying markets are in different currencies. One of the parties promises to make the payment based on a fixed rate, a floating rate interest rate (e.g. LIBOR), or the total return on an agreed equity benchmark (e.g. stock index) in a currency. In exchange, based on the same principal amounts, the other party will agree to make the payment of foreign equity return, denominated in another currency. As with interest rate swap, there is usually no initial exchange or re-exchange of principal amounts at maturity.

The unique feature of the swaps is that they provide the investors to acquire the returns of foreign equity markets without some constraints by investing directly, such as legal restrictions, transaction costs, withholding tax on dividends and exchange rate risk, especially. They give the possibility of gaining the foreign equity return that direct investment is not allowed, and also circumvent those legal and regulatory limitations. Therefore, this strategy makes the foreign investment much simpler, and the investors only focus on the purely performances of the underlying equity markets without considering the addition costs of direct investment.

An equity swaption is an option to enter into a swap at or before the maturity date. A payer swaption has a right to participate in the swap, paying a pre-specified fixed rate at some future dates and receiving particular equity return. Indeed, equity swaption is a call option on equity swap.

Chance and Rich (1998) use arbitrage-free replicating portfolios to derive valuation formulas for a variety of equity swaps. They indicate that at the start of the contract, the pricing models of equity swap and interest rate swap are alike. It is the dynamics of interest rate, not the dynamics of equity price, determines the swap value. Besides, the pricing of equity swaptions can be obtained in the same manner as that of interest rate swaptions. The levels of equity price during the life of the swaption are irrelevant to the swaption value.

According to their conclusion, the equity price processes have no impact on the pricing models of equity swap and swaption under certain conditions. The result is a puzzle for the practitioners. Kijma and Maromuchi (2001) have solved the puzzle in part by pricing equity swap in a stochastic interest rate economy. They demonstrate that the swap value also depends on the volatility of equity price, as well as the correlation of equity price and interest rate in the case of variable notional principal.

In this paper, we will consider the case if the swaps expand into international equity markets, and resolve the values of cross-currency equity swaps and swaptions with fixed notional principal. The traditional pricing formulas of equity swaps cannot be completely applied to the cross-currency way. Because investors have received the return on the underlying foreign market without currency risk, the currency hedging costs must be incorporated into the swap value. From the pricing models, we use a correction term to modify the foreign market bases. The results show that the volatilities of equity prices and exchange rate, the correlations between the foreign asset prices and exchange rate, all influence the values of equity swaps and swaptions, not only the dynamics of interest rates for domestic equity swaps.

The outline of this paper is as follows: the next section introduces the notation, terminology and assumptions underlying the economy. Section 3 derives the pricing models of cross-currency equity swaps and swaptions and the last is the conclusion.

## The Economy

This section introduces the dynamics of foreign exchange rate, forward interest rate and equity price, respectively. We will use the risk-neutral valuation technique developed by Harrion and Kreps (1981). Assume that our economy is complete and arbitrage-free, then there exists a unique spot martingale measure $\mathbf{P_d}$ for domestic market. By adjusting the risk premium, the original price processes of risky assets are transformed into $\mathbf{P_d}$, such that the present value of each traded asset is computed by discounting at the domestic risk-free rate[1].

We assume the trading takes place continuously over the time interval $[0,\tau]$, and we are currently at time $t \in [0,\tau]$. In our international economy, there are no taxes, transaction cost, and the investors can trade domestic and foreign securities without friction at each trading date. We let the process $\boldsymbol{W}(t) = \left(W_Q(t), W_{S_d}(t), W_{S_f}(t), W_{f_d}(t), W_{f_f}(t)\right)$ is a $k$-dimensional Brownian motions, which represent the random shocks of exchange rate $Q$, domestic and foreign equity price, $S_d$ and $S_f$, domestic and foreign forward interest rate, $f_d$ and $f_f$, respectively. The motions $\boldsymbol{W}$ are defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where the filtration $\mathbf{F}$ is assumed to be the

$\mathbf{P}$-augmentation of the natural filtration captured by those random shocks. We let the asset prices dynamics satisfy the following assumptions:

**Assumption 1 (Exchange Rate Dynamics)**

The exchange rate $Q(t)$ is denominated in units of domestic currency per units of foreign currency $f$. The price process is given by

$$dQ(t) = Q(t)\left(u_Q(t)dt + \sigma_Q(t)\cdot d\boldsymbol{W}(t)\right), \quad Q(0) > 0, \tag{1}$$

where $\sigma_Q \in \boldsymbol{R}^k$ denotes the volatility vector of exchange rate. The exchange rate process plays the role of converting the foreign market cash flow into units of domestic currency.

**Assumption 2 (Forward Interest Rate Dynamics)**

Applying the models of Heath, Jarrow and Morton (HJM, 1992), we let the forward interest rates which matures at time $T$ follow the stochastic difference equation

$$df_i(t,T) = \alpha_{f_i}(t,T)dt + \sigma_{f_i}(t,T)\cdot d\boldsymbol{W}(t), \; i = d, f, \; 0 \le t \le T \le \tau, \tag{2}$$

for domestic market $d$ and foreign market $f$, respectively. The drift term $\alpha_{f_i}(t,T)$ and volatility function $\sigma_{f_i}(t,T)$ of forward interest rates are assumed to be adapted with respect to the filtration $\mathbf{F} = \mathbf{F}^{\mathrm{W}}$ and joint measurable and uniformly bounded on $\{(t,T): 0 \le t \le T \le \tau\}$. By the setting of equation (2), the value of money market accounts with an initial investment of one unit of domestic currency and foreign currency $f$ satisfy

$$B_i(t) = \exp\left[\int_0^t f_i(u,u)du\right] = \exp\left[\int_0^t r_i(u)du\right], \quad i = d, f,$$

and the price of domestic and foreign pure discount bonds which mature at $T$, are given by

$$P_i(t,T) = \exp\left[-\int_t^T f_i(t,u)du\right], \quad i = d, f. \tag{3}$$

From equation (2) and (3), we have the dynamics of bond prices, as

$$dP_i(t,T) = P_i(t,T)\left(a_{f_i}(t,T)dt + b_{f_i}(t,T)\cdot d\boldsymbol{W}(t)\right), \tag{4}$$

where $a_{f_i}(t,T) = r_i(t) - \int_t^T \alpha_{f_i}(u,T)du + 1/2\left|b_{f_i}(t,T)\right|^2$ and $b_{f_i}(t,T) = -\int_t^T \sigma_{f_i}(u,T)du, i = d, f$.

**Assumption 3 (Equity Price Dynamics)**

In our economy, the dynamics of equity prices[2] for domestic and foreign market have the following expressions

$$dS_i(t)=S_i(t)\left(u_{S_i}(t)dt+\sigma_{S_i}(t)\cdot d\boldsymbol{W}(t)\right),\quad i=d,f\,, \tag{5}$$

where the drift terms and the volatilities of equity prices, $u_{S_i}(t)$ and $\sigma_{S_i}(t)$ are $\mathbf{F_t}$-adapted, jointly measure and uniformly bounded on $t\in[0,T]$.

To derive the swap and swaption values using the risk-neutral valuation framework, the domestic saving accounts are served as numeraire, such that the relative price processes of traded asset (in units of domestic currency) with respect to $B_d(t)$ are martingales. Then there exist a unique martingale measure $\mathbf{P_d}$, which is a probability measure on $(\Omega,\mathbf{F})$ equivalent to $\mathbf{P}$. We have the Radon-Nikodym derivative, satisfies

$$\frac{d\boldsymbol{P}_d}{d\boldsymbol{P}}=\exp\left[\int_0^t\eta(u)\cdot d\boldsymbol{W}(u)-\frac{1}{2}\int_0^t|\,\eta(u)|^2\,du\right], \tag{6}$$

where the market prices for risk $\eta(t)=\left(\eta_{S_d}(t),\eta_{f_d}(t),\eta_{S_f}(t),\eta_{f_f}(t),\eta_Q(t)\right),\ \eta(t)\in\boldsymbol{R}^k$, with respect to the random shocks $\boldsymbol{W}(t)$ is a solution to the following systems, as

$$\begin{aligned}
&a_{f_i}(t,T)dt+b_{f_i}(t,T)\cdot\eta_{f_i}(t)=0,\quad i=d,f,\\
&u_{S_i}(t)dt+\sigma_{S_i}(t)\cdot\eta_{S_i}(t)=0,\quad i=d,f,\\
&\left(u_Q(t)+r_d(t)-r_f(t)\right)dt+\sigma_Q(t)\cdot\eta_Q(t)=0.
\end{aligned} \tag{7}$$

From Girsanov's theorem, the process $\tilde{\boldsymbol{W}}$ on $(\boldsymbol{\Omega},\mathbf{F},\mathbf{P_d})$ such that

$$\int_0^t\tilde{\boldsymbol{W}}(u)du=\int_0^t\boldsymbol{W}(u)du-\int_0^t\eta(u)du$$

is also a $k$-dimensional Brownian motion under the measure $\mathbf{P_d}$.

By adjusting the risk premium of each traded asset, therefore, the domestic and foreign equity prices dynamics are

$$dS_d(t)=S_d(t)\left(r_d(t)dt+\sigma_{S_d}(t)\cdot d\tilde{\boldsymbol{W}}(t)\right), \tag{8}$$

$$dS_f(t)=S_f(t)\left(\left(r_f(t)-\sigma_Q(t)\sigma_{S_f}(t)\right)dt+\sigma_{S_f}(t)\cdot d\tilde{\boldsymbol{W}}(t)\right), \tag{9}$$

and the dynamics of domestic and foreign bond prices are

$$d\,P_d(t,T)=P_d(t,T)\left(r_d(t)dt+b_{f_d}(t,T)\cdot d\tilde{\boldsymbol{W}}(t)\right), \tag{10}$$

$$dP_f(t,T) = P_f(t,T)\Big(\big(r_f(t) - \sigma_Q(t) b_{f_f}(t,T)\big)dt + b_{f_f}(t,T)\cdot d\widetilde{W}(t)\Big). \tag{11}$$

The exchange rate dynamics also follow

$$dQ(t) = Q(t)\Big(\big(r_d(t) - r_f(t)\big)dt + \sigma_Q(t)\cdot d\widetilde{W}(t)\Big), \quad Q(0) > 0. \tag{12}$$

Comparing with equation (8) and (9), as well as equation (10) and (11), the differences are that we take exchange rate into consideration in the dynamics of foreign asset prices.

Secondly, under our Gaussian HJM framework of stochastic interest rate, the method of a forward risk adjustment is useful to obtain the arbitrage price of derivate assets. The arrangement provided that the forward price of any financial asset follow a martingale under the forward neutral probability associated with the settlement date of a forward contract. To arrive the result, the domestic discount bond prices are served as numeraire such that the price process of each traded assets follow forward martingale measure.

By the change of numeraire technique, we define a probability measure $\mathbf{P_d^T}$ on $(\Omega, \mathbf{F})$ is equivalent to $\mathbf{P_d}$ with the Radon-Nikodym derivative $\eta'(t)$, satisfies

$$\begin{aligned}\eta'(t) = \frac{d\boldsymbol{P}_d^T}{d\boldsymbol{P}_d} &= \frac{B_d(T)^{-1}}{E_{\boldsymbol{P}}\left[B_d(T)^{-1}\right]} = \frac{1}{B_d(T)P_d(0,T)} \\ &= \exp\left[\int_0^t b_{f_d}(u,T)\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t \left|b_{f_d}(u,T)\right|^2 du\right]\end{aligned}.$$

where $\mathbf{P_d^T}$ is called the forward martingale measure with respect to the settlement date $T$. From Girsanov's theorem, the process $\widetilde{\boldsymbol{W}}^T$ on $(\Omega, \mathbf{F}, \mathbf{P_d^T})$ given by

$$\int_t^T \widetilde{W}^T(u)du = \int_t^T \widetilde{W}(u)du - \int_t^T b_{f_d}(u,T)du \tag{13}$$

is a standard $k$-dimensional Brownian motion under $\mathbf{P_d^T}$.

As a result, if we let $F_{S_d}(t,T)$ be the forward equity price $S_d(t)$ at time $t$ for settlement at $T$, then the forward price satisfies

$$F_{S_d}(t,T) = \frac{S_d(t)}{P_d(t,T)}$$

under $\mathbf{P_d^T}$. The price process of $F_{S_d}(t,T)$ also follows

$$F_{S_d}(t,T)=F_{S_d}(0,T)\exp\left[\int_0^t\left(\sigma_{S_d}(u)-b_{f_d}(u,T)\right)\cdot d\widetilde{W}^T(u)-\frac{1}{2}\int_0^t\left|\sigma_{S_d}(u)-b_{f_d}(u,T)\right|^2du\right]. \quad (14)$$

Given the relevant price processes of each traded asset, we can derive the closed-form valuations of cross-currency equity swaps and swaptions under our Gaussian economy. The arbitrage prices are deduced in the next section.

## The Pricing Models of Equity Swaps and Swaptions

### The cross-currency equity swap with constant notional principal

We define a cross-currency equity swap in which starts at time $T_0$ and matures at $T_m$, $0\le T_0\le T_m\le\tau$, for $m$ periods. At each payment date $T_j$, $j=1,2,..m$, and $T_j-T_{j-1}=\delta$, the counterparty pays fixed rate $\overline{R}$ and receives the periodical foreign equity return $S_f(T_j)/S_f(T_{j-1})$ form time $T_{j-1}$ to $T_j$, where the payments are denominated in a currency of the principal amounts[3]. The swap value is obtained from the sum of the present values of payment spread at each settlement date. Then the value $V(T_0,\overline{R})$ at the starting time $T_0$ is given by

$$V(T_0,\overline{R})=\sum_{j=1}^{m}P_d(T_0,T_j)E\left[R_f(T_{j-1},T_j)-(1+\overline{R})\,\middle|\,\Phi_d^{T_0}\right]. \quad (15)$$

where $E(\cdot)$ is the conditional operator with respect to domestic information $\Phi_d^{T_0}$, and $R_f(T_{j-1},T_j)$ denotes the foreign equity return, $S_f(T_j)/S_f(T_{j-1})$.

To evaluate the expected return of the foreign equity by the forward-neutral valuation, we have to adjust the price process of foreign equity, such that follow $\mathbf{P}_{\mathbf{d}}^{\mathbf{T_j}}$ for each payment date $T_j$. By virtue of the process, the expected returns of foreign equity are deduced, equal

$$E\left[\frac{S_f(T_j)}{S_f(T_{j-1})}\,\middle|\,\Phi_d^{T_0}\right]=\left(1+f_f(T_0,T_{j-1},T_j)\right)G_f(T_0,T_{j-1},T_j),\ j=1,2,..m, \quad (16)$$

where $G_f(T_0,T_{j-1},T_j)$ is a correction term, defined as

$$G_f\left(T_0,T_{j-1},T_j\right)=\exp\left\{\int_{T_0}^{T_j}\left(b_{f_k}\left(u,T_j\right)-\sigma_{S_k}\left(u\right)\right)\left(\sigma_Q\left(u\right)+b_{f_k}\left(u,T_j\right)-b_{f_d}\left(u,T_j\right)\right)du \right.$$
$$\left. -\int_{T_0}^{T_{j-1}}\left(b_{f_k}\left(u,T_{j-1}\right)-\sigma_{S_k}\left(u\right)\right)\left(\sigma_Q\left(u\right)+b_{f_k}\left(u,T_j\right)-b_{f_d}\left(u,T_j\right)\right)du\right\}.$$

The construction of equation (16) is that if we want to derive the expected equity return for the time period: $\left[T_{j-1},T_j\right]$ at time $T_0$, the theoretical value can be inferred by the observed forward foreign equity prices that are matured at time $T_j$ and $T_{j-1}$, respectively. Applying the Itô's formula, the dynamics of forward equity price $F_{S_f}\left(t,T_j\right)$ are given by

$$dF_{S_f}\left(t,T_j\right)=F_{S_f}\left(t,T_j\right)\left(\left(b_{f_f}\left(t,T_j\right)-\sigma_{S_f}(t)\right)\left(\sigma_Q(t)-b_{f_f}\left(t,T_j\right)\right)dt \right. \\ \left. +\sigma_{S_f}(t)\cdot d\widetilde{\boldsymbol{W}}^{\mathbf{T_j}}(t)-b_{f_f}\left(t,T_j\right)\cdot d\widetilde{\boldsymbol{W}}^{\mathbf{T_j}}(t)-b_{f_d}\left(t,T_j\right)\cdot d\widetilde{\boldsymbol{W}}^{\mathbf{T_j}}(t)\right) \quad (17)$$

with respect to $\mathbf{P_d^{T_j}}$. By considering the dynamics of $F_{S_f}\left(t,T_{j-1}\right)$ under the same measure, and the correlations with the price processes of $F_{S_f}\left(t,T_j\right)$ and $F_{S_f}\left(t,T_{j-1}\right)$, the desired result is constructed.

The term $G_f\left(T_0,T_{j-1},T_j\right)$ is the correction for different market bases, if we let the dynamics of domestic and foreign interest rate are alike, and the volatility of exchange is zero, then the term disappear. Furthermore, the term is regarded as the hedging cost for domestic investors. To obtain the foreign equity return, the forward prices of equity and exchange rate can be used simultaneously to avoid the uncertain, where $G_f\left(T_0,T_{j-1},T_j\right)$ is reflected as the covariance risk of the two contracts.

Now, assume that the swap value is zero in equation (15), the swap coupon rate is

$$R=\frac{\sum_{j=1}^{m}P_d\left(T_0,T_j\right)\left[\left(1+f_f\left(T_0,T_{j-1},T_j\right)\right)G_f\left(T_0,T_{j-1},T_j\right)\right]}{\sum_{j=1}^{m}P_d\left(T_0,T_j\right)}-1. \quad (18)$$

In this model, we attach the term $G_f\left(T_0,T_{j-1},T_j\right)$ to differ from the plain vanilla swap for domestic market. We find that the major factors of the equity swap value are the term structures of domestic and foreign interest rate, which are determined by the volatilities of forward interest rates under our HJM case. If the volatility of interest rate is higher for foreign market and smaller for domestic market, respectively, then the swap rate will increase. But the volatilities of foreign equity prices and exchange rate, the correlations between the foreign asset prices and exchange rate will influence the swap rate by adding the extra term. As a result of Chance and Rich (1998), the equity price process is irrelevant to the swap coupon rate. If we expand the swap into international equity markets, however, the value also depends on the dynamics of foreign equity prices and exchange rate.

Kijima and Muromachi (2001) show that in a stochastic interest rate economy, the swap coupon rate must add a term to reflect the correlation between the equity price and interest rate when notional principal is varying according to the return on the underlying equity price. Our model is similar to their result, but the difference is that we consider the situation of foreign equity market. Therefore, if the extra conditions are incorporated into the structure of equity swap, then the swap value is not only determined by the interest rates processes, but also the other related asset prices dynamics.

Finally, if the investors pay the domestic equity return $R_d$ and receive the foreign equity return $R_f$ corresponding to the same period, then the value of the cross-currency two-ways equity swap at the initial date $T_0$ is given by

$$V\left(R_d, R_f, T_0\right)=\sum_{j=1}^{m} P_d\left(T_0, T_j\right)\left[\left(1+f_f\left(T_0, T_{j-1}, T_j\right)\right) G_f\left(T_0, T_{j-1}, T_j\right)-\left(1+f_d\left(T_0, T_{j-1}, T_j\right)\right)\right].$$

**The Pricing Model of the Cross-currency Equity Swaptions**

An payer equity swaption is an option to enter into an equity swap that pays the pre-specified fixed rate, and receives the periodical equity index return with given principal amounts. Chance and Rich's model show that if the strike rate is $\overline{R}$, the underlying swap starts at time $T_0$ and terminates at $T_m$ for $m$ periods. The European swaption value at time $t \leq T_0$ has the following expression

$$ES_t^{\overline{R}} = \sum_{j=1}^{m} P_d\left(t, T_j\right)\left(R-\overline{R}\right)^{+}, \tag{19}$$

where $R$ is the market swap rate determined at time $T_0$. In our model, $R$ denotes the swap rate for the cross-currency equity swap. Consequently, the level of equity price is irrelevant to the value of equity swaptions. In fact, it is identical to the price of interest rate swaptions, because the equity swaption value comes from differences between the swap rate and the strike rate at time $T_0$, if the former is higher than the latter. Then discount the differences during the life of the contract into time $t$. Thus, it is the term structure of interest rates from time $t$ to $T_m$ influences the swaption value, not the level of equity price. As Jarrow (1996) concludes that equity swaptions are more like—in fact identical to—options on bonds than like options on stocks.

However, from the pricing formula derived in equation (18), the dynamics of foreign equity prices and exchange rate will affect the swap value. Then we suggest that the swaption value may also be influence by those factors. Hence, we will deduce the value of cross-currency swaption under the same Gaussian forward-neutral framework and compare the differences with the interest rate swaption.

By rearranging the equation (19), we get

$$ES_t^{\bar{R}} = P_d\left(t,T_0\right)\sum_{j=1}^{m} P_d\left(T_0,T_j\right)\left(R-\bar{R}\right)^+ . \tag{20}$$

In other words, compute the sum of present values of the differences during the swap life first, and then discount it to time $t$. Now, if equilibrium, the swap rate at time $T_0$ is equivalent to expected returns of the underlying equity market during the contract's life. To obtain the arbitrage price of the cross-currency swaption, therefore, the swap rate can be substituted with the result of equation (18), as

$$ES_t^{\bar{R}} = P_d\left(t,T_0\right)E\left[\sum_{j=1}^{m} P_d\left(T_0,T_j\right)\left[\left(1+R_f\left(T_{j-1},T_j\right)\right)-\left(1+\bar{R}\right)\middle|\Phi_d^{T_0}\right]\right]^+ . \tag{21}$$

After adjusting the price process of each asset such that follows the measure $\mathbf{P_d^{T_0}}$, the swaption value is inferred. The next proposition demonstrates that how to obtain the result.

**Proposition: The Arbitrage Price of the Cross-currency Equity Swaption**

If the investor has the option to enter into a cross-currency equity swap, the underlying swap starts at time $T_0$ and terminates $T_m$ for $m$ periods, to pay the strike $\overline{R}$ and receive the foreign benchmark equity return $f$ for each settlement date $T_j$, $j = 1,2,..m,$. The arbitrage price of a European swaption at time $t \le T_0$ is given by

$$ES_t^{\overline{R}} = \int_{R^k} \left\{ \sum_{j=1}^{m} P_d(t,T_j)\left[\left(1+f_f(t,T_{j-1},T_j)\right)G_f(t,T_{j-1},T_j)n_k(x+y_j)-\left(1+\overline{R}\right)n_k(x+z_j)\right]^+ \right\} dx, \quad (22)$$

where $n_k$ is the standard $k$-dimensional Gaussian probability density, as

$$n_k(x) = (2\pi)^{-\frac{k}{2}} \mathbf{e}^{-\frac{|x|^2}{2}}, \quad \forall x \in \mathbf{R^k},$$

and the volatility vectors $y_1,..y_m, z_1,..z_m \in \mathbf{R^k}$ are such that for every $i, j = 1,2,..m$

$$y_i \cdot y_j = \int_t^{T_0} \left(\theta_{f_f}(u,T_i,T_{i-1}) + \theta_{f_d}(u,T_i,T_0)\right) \cdot \left(\theta_{f_f}(u,T_j,T_{j-1}) + \theta_{f_d}(u,T_j,T_0)\right) du,$$

$$y_i \cdot z_j = \int_t^{T_0} \left(\theta_{f_f}(u,T_i,T_{i-1}) + \theta_{f_d}(u,T_i,T_0)\right) \cdot \theta_{f_d}(u,T_j,T_0) du, \quad z_i \cdot z_j = \int_t^{T_0} \theta_{f_d}(u,T_i,T_0) \cdot \theta_{f_d}(u,T_j,T_0) du$$

with $\theta_k(u,T_j,T_{j-1}) = b_k(u,T_{j-1}) - b_k(u,T_j), k = f_d, f_f$.

**Proof**: See **Appendix.**

Under our Gaussian case, it is apparent that the equity prices and exchange rate dynamics will influence the swaption value adding the extra term $G_f(t,T_{j-1},T_j)$, although the volatilities of domestic and foreign forwrad interest rates are the significant factors of the option value. The finding is parallel to the cross-currency equity swap. Assume that we consider the swaption in a single currency, then our result is reduced to the model of interest rate swaption and the extra term disappears, which agrees with Chance and Rich (1998). Finally, we can derive variety forms of swaptions by the same way, for example, an equity swap with the notional principal is adjusted each payment date or denominated in another currency.

## Conclusion

The pricing of equity swap under risk-neutral framework is similar to that of interest rate swap corresponding to the same period, from previous studies. But if we extend the swaps to foreign equity markets, then the pricing model must add an extra term to reflect currency risk. The result is also applied to the cross-currency equity swaptions. We conclude that the dynamics of foreign equity prices and exchange rate also affect the values, not only the volatilities of interest rates.

## ENDOTES

1. The setup, see Amin and Jarrow (1991), Bingham and Kiesel. (1998).
2. The equity return often is a total return reflecting both dividend and share price movements, but the swap can be based on price movements only. In this paper, we set the dividend yield equal to zero, where the yield can treat as part of the capital gain.
3. For simplicity, we assume the notional principal is one dollar.

## APPENDIX

To derive the swaption value, we combine the results of equation (21) and equation (16), and find that

$$ES_t^{\bar{R}} = P_d\left(t,T_0\right)E\left[\sum_{j=1}^{n} P_d\left(T_0,T_j\right)\left(\left(1+f_f\left(T_0,T_{j-1},T_j\right)\right)G_f\left(T_0,T_{j-1},T_j\right)-\left(1+\bar{R}\right)\right)\middle|\Phi_d^{T_0}\right]^+ . \quad \text{(A.1)}$$

The expectation term consists of three parts: the domestic bond price, the foreign forward interest rate and the extra term, respectively. To extract the conditional expectation under forward-neutral valuation, we have to adjust the related price processes form time $T_0$ into $t$ and also satisfy the corresponding measure $\mathbf{P}_\mathbf{d}^{\mathbf{T}_0}$.

By the definition of forward interest rate, we have the relationship

$$P_d\left(T_0,T_j\right)=\frac{P_d\left(T_0,T_j\right)}{P_d\left(T_0,T_0\right)}=1+f_d\left(T_0,T_j,T_0\right). \quad \text{(A.2)}$$

Consequently, the price process of forward interest rate $f_d\left(T_0,T_j,T_0\right)$ has the following representation

$$\begin{aligned}&1+f_d\left(T_0,T_j,T_0\right)\\&=\left(1+f_d\left(t,T_j,T_0\right)\right)\exp\left\{\int_t^{T_0}\theta_{f_d}\left(u,T_0,T_j\right)\left(d\widetilde{W}(u)-b_{f_d}\left(u,T_0\right)du\right)-\frac{1}{2}\left|\theta_{f_d}\left(u,T_0,T_j\right)\right|^2du\right\}\end{aligned} . \quad \text{(A.3)}$$

where $\theta_{f_d}\left(u,T_0,T_j\right)=b_{f_d}\left(u,T_j\right)-b_{f_d}\left(u,T_0\right)$.

In order to satisfy the forward measure $\mathbf{P}_\mathbf{d}^{\mathbf{T}_0}$, we apply the methodology derived in equation (13), to adjust the price process of forward interest rate. Then we have the equality

$$1+f_d\left(T_0,T_j,T_0\right)=\left(1+f_d\left(t,T_j,T_0\right)\right)\exp\left\{\int_t^{T_0}\theta_{f_d}\left(u,T_0,T_j\right)\cdot d\widetilde{W}^T(u)-\frac{1}{2}\left|\theta_{f_d}\left(u,T_0,T_j\right)\right|^2du\right\} . \quad \text{(A.4)}$$

under $\mathbf{P}_\mathbf{d}^{\mathbf{T}_0}$. Hence, the domestic bond price process, which starts at time $T_0$ and matures at time $T_j$, has been transformed into the forward interest rate dynamics at time $t$ and also satisfies $\mathbf{P}_{\mathbf{d}}^{\mathbf{T}_0}$.

Secondly, by virtue of the same approach, we also have the dynamics of foreign forward interest rate $f_f\left(T_0, T_{j-1}, T_j\right)$ at time $t$,

$$\begin{aligned}&1+f_f\left(T_0,T_{j-1},T_j\right)=\left(1+f_f\left(t,T_{j-1},T_j\right)\right)\times\\&\exp\left\{\int_t^{T_0}\theta_{f_f}\left(u,T_j,T_{j-1}\right)\left(d\widetilde{W}^{T_0}(u)-\left(b_{f_d}\left(u,T_0\right)-\sigma_Q(u)-b_{f_f}\left(u,T_j\right)\right)du\right)-\frac{1}{2}\left|\theta_{f_f}\left(u,T_j,T_{j-1}\right)\right|^2du\right\}\end{aligned} \quad \text{(A.5)}$$

where $\theta_{f_f}\left(u,T_0,T_j\right)=b_{f_f}\left(u,T_j\right)-b_{f_f}\left(u,T_0\right)$. Then the dynamics of domestic and foreign forward interest rates with respect to $\mathbf{P}_{\mathbf{d}}^{\mathbf{T}_0}$ have been derived.

Through adding the process of $G_f\left(T_0,T_{j-1},T_j\right)$ into equation (A.5), the expected return of foreign equity can be rewritten as

$$\begin{aligned}&\left(1+f_f\left(T_0,T_{j-1},T_j\right)\right)G_f\left(T_0,T_{j-1},T_j\right)=\left(1+f_f\left(t,T_{j-1},T_j\right)\right)G_f\left(t,T_{j-1},T_j\right)\times\\&\qquad\exp\left\{\int_t^{T_0}\theta_{f_f}\left(u,T_j,T_{j-1}\right)\cdot d\widetilde{W}^{T_0}(u)+\theta_{f_f}\left(u,T_j,T_{j-1}\right)\theta_{f_d}\left(u,T_j,T_0\right)-\frac{1}{2}\left|\theta_{f_f}\left(u,T_j,T_{j-1}\right)\right|^2du\right\}\end{aligned}$$

Therefore, under $\mathbf{P}_{\mathbf{d}}^{\mathbf{T}_0}$, we get

$$ES_t^{\bar{R}}=E\left\{\sum_{j=1}^{n}P_d\left(t,T_j\right)\left[\left(1+f_f\left(t,T_{j-1},T_j\right)\right)G_f\left(t,T_{j-1},T_j\right)k_1-\left(1+\bar{R}\right)k_2\right]^+\right\} \quad \text{(A.6)}$$

with

$$\begin{aligned}&k_1=\exp\left\{\int_t^{T_0}\left(\theta_{f_f}\left(u,T_j,T_{j-1}\right)+\theta_{f_d}\left(u,T_j,T_0\right)\right)\cdot d\widetilde{W}^{T_0}(u)-\frac{1}{2}\int_t^{T_0}\left|\theta_{f_f}\left(u,T_j,T_{j-1}\right)+\theta_{f_d}\left(u,T_j,T_0\right)\right|du\right\}\\&k_2=\exp\left\{\int_t^{T_0}\theta_{f_d}\left(u,T_j,T_0\right)\cdot d\widetilde{W}^{T_0}(u)-\frac{1}{2}\int_t^{T_0}\left|\theta_{f_d}\left(u,T_j,T_0\right)\right|du\right\}\end{aligned}$$

Finally, applying the valuation formula of Brace and Musiela (1994), the European arbitrage price of cross-currency equity swaption is obtained.□

## Reference


Amin, K. I., & R. Jarrow, A. (1991). Pricing foreign currency options under stochastic interest rates. Journal of International Money and Finance, 10, 310- 29.

Amin, K. I., & Bodurtha, J. N. (1995). Discrete-time valuation of American opti-

ons with stochastic interest rate. Review of Financial Studies, 8, 193 - 234.
Brace, A., and Musiela, M. (1994). A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton. Mathematical Finance, 4, 259- 283.
Bingham, Nicholas H. and Rüdiger Kiesel. (1998). Risk-neutral valuation: Pricing and hedging of financial derivatives. New York: Springer Publishing.
Chance, D. M., & Rich, D. (1998). The pricing of equity swaps and swaptions. Journal of Derivatives, 5, 19 - 31.
Harrion, J. M., & Kerps, D. (1979). Martingales and arbitrage in multi-period securities markets. Journal of Economic Theory, 20, 381 - 408.
Harrion, J. M., & Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and Their Applications, 11, 215 - 60.
Heath, D., Jarrow, R., & Morton A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. Econometrica, 60, 77 - 105.
Jarrow, R., & Turnbull, S. (1996). Derivative Securities, Cincinasti: South Western Publishing.
Kijima, M., Muromachi, Y. (2001). Pricing equity swaps in a stochastic interest rate economy. Journal of Derivatives, 8, 19 - 35.
Marshall, J. E., Sorensen, E., & Tuncker, A. (1992). Equity Derivatives: The Plain Vanilla Equity Swap and its Variants. Journal of Financial Engineering, 1, 219 - 241.
Marshall, J., & Yuyuenyonwatana, R. (2000). Equity swaps: Structures, Uses, and Pricing. Handbook of Equity Derivatives, Francis and Toy and Whittaker ed., New York: Wiley Publishing.
Musiela, M., and Rutkowski, M. (1997). Martingale methods in financial modelling. New York: Springer Publishing.
Rich, D. (1995). Note on the valuation and hedging of equity swaps. Journal of Financial Engineering, 3, 323 - 34.
Schroder, M. (1999). Changes of numeraire for pricing Futures, Forward and Options. Review of Financial Studies, 12, 1143 - 1163.