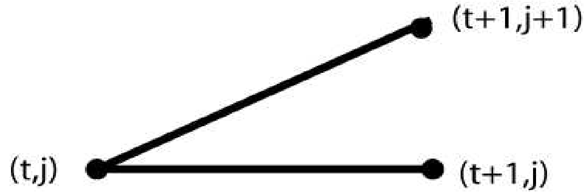
Arbitrage-free pricing of derivatives over a binomial tree

S H Man 10th May,2003

1 Some definitions

Lets fix some notation for the rest of this article. We assume that we have a binomial tree and the branching is given by the following diagram.



The tree starts at $(0,0)$. If we are at node (t,j), then it can go to node $(t+1,j+1)$ or $(t+1,j)$.
The following notation will be used throughout this article.

- $D(1), D(2), \ldots,$ be the (default-free) discount factors over $[0,1], [0,2], \ldots$ respectively. We can think of $D(t)$ as the time 0 value of a default-free zero-coupond bond with maturity t and face value of \$1.
- $r(t,j)$ denotes the (risk free) spot rate at (t,j) over $[t,t+1]$.
- $D(t,j)$ be the discount factor at (t,j) over $[t,t+1]$.
- $B(t)$ be the time t value of a deposit account, earning risk free interest, with $B(0)=1$.
- For $n \geq 1$, let
$$\Omega_n = \{(\omega_1, \omega_2, \ldots, \omega_n) \,|\, \omega_i = H \text{ or } T\}$$
Also if $0 \leq t \leq n$ and $\vec{\omega} \in \Omega_n$, we define $\#_t(\vec{\omega})$ to be the number of H in the first t entries of $\vec{\omega}$.

2 Some results on measureable function over a finite set

Through out this section, Ω is a finite set. Let $A_1, A_2, \ldots, A_k$ be a partition of Ω. Clearly,

$$\{\cup_{i\in I} A_i \,|\, I \subseteq \{1,2,\ldots,k\}\} \tag{1}$$

is a σ-algebra of Ω. We now show that every σ-algebra on Ω could be constructed in this way.

Lemma 2.1 *Suppose Ω is a finite set and $\mathcal{F}$ is a σ-algebra. For any $\omega \in \Omega$, define*

$$A_\omega = \bigcap_{\substack{\omega \in A \\ A \in \mathcal{F}}} A.$$

Then

i) $\{A_\omega\}_{\omega\in\Omega}$ *is a partition of* Ω.
ii) For any $A \in \mathcal{F}$ *and* $\omega \in \Omega$, $A_\omega \subseteq A$ *or* $A_\omega \cap A = \emptyset$.

Note that A_ω *is the smallest nonempty set in* $\mathcal{F}$ *that contains* ω. *Also, any element of* $\mathcal{F}$ *is a (disjoint) union of some* A_ω*s.*

Proof i) Let $\omega_1, \omega_2 \in \Omega$. It suffices to show $A_{\omega_1} \cap A_{\omega_2} = \emptyset$ or $A_{\omega_1} = A_{\omega_2}$. Suppose $\omega \in A_{\omega_1} \cap A_{\omega_2}$. Then $A_\omega \subseteq A_{\omega_1} \cap A_{\omega_2} \subseteq A_{\omega_1}$. Note that if $A_\omega \neq A_{\omega_1}$, then $A_{\omega_1} \backslash A_\omega$ would be in $\mathcal{F}$ with

$$\omega_1 \in A_{\omega_1} \backslash A_\omega \text{ and } A_{\omega_1} \backslash A_\omega \subsetneq A_{\omega_1}$$

This contradicts A_{ω_1} is the smallest element in $\mathcal{F}$ that contains ω_1.
ii) Let $A \in \mathcal{F}$, $\omega \in \Omega$. Suppose $\omega_1 \in A_\omega \cap A$. Then $A_{\omega_1} \subseteq A_\omega \cap A \subseteq A_\omega$. By (i), $A_{\omega_1} = A_\omega = A_\omega \cap A$. Hence $A_\omega \subseteq A$. □

Note that for any $A \in \mathcal{F}$, $A = \cup_{\omega \in A} A_\omega$.

The following result will be needed in later sections.

Theorem 2.2 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* X *be a random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}$ *be a* σ*-subalgebra of* $\mathcal{F}$. *Then, for any* $\omega \in \Omega$,

$$\mathbb{E}(X|\mathcal{G})(\omega) = \frac{\int_{A_\omega} X\, d\mathbb{P}}{\mathbb{P}(A_\omega)} = \frac{\sum_{\xi \in A_\omega} X(\xi)\mathbb{P}(\xi)}{\mathbb{P}(A_\omega)}$$

where A_ω *is as defined in Lemma 2.1 .*

Lemma 2.3 $f : (\Omega, \mathcal{F}) \longrightarrow \mathbb{R}$ *is a measureable function if and only if*

$$f = \sum_{i=1} d_i 1_{A_i}$$

for some $d_1, d_2, \ldots, d_n \in \mathbb{R}$ *and* A_i*s are pairwise disjoint elements of* $\mathcal{F}$. 1_A *is the indicator function on the set* A.

Proof ($\Rightarrow$) Let $d_1, d_2, \ldots, d_n$ be the distinct images of f. As f is measureable, by Lemma 2.1, each $f^{-1}(d_i)$ is a disjoint union of A_ωs . (A_ω is as defined in Lemma 2.1.) Then $f = \sum_{i=1} d_i 1_{f^{-1}(d_i)}$.

The converse is clear. □

Definition 2.4 *Let* $1 \leq k \leq n$ *be positive integers. Define*

$$\Omega_n = \{(\omega_1, \omega_2, \ldots, \omega_n) \mid \omega_i = H \text{ or } T\}$$

the sample space of tossing a coin n *times. For each* $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_k) \in \Omega_k$, *define*

$$A_{\vec{\xi}} = \{(\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n \,|\, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, k\}$$

Note that $\{A_{\vec{\xi}} \,|\, \vec{\xi} \in \Omega_k\}$ *is a partition of* Ω_n. *We denote this partition by* $P_{k,n}$. *The* σ*-algebra (on* Ω_n*) corresponds to* $P_{k,n}$ *is denoted by* $\mathcal{F}_k$. *We define* $\mathcal{F}_0 = \{\emptyset, \Omega_n\}$. *Note that* $\mathcal{F}_n$ *is the power set of* Ω_n *and*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n.$$

Corollary 2.5 *Let $1 \leq k \leq n$ be positive integers. Then the following statements are equivlatent.*
1) $f : (\Omega_n, \mathcal{F}_k) \longrightarrow \mathbb{R}$ is a measureable function.
2) For any $\vec{\xi} \in \Omega_k$, f is constant on $A_{\vec{\xi}}$. In other words, for any $\vec{\omega_1} = (\omega_{11}, \omega_{12}, \ldots, \omega_{1n}), \vec{\omega_2} = (\omega_{21}, \omega_{22}, \ldots, \omega_{2n}) \in \Omega_n$, $f(\vec{\omega_1}) = f(\vec{\omega_2})$ whenever $\omega_{1i} = \omega_{2i}$ for $i = 1, 2, \ldots, k$.

Proof ($\Rightarrow$) By Lemma 2.3 , $f = \sum_{i=1}^{m} d_i 1_{A_{\xi_i}}$ for some $d_i \in \mathbb{R}$ and $\xi_i \in \Omega_k$. It is clear that (2) is satisfied.

($\Leftarrow$) By assumption $f = \sum_{\xi \in \Omega_k} f(\xi) 1_{A_\xi}$. It follows from 2.3 that f is measureable. □

Corollary 2.6 *Suppose we have a map f : Binomial tree $\longrightarrow \mathbb{R}$. For $t = 0, 1, 2, \ldots, n$, f induces a $\mathcal{F}_t$ measuareble map*

$$(\Omega_n, \mathcal{F}_t) \longrightarrow \mathbb{R} : \vec{\omega} \mapsto f(t, \#_t(\vec{\omega})) \tag{2}$$

3 Arbitarge free pricing of a general binomial process

Let $\vec{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega_n$ and $t \leq n$. Recall that $\#_t(\vec{\omega})$ to be the number of H in the first t entries of $\vec{\omega}$. Let $\mathbb{P}$ be a probability on $(\Omega_n, \mathcal{F}_n)$. Define

$$A = \{(\xi_1, \ldots, \xi_t, H, \xi_{t+2}, \ldots, \xi_n) \,|\, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, t\} \tag{3}$$
$$A_t(\vec{\omega}) = \{(\xi_1, \xi_2, \ldots, \xi_n) \,|\, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, t\} \tag{4}$$

Let

$$p_t(\vec{\omega}) = \frac{\sum_{\vec{\xi} \in A} \mathbb{P}(\vec{\xi})}{\mathbb{P}(A_t(\vec{\omega}))} \tag{5}$$
$$q_t(\vec{\omega}) = 1 - p_t(\vec{\omega}) \tag{6}$$

We think of $p_t(\vec{\omega})$ as the probability, under $\mathbb{P}$, of going from node $(t, \#_t(\vec{\omega}))$ to $(t, \#_t(\vec{\omega}) + 1)$ in the binomial tree, given that we arrived at $(t, \#_t(\vec{\omega}))$ via $\vec{\omega}$. Note that $p_t(\vec{\omega})$ only depends on the first t entries of $\vec{\omega}$.

Lemma 3.1 *Let $0 \leq i \leq n-1$ be given. Suppose $X_{t+1} : (\Omega_n, \mathcal{F}_{t+1}) \longrightarrow \mathbb{R}$ is measureable. Then, for any $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n$*

$$\mathbb{E}_{\mathbb{P}}(X_{t+1} \,|\, \mathcal{F}_i)(\vec{\omega}) = X_{t+1}\big(\omega_1, \ldots, \omega_t, H\big) \cdot p_t(\vec{\omega}) \;+\; X_{t+1}\big(\omega_1, \ldots, \omega_t, T\big) \cdot q_t(\vec{\omega})$$

Proof Let t and $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n$ be given. By Theorem 2.2

$$\mathbb{E}(X_{t+1} \,|\, \mathcal{F}_t)(\vec{\omega}) = \frac{\sum_{\vec{\xi} \in A_{\vec{\omega}}} X_{t+1}(\vec{\xi}) \mathbb{P}(\vec{\xi})}{\mathbb{P}(A_{\vec{\omega}})} \tag{7}$$

where $A_{\vec{\omega}} = \{(\xi_1, \ldots, \xi_n) \,|\, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, t\}$. Let

$$A_1 = \{(\xi_1, \ldots, \xi_n) \,|\, \xi_{t+1} = H, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, t\}$$
$$A_2 = \{(\xi_1, \ldots, \xi_n) \,|\, \xi_{t+1} = T, \xi_i = \omega_i \text{ for } i = 1, 2, \ldots, t\}$$

Note that $A_{\vec{\omega}}$ is a disjoint union of A_1, A_2. Then (7) becomes

$$\begin{aligned}\mathbb{E}(X_{t+1}\,|\,\mathcal{F}_t)(\vec{\omega}) &= \frac{\sum_{\vec{\xi}\in A_1} X_{t+1}(\vec{\xi})\mathbb{P}(\vec{\xi})}{\mathbb{P}(A_{\vec{\omega}})} + \frac{\sum_{\vec{\xi}\in A_2} X_{t+1}(\vec{\xi})\mathbb{P}(\vec{\xi})}{\mathbb{P}(A_{\vec{\omega}})} \\ &= X_{t+1}(\omega_1, \ldots .\omega_t, H)\cdot p_t(\vec{\omega}) \,+\, X_{t+1}(\omega_1, \ldots .\omega_t, T)\cdot q_t(\vec{\omega})\end{aligned}$$

as X_{t+1} is $\mathcal{F}_{t+1}$-measureable. □

Theorem 3.2 *Suppose* $\{X_i\}_{i=0}^n$ *is a* $(\mathbb{P}, \{\mathcal{F}_i\}_{i=0}^n)$ *-martingale. If* $\{Y_i\}_{i=0}^n$ *is a* $(\mathbb{P}, \{\mathcal{F}_i\}_{i=0}^n)$ *-martingale, then for* $i = 1, 2, \ldots, n$

$$\phi_i = \frac{Y_i - Y_{i-1}}{X_i - X_{i-1}} : (\Omega_n, \mathcal{F}_{i-1}) \longrightarrow \mathbb{R}$$

is measurable and

$$Y_k = Y_0 + \sum_{i=1}^{k} \phi_i (X_i - X_{i-1}) \qquad \textit{for } k = 1, 2, \ldots, n$$

Also, for any $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n$,

$$\phi_i(\vec{\omega}) = \frac{N_i(\omega_1, \ldots, \omega_i, H) - N_i(\omega_1, \ldots, \omega_i, T)}{X_i(\omega_1, \ldots, \omega_i, H) - X_i(\omega_1, \ldots, \omega_i, T)}$$

Let B_t be the time t value of a deposit account with an initial investment of \$1 at time 0. Recall $D(t, j)$ is the discount factor over $[t, t+1]$ at (t, j) of the binomial tree. By Corollary (2.6), for $t \geq 0$,

$$D_t : (\Omega_n, \mathcal{F}_t) \longrightarrow \mathbb{R} : \vec{\omega} \mapsto D(t, \#_t(\vec{\omega})) \tag{8}$$

is $\mathcal{F}_t$-measureable. Then

$$B_t = D_0^{-1} D_1^{-1} \ldots D_{t-1}^{-1} \tag{9}$$

Let S be a security and its value at time t is S_t and S_t is $\mathcal{F}_t$-measurable.

Theorem 3.3 *Let* V *be a derivative on* S *with maturity* n*. Suppose*
1) we are given V_n *and it is* $\mathcal{F}_n$*-measureable,*
2) $\{B_t^{-1} S_t\}_{t=0}^n$ *is a* $\mathbb{P}$*-martingale.*
Then

$$V_0 = \mathbb{E}_{\mathbb{P}}((1+r)^{-n} V_n)$$

Proof Let $N_t = \mathbb{E}_{\mathbb{P}}(\frac{V_n}{B_n}\,|\,\mathcal{F}_t)$ for $t = 0, 1, 2, \ldots, n$. Note that

$$N_0 = \mathbb{E}_{\mathbb{P}}(\frac{V_n}{B_n}\,|\,\mathcal{F}_0) = \mathbb{E}_{\mathbb{P}}(\frac{V_n}{B_n}) \tag{10}$$

$$N_n = \mathbb{E}_{\mathbb{P}}(\frac{V_n}{B_n}\,|\,\mathcal{F}_n) = \frac{V_n}{B_n} \tag{11}$$

By Theorem , for $t = 1, 2, \ldots, n$,

$$\phi_t = \frac{N_t - N_{t-1}}{B_t^{-1} X_t - B_{t-1}^{-1} X_{t-1}} = \frac{(N_t - N_{t-1}) B_t}{X_t - B_t B_{t-1}^{-1} X_{t-1}} \tag{12}$$

We now construct a series of self-financing portfolios with the same payoff as V at time n. At time $t = 0, 1, \ldots, n-1$, form the portfolio Π_t :

- ϕ_{t+1} units of the security
- $N_t - \frac{\phi_{t+1}X_t}{B_t}$ units of B_t

Value of Π_t at time $[t, t+1)$:

$$\phi_{t+1}X_t + (N_t - \frac{\phi_{t+1}X_t}{B_t})B_t = N_tB_t$$

Value of Π_t at time $[t+1, t+2)$:

$$\begin{aligned} & \phi_{t+1}X_{t+1} + (N_t - \frac{\phi_{t+1}X_t}{B_t})B_{t+1} \\ = & \phi_{t+1}X_{t+1} + N_tB_{t+1} - \phi_{t+1}X_t\frac{B_{t+1}}{B_t} \\ = & N_tB_{t+1} + \phi_{t+1}(X_{t+1} - X_t\frac{B_{t+1}}{B_t}) \\ = & N_{t+1}B_{t+1} \qquad \text{by (12)} \end{aligned}$$

The above shows that starting with an initial value N_0, the above series of portfolios are self-financing. Note that the value of Π_{n-1} at time n is V_n. As the series of portfolios have the same paid off as V at time n. The value of V and the value of Π_0 must be the same at time 0 to avoid arbitarge. ie

$$V_0 = \text{value of } \Pi_0 \text{ at time } 0 \ = N_0 = \mathbb{E}_{\mathbb{P}}(\frac{V_n}{B_n})$$

□

In fact, we have proved the following.

Theorem 3.4 *(Assumption as in (3.3)) For* $t = 0, 1, \ldots, n$

$$V_t = \mathbb{E}_{\mathbb{P}}(B_n^{-1}B_{t_0}V_n|\mathcal{F}_t)$$

Proof Let $1 \leq t_0 \leq n-1$ be given. At time $t = t_0, t_0+1, \ldots, n-1$, form the portfolio Π_t as described in Theorem 3.3. As the value of Π_{n-1} have the same payoff as V at time n, their values at time t_0 must be the same. i.e

$$\begin{aligned} V_{t_0} &= \text{value of } \Pi_{t_0} \text{ at time } t_0 \\ &= B_{t_0}N_{t_0} \\ &= B_{t_0}\mathbb{E}_{\mathbb{P}}(B_n^{-1}V_n \,|\, \mathcal{F}_{t_0}) \\ &= \mathbb{E}_{\mathbb{P}}(B_t^{-1}B_{t_0}V_n \,|\, \mathcal{F}_{t_0}) \quad \text{ as } B_{t_0} \text{ is } \mathcal{F}_{t_0}\text{-measurable} \end{aligned}$$

□

In view of the above results, in order to price a derivative on S, we need to find a probability that turns $\{B_t^{-1}S_t\}_t$ into a $\mathbb{P}$-martingale.

4 Arbitarge-free pricing of binomial process with constant spot rates

Recall that we have a binomial tree and

- (t, j) denotes the node at time t and state j
- $r(t, j)$ is the interest rate over $[t, t+1]$ at (t, j)
- $D(t, j)$ is the discount factor at (t, j)over $[t, t+1]$.

Note that

$$D(t,j) = \begin{cases} e^{-r(t,j)} & \text{if interest is compounded continuously} \\ \frac{1}{1+r(t,j)} & \text{if simple interest is used} \end{cases}$$

and

$$B(t) = \begin{cases} e^{rt} & \text{if interest is compounded continuously} \\ (1+r)^t & \text{if simple interest is used} \end{cases}$$

Let S be a security and and we are given S_t, a $\mathcal{F}_t$-measureable function, the value of S at time t, for $t = 0, 1, \ldots$.

We now define a probability $\mathbb{P}$ on $(\Omega_n, \mathcal{F}_n)$ such that $\{B_t^{-1} S_t\}_{t=0}^n$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t=0}^n)$ - martingale.
For any $\vec{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega_n$, define $p_t(\vec{\omega})$, $q_t(\vec{\omega}) = 1 - p_t(\vec{\omega})$, to be the solution of

$$S_t(\vec{\omega}) = D_t(\vec{\omega})\Big(p_t(\vec{\omega}) \cdot S_{t+1}(\omega_1, \ldots, \omega_t, H) \; + \; q_t(\vec{\omega}) \cdot S_{t+1}(\omega_1, \ldots, \omega_t, T)\Big) \tag{13}$$

Define

$$f_{t+1}(\vec{\omega}) = \begin{cases} p_t(\vec{\omega}) & \text{if } \omega_{t+1} = H \\ p_t(\vec{\omega}) & \text{if } \omega_{t+1} = T \end{cases}$$

Note that $f_{t+1}(\vec{\omega})$ only depends on the first $k+1$ entries of $\vec{\omega}$.
Define a probability $\mathbb{P}$ on $(\Omega_n, \mathcal{F}_n)$ as follows. For any $\vec{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega_n$, define

$$\mathbb{P}(\vec{\omega}) = f_1(\vec{\omega}) \cdot f_2(\vec{\omega}) \cdots f_n(\vec{\omega}) \tag{14}$$

It is easy to check that (5) is satisfied.

Theorem 4.1 *$\{B(t)^{-1} S_t\}_{t=0}^n$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t=0}^n)$ -martingale.*
i.e. $E_{\mathbb{P}}(B_{t+1}^{-1} S_{t+1} \,|\, \mathcal{F}_t) = B_t^{-1} S_t$

Proof Let $0 \le t \le n-1$ and $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n$. Then

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}(B_{t+1}^{-1} S_{t+1} \,|\, \mathcal{F}_t)(\vec{\omega}) \\ = \; & B_{t+1}^{-1}(\vec{\omega}) \cdot \mathbb{E}_{\mathbb{P}}(S_{t+1} \,|\, \mathcal{F}_t)(\vec{\omega}) \qquad \text{as } B_{t+1} \text{ is } \mathcal{F}_t\text{-measureable} \\ = \; & B_{t+1}^{-1}(\vec{\omega}) \cdot \Big(S_t(\omega_1, \ldots, \omega_i, H) \cdot p_t(\vec{\omega}) \; + \; S_t(\omega_1, \ldots, \omega_t, T) \cdot q_t(\vec{\omega})\Big) \text{ by Lemma 3.1} \\ = \; & B_{t+1}^{-1}(\vec{\omega}) \cdot D_t(\vec{\omega}) \cdot S_t(\vec{\omega}) \qquad \text{by 14} \\ = \; & B_t^{-1}(\vec{\omega}) \cdot S_t(\vec{\omega}) \qquad \square \end{aligned}$$

We can now price any derivatives on S using Theorem 3.3.
Example (Notation as above). Suppose we have the following.

- $S(0,0) = S_0$.
-
$$S_{t+1} = \begin{cases} dS_t & \text{if upward branch is taken} \\ uS_t & \text{otherwise} \end{cases}$$

  where d, u are constants with $0 < d < 1 < u$. It follows that

$$S_t(\vec{\omega}) = S_0 \binom{n}{\#_t(\vec{\omega})} u^{\#_t(\vec{\omega})} d^{n-\#_t(\vec{\omega})}$$

  for any $\vec{\omega} \in \Omega_n$.

- $r(t,j) = r$, a constant.

As the spot rates are constant, the $D(t,j)$s are also constant. We put $D = D(t,j)$. Then, for any $\vec{\omega} \in \Omega_n$, (13) gives

$$p_t(\vec{\omega}) = \frac{(1 - dD)}{D(u-d)}$$

which is independent of $\vec{\omega}$. In this case, the probability of going from $(t, \#_t(\vec{\omega}))$ to $(t+1, \#_t(\vec{\omega})+1)$ is independent of $\vec{\omega}$. Note that $B_t = D^{-t}$. Let V be a derivative on S, by Theorem 3.4 is $V_t = \mathbb{E}_{\mathbb{P}}(D^{n-t}V_n \,|\, \mathcal{F}_t)$.

5 Arbitarge free pricing of interest rate derivatives over a binomial tree

Recall that we have a spot rate binomial tree. Let X denote the (default-free) zero-coupond bond with maturity $n+1 (n \geq 1)$ and face value of \$1. By Theorem (3.4), in order to price a derivative on X, we need to find a probability on $(\Omega_n, \mathcal{F}_n)$ such that

$$\{B_t^{-1} X_t\}_{t=0}^{n} \qquad \text{is a } \mathbb{P}\text{-martingale.} \tag{15}$$

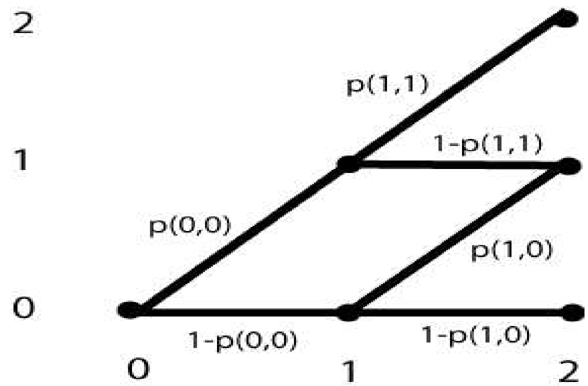
We can not just copy the argument in section 4 because we do not know X_t, the price of X at time t. In the following, we shall turn $\{B_t^{-1} X_t\}_{t=0}^{n}$ into a martingale in two steps.
Step 1 Find a probability $\mathbb{P}$ on $(\Omega_n, \mathcal{F}_n)$ such that

$$\begin{aligned} D(n+1) &= \mathbb{E}_{\mathbb{P}}(B_n^{-1} D_n) \\ &= \mathbb{E}_{\mathbb{P}}(D_0 D_1 \cdots D_n) \end{aligned} \tag{16}$$

Step 2 Use $\mathbb{P}$ from Step 1 to define X_t for $t = 0, 1, \ldots, n-1$ such that $\{B_t^{-1} X_t\}_{t=0}^{n}$ is a $\mathbb{P}$-martingale.

Note that if our spot rate binomial tree is generated from Ho-Lee or BDT model, then $\mathbb{P}$ is known (see [1, Chapter 15]). We shall construct a $\mathbb{P}$ that satisfies (16) by induction on n. One application of the construction described below is to verify spot rate binomial trees generated from Ho-Lee or BDT model. Lets do an example to illusturate how we might go about finding $\mathbb{P}$ that satisifies (16).

Example Case n=3 Suppose the risk neutral probability at (t,j) of going to $(t+1, j+1)$ is $p(t,j)$. (Then the risk neutral probability at (t,j) of going to $(t+1,j)$ is $1 - p(t,j)$.)



Then, for $n = 0, 1$, (16) gives

$$D(2) = D(0,0)D(1,1)p(0,0) + D(0,0)D(1,0)(1 - p(0,0)) \tag{17}$$

and

$$\begin{aligned} D(3) &= D(0,0)D(1,1)D(2,2)p(0,0)p(1,1) + D(0,0)D(1,1)D(2,1)p(0,0)(1 - p(1,1)) \\ &+ D(0,0)D(1,0)D(2,1)(1 - p(0,0))p(1,0) \\ &+ D(0,0)D(1,0)D(2,0)(1 - p(0,0))(1 - p(1,0)) \end{aligned} \tag{18}$$

We can deduce $p(0,0)$, from (17). If we assume $p(1,1) = p(1,0)$, then we can get $p(1,1)$ from (18). We now construct $\mathbb{P} : (\Omega_2, \mathcal{F}_2) \longrightarrow [0,1]$ in the same as (14). Define

$$f_1(\vec{\omega}) = \begin{cases} p(0,0) & \text{if } \omega_1 = H \\ 1 - p(0,0) & \text{if } \omega_1 = T \end{cases}$$

$$f_2(\vec{\omega}) = \begin{cases} p(1,1) & \text{if } \omega_2 = H \\ 1 - p(1,1) & \text{if } \omega_2 = T \end{cases}$$

We define $\mathbb{P} = f_1 f_2$. (End of Example)

We now describe the construction of a probability $\mathbb{P}_t : (\Omega_t, \mathcal{F}_t) \longrightarrow [0,1]$, where $t = 1, 2, \cdots, n$ such that

$$D(t+1) = \mathbb{E}_{\mathbb{P}_t}(D_0 D_1 \cdots D_t) \tag{19}$$

Case $n = 1$ Let p_0 be the solution of

$$D(2) = D(0,0)p_0 D(1,1) + D(0,0)(1 - p_0)D(1,0) \tag{20}$$

Define $\mathbb{P}_1 : \Omega_1 \longrightarrow [0,1]$ by

$$\mathbb{P}_1(\omega) = \begin{cases} p_0 & \text{if } \omega = H \\ 1 - p_0 & \text{if } \omega = T \end{cases} \tag{21}$$

where $\omega \in \Omega_1$ and By (20), (19) is satisfied for $t = 1$.

Inductive step Suppose we have constructed $\mathbb{P}_t$ and (19) holds for some $1 \leq t < n$. Define $\mathbb{P}_{t+1} : \Omega_{t+1} \longrightarrow [0,1]$ by

$$\mathbb{P}_{t+1}(\vec{\omega}) = \begin{cases} \mathbb{P}_t(\vec{\omega})p_t & \text{if } \omega_{t+1} = H \\ \mathbb{P}_t(\vec{\omega})(1 - p_t) & \text{if } \omega_{t+1} = T \end{cases} \tag{22}$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_{t+1}) \in \Omega_{t+1}$ and p_t is to be determined from (19). $\mathbb{P}_t(\vec{\omega})$ is defined to be $\mathbb{P}_t$ of the first t entries of $\vec{\omega}$.
We shall choose p_t to satisfy (19) :

$$D(t+2) = \sum_{\vec{\omega} \in \Omega_{t+1}} \mathbb{P}_{t+1}(\vec{\omega}) D_0(\vec{\omega}) D_1(\vec{\omega}) \ldots D_{t+1}(\vec{\omega}) \tag{23}$$

Let

$$\begin{aligned} A &= \{(\omega_1, \ldots, \omega_{t+1}) \in \Omega_{t+1} \,|\, \omega_{t+1} = H\} && (24) \\ B &= \{(\omega_1, \ldots, \omega_{t+1}) \in \Omega_{t+1} \,|\, \omega_{t+1} = T\} && (25) \end{aligned}$$

Then (23) becomes

$$\begin{aligned} D(t+2) &= \Big(\sum_{\vec{\omega} \in A} \mathbb{P}_t(\vec{\omega}) D_0(\vec{\omega}) D_1(\vec{\omega}) \ldots D_{t+1}(\vec{\omega}) \Big) p_t \\ &+ \Big(\sum_{\vec{\omega} \in B} \mathbb{P}_t(\vec{\omega}) D_0(\vec{\omega}) D_1(\vec{\omega}) \ldots D_{t+1}(\vec{\omega}) \Big) (1 - p_t) && (26) \end{aligned}$$

In the above equation, the coefficients of p_t and $(1 - p_t)$ are known. Hence we can work out p_t . As all the steps above are reversible, we have constructed a $\mathbb{P}_{t+1}$ that satisfies (19). By induction, we can find a $\mathbb{P}$ that satisfies (16).

We now proceed with Step 2. In the following, let $\mathbb{P}$ satisfies (16). Recall that X is a (default-free) zero coupond bond with maturity $n+1$ and face value \$1. Define $X_0 = D(n+1)$, $X_n = D_n$. For $0 \leq t < n$ and $\vec{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega_n$ define

$$X_t(\vec{\omega}) = D_t(\vec{\omega}) \Big(p_t(\vec{\omega}) \cdot X_{t+1}(\omega_1, \ldots, \omega_t, H) \,+\, q_t(\vec{\omega}) \cdot X_{t+1}(\omega_1, \ldots, \omega_t, T) \Big) \tag{27}$$

where $p_t(\vec{\omega}), q_t(\vec{\omega})$ are as defined in (13).

Theorem 5.1 $\{B_t^{-1} X_t\}_{t=0}^n$ *is a* $(\mathbb{P}, \{\mathcal{F}_t\}_{t=0}^n)$ *-martingale.*
i.e. $E_{\mathbb{P}}(B_{t+1}^{-1} S_{t+1} \,|\, \mathcal{F}_t) = B_t^{-1} S_t$

Proof The same argument in the proof of Theorem 4.1. $\square$

References

[1] R Jarrow and S Turnbull, *Derivative Securities*, South-Western College Publishing