On binomial expectations and option pricing

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Abstract :

We show how a discrete random variable on a finite probability space endowed with a binomial distribution may be close to a random variable on the continuum, in a way which respects the expectations. As an application, we approximate the random variables of a discrete geometric binomial process by continuous exponentials, and thus derive an option price formula, which contains the formula of Black and Scholes as a special case.

Keywords : binomial distribution, standard normal distribution, Riemann-sum, shadow, discrete geometric Brownian motion, option pricing.

1 Introduction

The present paper studies continuous approximations of discrete expressions in the context of elementary probability theory. The main result, Theorem 2.1, is a sort of extension of the De Moivre-Laplace central limit theorem, and concerns the approximation of the expectation of a random variable with respect to a binomial distribution by an expectation with respect to the standard normal distribution.

Our study is motivated by the derivation of the Black-Scholes formula (see [4]) for the pricing of European call options. In [5], J. C. Cox, S. A. Ross and M. Rubinstein presented an option pricing formula in the form of a discrete binomial expectation, and then they showed that in the limit it converged to the Black-Scholes formula.

As a consequence of our main theorem we obtain a pricing formula for continuous options, of which the Black-Scholes formula is a special case. Our derivation is both more direct and more general than the derivation of Cox, Ross and Rubinstein: we reduce their sum formula to a Riemann-sum of the Black-Scholes integral formula. However, our setting is still their simple discrete pricing model, and thus avoids entirely the complications of limits of stochastic processes, continuous stochastic processes and measure theory. Instead, we apply nonstandard analysis, and following N. G. Cutland, E. Kopp and W. Willinger [6], we assume that the time steps of the discrete model are infinitesimal. With respect to their approach to option pricing, we obtained a further simplification, by avoiding the transitions between a standard and a nonstandard model, and Loeb-measure theory.

Notations and a presentation of the main result: We start by introducing some conventions and notations, and an informal presentation of the results.

Let

$$B_{N,p}(j) = \begin{pmatrix} N \\ j \end{pmatrix} p^j (1-p)^{N-j}$$

be the j^{th} binomial coefficient and put

$$\begin{array}{rcl} \mu_p & = & N \cdot p \\ \sigma_p & = & \sqrt{N \cdot p(1-p)} \\ x_j & = & \dfrac{j - \mu_p}{\sigma_p} \\ \Omega_p & = & \{x_j \mid j = 0, 1, \ldots, N\}. \\ dx_j & = & x_{j+1} - x_j \end{array} \tag{1.1}$$

Notice that the x_j are "normalized" with respect to the probability distribution $B_{N,p}(j)$: their mean is 0 and their standard deviation equals 1. For large N we have the well-known approximation

$$B_{N,p}(j) \sim \frac{1}{\sqrt{2\pi}\sigma_p} \cdot \mathrm{e}^{-\left(\frac{j-\mu_p}{\sigma_p}\right)^2/2} = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-x_j^2/2} dx_j$$

It may be expected that the approximation carries over to sums:

$$\sum_{i \leq j} B_{N,p}(i) \sim \frac{1}{\sqrt{2\pi}} \cdot \sum_{i \leq j} e^{-x_i^2/2} dx_i$$

Thus we sketched a derivation of the De Moivre-Laplace central limit theorem

$$\sum_{i \leq j} B_{N,p}(i) \sim \mathcal{N}\left(\frac{j-\mu_p}{\sigma_p}\right)$$

where $\mathcal{N}$ is the normal distribution function given by

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-x^2/2} dx. \tag{1.2}$$

Our main result concerns expectations of the form

$$\mathrm{E}(h) = \sum_{i=0}^{N} h(x_i) B_{N,p}(i) \tag{1.3}$$

where h is a discrete random variable defined on the x_i's.

We show that under a suitable condition the above reasoning can be extended to this sum, leading to the approximations

$$\begin{aligned} \sum_{i=0}^{N} h(x_i) B_{N,p}(i) &\sim \frac{1}{\sqrt{2\pi}} \cdot \sum_{i=0}^{N} h(x_i) \mathrm{e}^{-x_i^2/2} dx_i \\ &\sim \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{h}(x) \cdot \mathrm{e}^{-x^2/2} dx \end{aligned}$$

where $\overline{h}$ is a continuous real function, closely related to h. So, indeed we transformed an expectation with respect to the binomial distribution into an expectation with respect to the standard normal distribution. We remark that our formal nonstandard proof will be very similar to the observations above.

Discrete arithmetic and geometric Brownian motions: Our application concerns the approximation of the expectation of a random variable with respect to a discrete geometric binomial process $S(t,x)$. This process will be defined on a *binomial cone* . Let $T > 0$, $N \in \mathbb{N}$, and $dt > 0$ be such that $N dt = T$. Then $W_{T,dt}$ is the cone given by

$$W_{T,dt} = \left\{ (t,x) \in [0,T] \times \mathbb{R} \;\middle|\; \begin{array}{c} \exists m,n \in \mathbb{N}, \quad 0 \leq m \leq n \leq N \\ \text{and} \\ t = ndt, \quad x = (-n+2m)\sqrt{dt} \end{array} \right\} \tag{1.4}$$

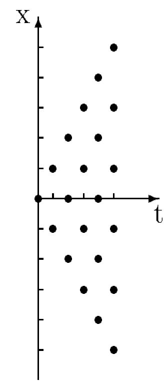

Figure 1: A binomial cone for $N = 5$.

We call dt the *period* of the cone and

$$\mathbf{T} \equiv \{0, dt, 2dt, \ldots, N dt = T\} \tag{1.5}$$

the *time line* of the cone. Notice that $W_{T,dt}$ is the union of all trajectories of the discrete arithmetic Brownian motion ("Wiener walk") on the time line $\mathbf{T}$. Sometimes we simply write W_T instead of $W_{T,dt}$. We write $W_T(t)$ for the vertical sections of the cone; they correspond to the values reached at time t by the sample paths of the discrete arithmetic Brownian motion. Usually dt is infinitesimal, and then we speak also of an *infinitesimal binomial cone*. Notice that in this case the vertical step $\sqrt{dt}$, though still infinitesimal is infinitely large with respect to the horizontal step dt. See also Figure 1.

The process $S(t,x)$, called the *discrete geometric Brownian motion*, is defined by induction on W_T. Let $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $0 < p < 1$. We put

$$S(0,0) = S_0$$

and for $t \in \mathbf{T}$, $t < T$

$$\left\{ \begin{array}{rcl} S(t+dt,\ x+\sqrt{dt}) & = & S(t,x) \cdot (1 + \mu dt + \sigma\sqrt{dt}) \\ S(t+dt,\ x-\sqrt{dt}) & = & S(t,x) \cdot (1 + \mu dt - \sigma\sqrt{dt}) \end{array} \right. \tag{1.6}$$

Then indeed the process is defined on $W_{T,dt}$. We assume that the upper increment of (1.6) has conditional probability p, and the lower increment has conditional probability $1-p$, and that the increments are independent in time. Then $S(t,x)$ is properly defined as a stochastic process, and up to elementary transformations its random variables $S(t) \equiv S(t,\cdot)$ have binomial distributions. In particular

$$Pr\left\{S(T) = S_0\left(1+\mu dt + \sigma\sqrt{dt}\right)^j\left(1+\mu dt - \sigma\sqrt{dt}\right)^{N-j}\right\} = B_{N,p}(j).$$

Note that if $p = 1/2$, then μ is the relative conditional expectation, or *drift rate* of the process and σ^2 its relative conditional variance, or *volatility*.

Expectations and option pricing: In the economic context of option pricing, the process $S(t,x)$ endowed with the conditional probability $p = \frac{1}{2}$ is considered as a model describing the possible movements in time of the price of a share of some stock; trading is allowed at the times $\{0,\, dt,\, 2dt,\, \ldots,\, T\}$, the drift rate of the stock price being equal to μ, and its volatility σ. Given a real-valued function f, the random variable $f(S(T))$ models a claim on that share at the future time T. For instance, let $K > 0$. Then the claim

$$f(S(T)) = (S(T) - K)^+$$

is called *the European call option with exercise date T and with striking price K*. It models the payoff of a contract giving its owner the right to buy the share S at time T for the price K.

In fact, we described a stochastic process which is suitable for the discrete option pricing model of Cox, Ross and Rubinstein. They argue (see also [4]) that if r is the risk-free rate of interest, the correct price C_{dt} of the claim f must be the Present Value (henceforth PV_r) of the expectation of the random variable $f(S(T))$ in a risk-neutral world (that is, the drift rate μ of the process S must be r). Let then

$$\mathrm{E}_r f(S(T)) \tag{1.7}$$

denote the expectation of the random variable $f(S(T))$ in a risk-neutral world. Then

$$C_{dt} = PV_r(\mathrm{E}_r f(S(T))) \tag{1.8}$$

Recall that the present value in a risk-neutral world of an asset A equals its future value $A(T)$ at time T discounted at the risk-free rate of interest. That is to say

$$PV_r(A) = A(T)/(1+rdt)^{T/dt}$$

If the process $S(t,x)$ is in a risky world, (that is, its drift rate μ is different from r) then it is always possible to adjust its conditional probability p to some value $p(r)$ which will change its drift rate to the prescribed risk-free rate of interest $r \in \mathbb{R}$. Note that $p(r)$ must satisfy

$$p(r)(1+\mu dt+\sigma\sqrt{dt})+(1-p(r))(1+\mu dt-\sigma\sqrt{dt}) = 1+rdt$$

so

$$p(r) = \frac{1}{2} + \frac{r-\mu}{2\sigma}\sqrt{dt} \tag{1.9}$$

In the remaining, the conditional probability of the process $S(t,x)$ is $p(r)$, but for convenience, we will write it p.

The expectation (1.7) can be written in the form (1.3); indeed define the affine transformation $\nu_p : \Omega_p \to W_T(T)$ by

$$\nu_p(x) = 2\sqrt{T}\left[\sqrt{p(1-p)}x + \left(p-\frac{1}{2}\right)\frac{\sqrt{T}}{\sqrt{dt}}\right] \tag{1.10}$$

Then we have

$$\mathrm{E}_r f(S(T)) = \sum_{j=0}^{N} f(S(T,\ \nu_p(x_j))B_{N,p}(j) \tag{1.11}$$

Due to the rapid trading at stock markets, economists prefer a market model with a continuous time line: the Black-Scholes market, for which the option price C_0 becomes the Black-Scholes formula [4]. Now both the formulation of a Black-Scholes market model, and a derivation of the Black-Scholes formula within such a model are very intricate (see [9] for a survey). Instead, as argued by Cutland, Kopp and Willinger in [6], the Cox-Ross-Rubinstein model is a good alternative, provided the period dt is infinitesimal: it expresses rapid trading, it has the simplicity of a discrete model, and the option price C_{dt} almost does not depend on the length of dt. In fact, the difference between C_{dt} and the Black-Scholes price C_0 is infinitesimal under some natural conditions on the order of magnitude of the parameters involved. Indeed, using the approximation of ν_p given in Lemma 2.4, that of $S(T)$ given in Proposition 2.5, and that of the binomial expectation stated in the main theorem (Theorem 2.1), we prove that

$$PV_r\left(\mathrm{E}_r f(S(T))\right) \simeq \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f\left(S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}\right) \cdot \mathrm{e}^{-x^2/2} dx \tag{1.12}$$

The integral of the right-hand side of (1.12) is the Feynman-Kac formula (see [9]).

There are three main differences between the work of Cutland, Kopp and Willinger [6] and our approach. First to estimate $S(T)$ they use a nonstandard Itô-calculus, while we use a "method of lines." Second to relate the discrete and the continuous they use the Loeb-measure and Loeb-spaces [16], while we use Riemann-sums, such as sketched above, and the external numbers of [13] and [14]. Third, their setting is Robinsonian nonstandard analysis [20], while our setting is axiomatic nonstandard analysis IST [18]. The main difference is that in the latter approach the infinitesimals are included within the set of real numbers $\mathbb{R}$, while in the former approach they are included in a nonstandard extension of $\mathbb{R}$.

Outline of this paper: This paper has the following structure. In Section 2 we state and prove our main theorem on the approximation of binomial expectations by standard normal expectations. We also show how some expectations in a somewhat more general setting may be reduced to the main theorem, by a lemma of Girsanov type.

In Section 3 we approximate the discrete random variables of the geometric binomial process $S(t, x)$ by continuous exponential functions, and then we state and prove the option pricing formula of continuous or nearly continuous claims, of which the Black-Scholes formula is a consequence.

The reader is referred to [7] or [17] for an introduction to the axiomatic nonstandard analysis IST, to [1], [2] or [19] for an account of discrete stochastic processes from a nonstandard point of view, and to [9] or [11] for the classical option pricing theory.

To simplify our approximative and asymptotic calculations we use a sort of nonstandard $\mathcal{O}$-calculus; *i.e.* the calculus of external numbers and external intervals of [13] and [14]. We recall here some notations: the symbol $\oslash$ designates the external set of infinitesimals, the symbol $\mathcal{L}$ the external set of limited numbers, the symbol @ the external set of positive appreciable numbers and the symbol $\not\infty$ the external set of all positive infinitely large numbers.

2 Binomial expectations

2.1 Preliminaries

Our main theorem relates the expectation of a discrete, nearly continuous random variable to the expectation of a properly continuous random variable. Nonstandard analysis makes it possible to express near-continuity of a discrete function through the notion of S-*continuity*, and to describe the transition from the discrete to the continuous by the notion of shadow.

If the difference between two real numbers u and v is infinitesimal, we write $u \simeq v$. Otherwise, we write $u \not\simeq v$. We recall the notion of *S-continuity*: a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is *S-continuous* on D if for all $x, y \in D$

$$x \simeq y \Rightarrow f(x) \simeq f(y)$$

Standard continuous functions are S-continuous (see robinson-ans), but we will see examples of discrete (nonstandard) functions which are S-continuous.

Grosso modo, if A is a set, the *shadow* or *standard part* of A is the standard set ${}^oA$ which is most close to it. We do not state the formal definition of the shadow, which uses the concept of Standardization [18]. Instead we refer to [7], and give some examples, which illustrate how this notion may relate the discrete and the continuous.

- Let $dt > 0$ be infinitesimal, then

$$^o dt = 0$$

- Let T be standard, and assume $T = N dt$, where $N \in \mathbb{N}$ is infinitely large. The shadow of the discrete "infinitesimal time line (1.5)" is the continuous time interval $[0, T]$:

$$^o\mathbf{T} = [0, T].$$

- Assume that $0 \underset{\not\simeq}{<} p \underset{\not\simeq}{<} 1$, and consider the set Ω_p of (1.1). Again, the difference $1/\sqrt{p(1-p)N}$ of two successive members of Ω_p is infinitesimal; note also that Ω_p contains negative and positive unlimited numbers. The shadow of Ω_p is the whole continuum:

$$^o\Omega_p = \mathbb{R}$$

• Let $r > 0$ be a standard real number, and consider the discrete function $f : \mathbf{T} \to \mathbb{R}$ defined by

$$f(t) = (1 + rdt)^{t/dt}$$

Note that we have the Euler approximation

$$(1 + rdt)^{t/dt} \simeq \mathrm{e}^{rt}$$

Clearly f is S-continuous on $\mathbf{T}$. The shadow of f is the continuous function ${}^o f : [0, T] \to \mathbb{R}$ given by

$$f(t) = \mathrm{e}^{rt} \tag{2.1}$$

As will be shown in this paper, the shadow of the random variables of the geometric binomial process $S(t, x)$ are also continuous exponentials.

• In general, let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called of *class* S^o if it is S-continuous and takes limited values at limited arguments of D. The theorem of the continuous shadow of Robinson (see Theorem 4.5.10 of [20], and [7]) states that such a function has a shadow

$${}^o f : {}^o D \to \mathbb{R}$$

which is standard continuous, and moreover satisfies for all $t \in {}^o D \cap D$

$$({}^o f)(t) \simeq f(t)$$

• The last example concerns Riemann-sums. Let $\mathrm{st}\, a, b,\ a < b$. Let $dx > 0$, $N \in \mathbb{N}$ be such that $b - a = Ndx$, and consider a function

$$f : \{a,\ a + dx,\ \ldots a + Ndx = b\} \to \mathbb{R}$$

Assume $dx \simeq 0$ and f is of class S^o. Then

$$\sum_{j=0}^{N} f(a + jdx)dx \simeq \int_a^b ({}^o f)(y)dy$$

We may extend the approximation to the "external integration" of [13] or [14]; *i.e.* if f is at least defined for all j such that jdx is limited, we have, in the sense of external numbers

$$\sum_{jdx = \mathcal{L}} f(jdx)dx \equiv \int_{\mathcal{L}} ({}^o f)(y)dy$$

In fact our main theorem will be proved along these lines.

Before presenting the main theorem, we formulate some nonstandard growth conditions. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is of exponential order at $+\infty$ if there are numbers A, K, C such that for all $|x| > A$

$$|f(x)| \leq Ke^{Cx}$$

A function f is said to be of *S-exponential order* if the above numbers may be taken standard. Expressed in terms of external numbers this becomes

$$f(x) = \mathcal{L}e^{@x}$$

for all positive, infinitely large x. It is an elementary nonstandard exercise to prove the following property: if $f : D \subset \mathbb{R} \to \mathbb{R}$ is of class S^o, and of S-exponential order, its shadow is of exponential order.

In the same spirit, a function $f :]0, \infty[\to \mathbb{R}$ is said to be *of rational growth in* 0 *or in* $+\infty$ if there are constants $A > 0$, K, r such that $f(x) \leq Kx^r$ for $x \geq A$ and constants $B > 0$, L, s such that $f(x) \leq Lx^s$ for $x \leq B$. If these constants may be taken standard, then f is said to be of *S-rational growth* at 0 and $+\infty$. Also, if f is of class S^o and of S-rational growth, its shadow is of rational growth.

2.2 The main theorem

Theorem 2.1 (Main Theorem) *Let $N \simeq +\infty$, $0 \lneq p \lneq 1$ and Ω_p be the probability space given by* (1.1) *and endowed with the binomial distribution $B_{N,p}$. Let $h : \Omega_p \to \mathbb{R}$ be a random variable of class S^o, and of S-exponential order in $+\infty$. Then*

$$E(h) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {}^o h(x) \cdot e^{-x^2/2} dx \tag{2.2}$$

The proof of the theorem will be divided in two parts: an approximation of the binomial coefficients, and the transition of a Riemann-sum into the Riemann-integral.

Proposition 2.2 *Let $N \simeq +\infty$, $0 \lneq p \lneq 1$, and $0 \leq j \leq N$. Then*

1. *for all* $j = \mu_p + \mathcal{L}\sigma_p$

$$(2.3) \qquad B_{N,p}(j) = \frac{1+\oslash}{\sigma_p \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2} \cdot \left(\frac{j-\mu_p}{\sigma_p}\right)^2\right]$$

2. *for all* $j = \mu_p \pm \not\infty \cdot \sigma_p$

$$(2.4) \qquad B_{N,p}(j) = \exp\left(-\not\infty \left|\frac{j-\mu_p}{\sigma_p}\right|\right)$$

Proof : A straightforward calculation yields

$$(2.5) \qquad \frac{B_{N,p}(j+1)}{B_{N,p}(j)} = \frac{1-(j-\mu_p)p/\sigma_p^2 + \mathcal{L}/\sigma_p^2}{1+(j-\mu_p)(1-p)/\sigma_p^2 + \mathcal{L}/\sigma_p^2}$$

To show (1), note that (2.5) may be simplified to

$$\frac{B_{N,p}(j+1)}{B_{N,p}(j)} = 1 - \frac{[j-\mu_p]}{\sigma_p^2} + \frac{\mathcal{L}}{\sigma_p^2}$$

So

$$\begin{aligned}(2.6)\ \frac{B_{N,p}(j)}{B_{N,p}([\mu_p])} &= \exp \sum_{i=0}^{[j-\mu_p]} \log\left(1 - \frac{i}{\sigma_p^2} + \frac{\mathcal{L}}{\sigma_p^2}\right) \\ &= \exp \sum_{i=0}^{[j-\mu_p]} \left(-\frac{i}{\sigma_p^2} + \frac{\mathcal{L}}{\sigma_p^2}\right) = \exp\left(-\frac{1}{2} \cdot \left(\frac{j-\mu_p}{\sigma_p}\right)^2 + \oslash\right)\end{aligned}$$

To estimate the term $B_{N,p}([\mu_p])$, note first that by the Mass Concentration Lemma [3] we have

$$\sum_{j=\mu_p+\mathcal{L}\sigma_p} B_{N,p}(j) = 1 + \oslash.$$

So

$$\begin{aligned}(2.7) \qquad \frac{1}{B_{N,p}([\mu_p])} &= \sum_{j=\mu_p+\mathcal{L}\sigma_p} \frac{B_{N,p}(j)}{B_{N,p}([\mu_p])} \\ &= \sigma_p \sum_{j=\mu_p+\mathcal{L}\sigma_p} \exp\left(-\frac{1}{2} \cdot \left(\frac{j-\mu_p}{\sigma_p}\right)^2 + \oslash\right) \cdot \frac{1}{\sigma_p} \\ &= \sigma_p \left(\int_{\mathcal{L}} e^{-x^2/2} dx + \oslash\right) \equiv \sigma_p \cdot \sqrt{2\pi}(1+\oslash)\end{aligned}$$

Now (2.6) and (2.7) imply (2.3).

To show (2), assume first that $j > \mu_p + \mathcal{L}\sigma_p$. Then (2.5) may be simplified to

$$\frac{B_{N,p}(j+1)}{B_{N,p}(j)} < 1 + \frac{\mathcal{L}}{\sigma_p}.$$

Hence, using the fact that all $B_{N,p}(j)$ are infinitesimal

$$B_{N,p}(j) < B_{N,p}\left(\left[\frac{j+\mu_p}{2}\right]\right) \cdot \left(1 + \frac{\mathcal{L}}{\sigma_p}\right)^{(j-\mu_p)/2} < \exp\left(\mathcal{L} \cdot \frac{j-\mu_p}{\sigma_p}\right)$$

This implies (2.4). The case where $j < \mu_p + \mathcal{L}\sigma_p$ is treated similarly.

Comment : There are many proofs, both classical and nonstandard, of the proposition above, or closely related formulae. See for instance [3], [8], [15] or [10]. The above proof has the advantage of being straightforward, of avoiding the use of Stirling's formula, and of estimating the tails of the binomial distribution.

Proof :[of Theorem 2.1] Using Proposition 2.2 and the additivity of external integration proved in [13] and [14], we obtain

$$\begin{aligned}
\mathrm{E}(h) &= \sum_{j=0}^{N} B_{N,p}(j)h(x_j) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{j=\mu_p+\mathcal{L}\sigma_p} \left(\mathrm{e}^{-x_j^2/2}h(x_j) + \oslash\right) dx_j \\
&\qquad\qquad + \sum_{j=\mu_p \pm \not{\infty}\sigma_p} \mathrm{e}^{-\not{\infty}|x_j|}\mathrm{e}^{@|x_j|}
\end{aligned}$$

To estimate the first sum we use an approximation by the Riemann-integral and to get a (rough) estimate of the second sum, we use the well-known integral-majoration of decreasing series. We find

$$\begin{aligned}
\mathrm{E}(h) &= \frac{1}{\sqrt{2\pi}} \sum_{j=\mu_p+\mathcal{L}\sigma_p} h(x_j) \cdot \mathrm{e}^{-x_j^2/2} dx_j + \oslash \cdot \sum_{j=\mu_p+\not{\infty}\sigma_p} \mathrm{e}^{-|x_j|} dx_j \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathcal{L}} {}^{o}h(x) \cdot \mathrm{e}^{-x^2/2} dx + \oslash + \oslash \cdot \int_{|x|>\mathcal{L}} \mathrm{e}^{-|x|} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {}^{o}h(x) \cdot \mathrm{e}^{-x^2/2} dx + \oslash
\end{aligned}$$

Comment : There are of course many alternative versions of the above theorem. What matters, is that the mass (see [3]) of the random variable h with respect to the binomial distribution is included within $\mathcal{L}$, and that on this set, the sum $\Sigma B_{N,p}(j)h(x_j)$ acts as a Riemann-sum. This is for instance the case when h is the restriction to Ω_p of a standard Riemann integrable function which is bounded on every standard interval. Note that for st A, and $h = \chi_{]-\infty,A]}$ we thus obtain the De Moivre-Laplace central limit theorem

$$\sum_{(j-\mu_p)/\sigma_p \leq A} B_{N,p}(j) \simeq \mathcal{N}(A)$$

where $\mathcal{N}$ is given by (1.2).

Further, the main theorem is a consequence of a fundamental theorem of Loeb-measure theory. Indeed,*mutatis mutandis*, the finite sequence

$$\{ B_{N,p}(j) \cdot h(x_j)/dx_j \ \mid \ \ j = 0, 1, \ldots, N\}$$

is a "lifting" of the standard Lebesgue integrable function ${}^{o}h(x) \cdot \mathrm{e}^{-x^2/2}$, and then, say, Theorem IV.1.16 of [12] applies.

2.3 Relative normalisation

The main theorem concerns random variables on the probability space

$$\Omega_p = \{x_1, \ldots, x_N\},$$

i.e. the space on which the probability distribution $B_{N,p}$ is normalised. We will consider here a more general case of functions on a space which are easily transformed into random variables on the normalised probability space Ω_p.

Definition 2.3 *Let $N \in I\!N$ and $\Omega \equiv \{y_1, \ldots, y_N\}$ be a finite set. The transformation $\nu_p : \Omega_p \to \Omega$ defined by*

$$\nu_p(x_j) = y_j$$

where ν_p is given by (1.10)*, is called a* normalisation *of Ω with respect to the binomial distribution $B_{N,p}$.*

Let $f : \Omega \to I\!R$ be a function. The random variable $f_p : \Omega_p \to I\!R$ defined by

$$f_p(x) = f(\nu_p(x))$$

is called the relative normalisation *of f with respect to the binomial distribution $B_{N,p}$.*

If $\Omega \subset \mathbb{R}$ consists of N equidistant points, the normalisation of Ω is an affine transformation. Next lemma gives an approximation of this transformation in case Ω is (the image of) the random variable $W_T(T)$ of the binomial cone given by (1.4); *i.e.* the case of N equidistant points at distance $2\sqrt{dt}$ and of mean 0.

Lemma 2.4 *Let $T > 0$ be appreciable, $dt > 0$ be infinitesimal, and $N \in \mathbb{N}$ be such that $N dt = T$. Assume $p = \frac{1}{2} + \alpha\sqrt{dt}$, where α is limited. Let ν_p be the normalisation of $W_T(T)$ with respect to the binomial distribution $B_{N,p}$. Then for all $x \in \Omega_p$*

$$\nu_p(x) \simeq 2\alpha T + \sqrt{T}x$$

Proof : Notice that at most $x = \mathcal{L}/\sqrt{dt}$. Hence

$$\begin{aligned} \nu_p(x) &= 2\alpha T + \sqrt{T(1-4\alpha^2 dt)}x = 2\alpha T + \sqrt{T}x + \mathcal{L}dt \cdot x \\ &= 2\alpha T + \sqrt{T}x + \oslash \end{aligned}$$

The lemma may be seen as a very simple case of the classical Girsanov Theorem.

Notice that under the above conditions, a real function f defined on Ω is of class S^o and of S-exponential order if and only if its relative normalisation f_p is of class S^o and of S-exponential order.

ithmetic

Proposition 2.5 *Let $T > 0$ be appreciable, and W_T be a binomial cone. Let $S(t,x)$ be the discrete geometric Brownian motion on W_T with initial value $S_0 > 0$, drift rate μ and volatility σ^2. Assume that S_0 and σ are appreciable, and that μ is limited. Then for all limited x*

$$S(T,x) \simeq S_0 \cdot e^{(\mu - \sigma^2/2)T + \sigma x}$$

Furthermore $S(T,\cdot)$ is of S-exponential order at $\pm\infty$.

Proof : Let N be any unlimited integer, and let $dt = T/N$. (For convenience, we will suppose that N is even.) We estimate $S(T,x)$ by going first horizontally from $S(0,0)$ to $S(T,0)$, and then vertically from $S(T,0)$ to $S(T,x)$.

For all (t,x) on W_T

$$\begin{aligned} S(t+2dt,x) &= S(t,x) \cdot \left(1 + \mu dt + \sigma\sqrt{dt}\right) \cdot \left(1 + \mu dt - \sigma\sqrt{dt}\right) \\ &= S(t,x) \cdot \left(1 + (2\mu - \sigma^2 + \oslash)dt\right) \end{aligned}$$

Hence

$$S(T,0) = S(0,0)\cdot(1+(2\mu-\sigma^2+\oslash)dt)^{T/(2dt)} \simeq S_0\cdot \mathrm{e}^{\left(\mu-\sigma^2/2\right)T}+\oslash$$

Also

$$\begin{aligned} S\left(t,x+2\sqrt{dt}\right) &= S(t,x)\cdot\frac{1+\mu dt+\sigma\sqrt{dt}}{1+\mu dt-\sigma\sqrt{dt}} \\ &= S(t,x)\cdot\left(1+(2\sigma+\oslash)\sqrt{dt}\right) \end{aligned}$$

So

$$S(T,x) = S(T,0)\cdot\left(1+(2\sigma+\oslash)\sqrt{dt}\right)^{x/\left(2\sqrt{dt}\right)} = S(T,0)\cdot\mathrm{e}^{(\sigma+\oslash)x}$$

Hence for all limited x

$$S(T,x)\simeq S_0\cdot\mathrm{e}^{\left(\mu-\sigma^2/2\right)T+\sigma x}$$

and for all (T,x) on W_T

$$S(T,x) = \mathcal{L}\cdot\mathrm{e}^{@x}$$

which means that $S(T,\cdot)$ is of S-exponential order.

The final theorem gives an infinitesimal approximation of the price C_0 of a claim $f(S(T))$ in a Black-Scholes market, using the Cox-Ross-Rubinstein model (see formulae (1.8) and (1.11)) with infinitesimal trading periods.

Theorem 2.6 (Option Pricing Formula) *Let $T>0$ be appreciable and W_T be an infinitesimal binomial cone. Let $S(t,x)$ be the discrete geometric Brownian motion on W_T with appreciable initial value $S_0>0$, limited drift rate μ and appreciable volatility σ^2. Let r be a limited risk-free rate of interest. Let $f: S(T)\to I\!R$ be a S-continuous claim of S-rational growth at 0 and at $+\infty$, and let $C_0 = {}^o\left(PV_r(E_r f(S(T)))\right)$ be the shadow of its price. Then*

$$C_0 \simeq \frac{e^{-rT}}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} {}^o f\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}\right)\cdot e^{-x^2/2}dx \tag{2.1}$$

Proof : Let dt be the infinitesimal time period associated to the cone W_T. Let $p := p(r)$ be the conditional probability which changes the drift rate of $S(t,x)$ from μ into r. By (1.9)

$$p(r) = \frac{1}{2}+\frac{r-\mu}{2\sigma}\sqrt{dt}$$

Notice that $(r-\mu)/2\sigma$ is limited. Hence by Proposition 2.5 and Lemma 2.4, the relative normalisation of the random variable $S(T,\cdot)$ with respect to the binomial distribution $B_{N,p}$ satisfies, for limited x,

$$\begin{aligned} S_p(T,x) &\simeq S_0 \cdot \exp\left((\mu - \sigma^2/2)T + \sigma^2\left(\frac{r-\mu}{2\sigma}\right)T + \sigma\sqrt{T}x\right) \\ &\simeq S_0 \cdot \exp\left((r-\sigma^2/2)T + \oslash\sqrt{T}x\right). \end{aligned}$$

Clearly $(foS)_p = foS_p$ is of class S^o and of S-exponential order in $\pm\infty$. Hence by Theorem 2.1

$$\mathrm{E}_r f(S(T)) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {}^o f\left(S_0 \cdot \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}\right) \cdot \mathrm{e}^{-x^2/2} dx$$

Because $C_0 \simeq PV_r\left(\mathrm{E}_r f(S(T))\right)$, formula (2.1) follows from (2.3).

We notice that the pricing formula (2.1) corresponds to the classical formula in case f is (the restriction to $S(T)$ of) a standard continuous function of rational growth. In particular, we thus obtain the Black-Scholes formula for the European option.

Corollary 2.7 (Black-Scholes formula) *Assume the conditions of Theorem 2.6 are satisfied. Let C_0 be the shadow of the price of a the European call option $(S(T)-K)^+$ with striking price K and exercise date T. Put*

$$x_0 = \frac{\log(S_0/K) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}$$

Then

$$C_0 \simeq S_0 \cdot \mathcal{N}\left(x_0 + \sigma\sqrt{T}\right) - Ke^{-rT} \cdot \mathcal{N}(x_0). \tag{2.2}$$

∎

Notice that (2.1) and (2.2) become identities if S_0, K, r, σ and T are standard. The formula (2.2) is a straightforward standard transformation of formula (2.1), which we omit.

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