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Solving dynamic portfolio problems using stochastic programming

We consider a dynamic stochastic programming problem (MRP) in multistage recourse form in which the portfolio manager seeks the maximization of terminal wealth under a set of linear constraints:

$$\textbf{(MRP)} \qquad \max_{x_1}\{f_1(x_1) + I\!E_{\boldsymbol{\omega_2}}[\max_{x_2}(f_2(x_2(\boldsymbol{\omega}^2)) + \ldots + I\!E_{\boldsymbol{\omega}_T|\boldsymbol{\omega}^{T-1}}[\max_{x_T} f_T(x_T(\boldsymbol{\omega}^T))])]\} \tag{1}$$

s.t.

$$\begin{array}{llllll}
A_1 x_1 & & & & = & b_1 \\
B_2(\boldsymbol{\omega}^2)x_1 & + & A_2(\boldsymbol{\omega}^2)x_2(\boldsymbol{\omega}^2) & & = & b_2(\boldsymbol{\omega}^2) \quad a.s \\
 & & B_3(\boldsymbol{\omega}^3)x_2(\boldsymbol{\omega}^2) \quad + \quad A_3(\boldsymbol{\omega}^3)x_3(\boldsymbol{\omega}^3) & & = & b_3(\boldsymbol{\omega}^3) \quad a.s. \\
 & & & \ddots & & \vdots \\
 & & & B_{T+1}(\boldsymbol{\omega}^{T+1})x_T(\boldsymbol{\omega}^T) & = & b_{T+1}(\boldsymbol{\omega}^{T+1}) \; a.s.
\end{array}$$

$$l_1 \le x_1 \le u_1$$
$$l_t(\boldsymbol{\omega}^t) \le x_t(\boldsymbol{\omega}^t) \le u_t(\boldsymbol{\omega}^t) \quad a.s. \quad t = 2,\ldots,T.$$

In (MRP) $\boldsymbol{\omega} := \{\boldsymbol{\omega}_t\}_{t=1}^{T+1}$ is a discrete, autocorrelated random data process, representing financial returns and borrowing conditions, defined in a canonical probability space $(\Omega, \mathcal{F}, P)$. The time set is finite with *horizon* T, and the decision process $\mathbf{x} := \{\mathbf{x}_t\}_{t=1}^{T}$ takes values in $X \subseteq I\!R^n$, for $X := \Pi_{t=1}^{T} X_t, \quad X_t \subseteq I\!R^{n_t}, \; n := \sum_{t=1}^{T} n_t$ and is *adapted* or *nonanticipative*, i.e. $\mathbf{x}_t = \{\mathbf{x}_t|\mathcal{F}_t\} \quad a.s.$, with respect to the filtration $\mathcal{F}_t := \sigma\{\omega^t\}$ defined by the data process *histories* $\boldsymbol{\omega}^t := (\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_t)$, $t = 2,\ldots,T+1$. Here $\omega^t := \{(\xi_s, B_s, A_s, b_s) : s = 1,\ldots,t\}$, with $f_t : \Omega \times X_t \to I\!R$ a suitable map, and $A_t \in I\!R^{m_t \times n_t}$, $B_t \in I\!R^{m_t \times n_t}$ and $b_t \in I\!R^{m_t}$.

Applying the dynamic programming recursion to (MRP), for $t = 1, 2\ldots, T$, yields a set of *nodal* stochastic programs:

$$\pi_t(\omega^t) := \max_{x_t \in X_t} \{f_t(\omega^t, x^{t-1}, x_t) + I\!E v_{t+1}(\boldsymbol{\omega}^t, x^t)|\mathcal{F}_t\}\,, \tag{2}$$

where for each $t = 1,\ldots,T$ we have a so called *here-and-now* problem at the node defined by the data path history ω^t. Notice that from (2) it follows that $\boldsymbol{\pi}_t = \{\boldsymbol{\pi}_t|\mathcal{F}_t\} \; a.s.$. The nonanticipative condition can be relaxed to *perfect foresight* in order to derive the corresponding set of *distribution* problems:

$$\phi_t(\omega^t) := I\!E[\max_{x_t \in X_t}\{f_t(\omega^t, x^{t-1}, x_t) + v_{t+1}(\boldsymbol{\omega}^t, x^t)\}|\mathcal{F}_{T+1}] \;=\; I\!E[\max_{x_t \in X_t}\{f_t(\omega^t, x^{t-1}, x_t) + v_{t+1}(\boldsymbol{\omega}^t, x^t)\}] \tag{3}$$

We define in this way the *local expected value of perfect information (EVPI)* (cf. [6,8]) as $\eta_t(\boldsymbol{\omega}^t) := \phi_t(\boldsymbol{\omega}^t) - \pi_t(\boldsymbol{\omega}^t)$, for each $t = 1,\ldots,T$. For the important properties of this process, adopted in [6,8] as an *importance sampling* criterion for the selection of a sample set of *relevant representative* datapath (*scenarios*) in a sequential sampling procedure, we refer to [6]. The EVPI sampling procedure discussed in [6,8], by concentrating only on criterion-relevant scenarios, allows the solution of very large problems by *implicitly* taking very large numbers of scenarios into account – an important consideration for confidence in the calculated solutions of financial planning problems.

We report here a pension fund manager's annual planning problem over a ten year horizon with four investment categories (ordinary shares, consol bonds, index-linked securities, real estate) and five funds. The returns on these investment categories and the pension payments at the end of every year are random, with a set of recourse *portfolio-rebalancing* decisions associated with each realization.

An integrated stochastic programming system for the solution of complex financial-planning problems requires:-

- A correct representation of the decision problem, as in the multistage recourse form (MRP), with implicit or explicit characterization of the nonanticipativity condition [2].
- A scenario generator for the simulation of the vector stochastic process $\boldsymbol{\omega}$ in $(\Omega, \mathcal{F}, P)$ representing the coefficient data of the model [2].
- The generation of the stochastic program for numerical solution, for which we use here the *STOCHGEN* library [4] incorporating H.J. Greenberg's MODLER [10].

Table 1 presents a set of results for the 10-year horizon asset-liability problem with an increasing number of scenarios based on the following solution methods:-

- Direct solution of the deterministic equivalent problem by a *primal-dual interior point* method (CPLEX Version 3.0, 1992).

Number of scenarios	1024	1536	1920	2304	2688
Tree structure	$4^1\ 2^8$	$4^1\ 3^1\ 2^7$	$5^1\ 3^1\ 2^7$	$6^1\ 3^1\ 2^7$	$7^1\ 3^1\ 2^7$
Matrix dimension	$134K \times 256K$	$201K \times 384K$	$251K \times 480K$	$302K \times 576K$	$352K \times 672K$
Entries	689, 156	1, 033, 268	1, 315, 528	1, 578, 528	1, 841, 520
Density	0.002007%	0.001338%	0, 001089%	0, 0009073%	0, 000778%
Objective	1791.18	1693.47	1769.45	1664.66	1679.92
Root EVPI	39.2%	45.4%	37%	47.5%	45%
CPLEX barrier	618.2″	898.54″	1305.7″	unsolved	unsolved
No of iterations	75	74	82		
MSLiP	367.94″	557.71″	704.03″	933.69″	1006.59″
No of iterations	11	12	12	16	15
MSLiP-OSL	347.85″	557.41″	758.8″	1192.18″	1070.84″
No of iterations	10	11	12	17	13
EVPI sequential sampling	(3 iterations,	15 independent trials)			
Average *selected* scenarios	584	845	961	1183	2015
Average bias in the obj	1.7%	0.2%	3.8%	4.6%	0.8%
Average solution time	529″	808″	897″	1240″	2141″

Table 1: Numerical results for a long term asset-liability problem

- Taking advantage of the particular structure of the deterministic equivalent problem of (1), the original problem is decomposed into a sequence of subproblems and solved with *nested Benders decomposition* (MSLiP and MSLiP-OSL Version 8.3, 1995).
- The problems are sequentially approximated using the expected value of perfect information attached to nodes of the associated data scenario tree. The *EVPI sampling* algorithm is interfaced – so far in a non-optimized manner – both with the STOCHGEN model generation library to generate sampled problem estimates from pregenerated data scenarios, and with the decomposition algorithm (MSLiP-OSL) for their solution.

The numerical results are obtained on an IBM RS6000/590 with 256MB of RAM running under AIX 4.1.

MSLiP-OSL instantiates nested Benders decomposition with the OSL simplex solver and provides an accurate and stable solver for large and very large problems with high EVPI such as those reported in the table (see also [3,8]). It is compared with Gassmann's original MSLiP solver and with CPLEX barrier which proved to be the best available interior point algorithm for large problems with sparse structure.

1. References

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