Richardson Extrapolation Techniques for Pricing American-style Options

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Abstract

In this paper we re-examine the Geske-Johnson formula (1984) and extend the analysis by deriving a modified Geske-Johnson formula that can overcome the possibility of non-uniform convergence encountered in the original Geske-Johnson formula. Furthermore, we propose a numerical method, the repeated Richardson extrapolation, which is able to estimate the interval of true option values when the accelerated binomial option pricing models are used to value American-style options. We also investigate the possibility of combining the Binomial Black and Scholes method proposed by Broadie and Detemple (1996) with the repeated Richardson extrapolation technique. From the simulation results, our modified Breen accelerated binomial model is as fast, but on average more accurate than, the Breen accelerated binomial model. We lastly illustrate that the repeated Richardson extrapolation approach can estimate the interval of true American option values extremely well.

1 Introduction

In an important contribution, Geske and Johnson (1984) showed that it was possible to value an American-style option by using a series of options exercisable at one of a finite number of exercise points (known as Mid-Atlantic or Bermuda options). They employed Richardson extrapolation techniques to derive an efficient computational formula using the values of Bermuda options. The Richardson extrapolation techniques were afterwards used to enhance the computational efficiency and/or accuracy of American option pricing in two directions in the literature. First, one can apply the Richardson extrapolation in the number of time steps of binomial trees to price options. For example, Broadie and Detemple (1996), Tian (1999), and Heston and Zhou (2000) apply a two-point Richardson extrapolation to the binomial option prices. Second, the Richardson extrapolation method has been used to approximate the American option prices with a series of options with an increasing number of exercise points. The existing literature includes Breen (1991) and Bunch and Johnson (1992).

Two problems are recognized to exist with this methodology. First, as pointed out by Omberg (1987), there may in the case of some options be the problem of non-uniform convergence.[1] In general, this arises when a

[1]In the Geske-Johnson formula, they defined $P(1)$, $P(2)$ and $P(3)$ as follows: (i) $P(1)$ is a European option, permitting exercise at time T, the maturity date of the option; (ii) $P(2)$ is the value of a Mid-Atlantic option, permitting exercise at time $T/2$ or T; (iii) $P(3)$ is the value of a Mid-Atlantic option, permitting exercise at time $T/3$,$2T/3$, or T. Omberg (1987) showed a plausible example of a non-uniform convergence with a deep-in-the-money put option written on a low volatility, high dividend stock going ex-dividend once during the term of the option at time $T/2$. In this case, there is a high probability that the option will be exercised at time $T/2$ immediately after the stock goes ex-dividend. Thus, $P(2)$

Mid-Atlantic option with n exercise points has a value that is less than that of an option with m exercise points, where $m < n$. A second problem with the Geske-Johnson method is that it is difficult to determine the accuracy of the approximation. How many options or how many exercise points have to be considered in order to achieve a given level of accuracy?

In this paper we examine these two problems in the context of binomial approximations of the underlying lognormal process generating asset prices. Breen (1991) presented what he termed an accelerated binomial model (henceforth refered to as the AB model) in which each of the options with 1, 2, 3,..,n exercise points is priced with a binomial model. This technique is of interest, because it allows an evaluation of options where n is relatively large, or where more than one variable affects the option's value. In these cases the use of a multivariate lognormal distribution may become impractical. However, owing to employing the Geske-Johnson's formula, Breen's accelerated binomial model may also encounter the problem of non-uniform convergence. We solve these problems by introducing two techniques from Richardson's numerical approximation. In place of the arithmetic time steps used by Geske-Johnson, we employ geometric time steps.

Secondly, we employ a technique known as repeated Richardson approximation.[2] This helps to specify the accuracy of an approximation to the unknown true option price, helping to determine the smallest value of n that can solely be used in an option price approximation. We also investigate the possibility of combining the Binomial Black and Scholes (hereafter known as

will lead to be greater than $P(3)$, and the problem of non-uniform convergence emerges.

[2] We will discuss the repeated Richardson approximation techniques in details in section 4.

BBS) method proposed by Broadie and Detemple (1996) with the repeated Richardson extrapolation technique.

The plan of this paper is as follows. In section 2 we briefly review the literature on the approximation of American-style option prices with a series of options with an increasing number of exercise points. This allows us to specify the incremental contribution of our paper. In section 3 we introduce our technique based on geometric time steps to derive a modified Geske-Johnson formula which can overcome the problem of non-uniform convergence encountered in the original Geske-Johnson formula. Section 4 employs the technique of the repeated Richardson approximation to determine the accuracy of the approximated option prices. This method can tell us how many exercise points have to be considered to achieve a given level of accuracy. In section 5 we illustrate and discuss the numerical results. Conclusions and possible applications of our technique are discussed in section 6.

2 Literature Review

In their original paper, Geske-Johnson show that an American put option can be calculated to a high degree of accuracy using a Richardson approximation. If $P(n)$ is the price of a Mid-Atlantic option exercisable at one of n equally-spaced exercise dates, then, for example, using $P(1)$, $P(2)$ and $P(3)$, the price of the American put is approximately

$$P(1,2,3) = P(3) + \frac{7}{2}(P(3) - P(2)) - \frac{1}{2}(P(2) - P(1)), \tag{1}$$

where $P(1,2,3)$ denotes the approximated value of the American option using the values of options with 1, 2 and 3 possible exercise points.

In a subsequent contribution, Bunch and Johnson (1992) suggest a modification of the Geske-Johnson method based on the use of an approximation

$$P(1,2) = P^{max}(2) + (P^{max}(2) - P(1)), \tag{2}$$

where $P^{max}(2)$ is the option's value exercisable at one of two points at time, when the exercise points are chosen so as to maximize the option's value. They showed that if the time steps are chosen so as to maximize $P(2)$,[3] then accurate predictions of the American put price can be made with greater computational efficiency than in the case of the original Geske-Johnson method.

Omberg (1987) and Breen (1991) consider the Geske-Johnson method in the context of binomial computations. Omberg (1987) shows that there may be a problem of non-uniform convergence since $P(2)$ in equation (1) is computed using exercise points at time T and T/2, where T is the time to maturity of option, and $P(3)$ is computed using exercise points at time $T, T/3$, and $2T/3$. Therefore $P(3)$ is not always greater than $P(2)$. Although Breen (1992) also points out the above mentioned problem of non-uniform convergence, he still suggests and tests a binomial implementation of the original Geske-Johnson formula.

It is well known that convergence of a binomial option price to the true price is not monotonic, but oscillatory, in the step size (see for example, Broadie and Detemple (1996) and Tain (1999)). The chaotic convergence limits the use of extrapolation techniques to enhance the rate of conver-

[3]Bunch and Johnson suggest that the time of the first exercise point of $P(2)$ can be chosen by examining seven time spaces at $T/8, 2T/8, 3T/8, 4T/8, 5T/8, 6T/8$ and $7T/8$ and the time of the second exercisable point is usually allocated at time T, the maturity date of the option.

gence. As a result, several papers in the literature have modified the CRR binomial model to produce smooth convergence. Among them, Broadie and Detemple (1996) propose a method terms Binomial Black and Scholes (hereafter BBS) model which gives smooth convergence prices. The BBS method is a modification to the binomial method where the Black-Scholes formula replaces the usual "continuation value" at the time step just before option maturity. The BBSR method adds the Richardson extrapolation to the BBS method. In particular, the BBSR method with n steps computes the BBS prices corresponding to $m = n/2$ steps (say P_m) and n steps (say P_n) and then sets the BBSR price to $P = 2P_n - P_m$.

3 The Modified Geske-Johnson formula

From the previous review of the Geske-Johnson approximation method, we find that it is possible for the condition, $P(1) < P(2) > P(3)$, to occur. Thus, the problem of non-uniform convergence will emerge. To solve this problem, we follow Omberg's suggestion to construct the approximating sequence so that each opportunity set includes the previous one, and therefore is at least as good, by using geometric time steps $[1, 2, 4, 8, ...]$ generated by successively doubling the number of uniformly-spaced exercise dates, rather than the arithmetic time steps $[1, 2, 3, 4, ..]$ employed by Geske-Johnson.

Let $F(h)$ be the value of the function of interest when a step size of h is used. Here, we assume that $F(h)$ is a smooth function and the derivatives, $F^{'}(h)$, $F^{''}(h)$,... exist. We wish to find $F(0)$, which is the true option value in the case of the option valuation.

Suppose the time steps follow a geometric series

$$h, q^{-1}h, q^{-2}h, q^{-3}h, ...,$$

where $q > 0$. Suppose also that $F(h)$ takes the form

$$F(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + O(h^{\gamma_3}), \tag{3}$$

where $\gamma_1 < \gamma_2 < \gamma_3$, $\lim_{h\to 0} F(h) = a_0$ and $O(h^{\gamma_3})$ is the term with order equal to or higher than γ_3. We then can also write

$$F(q^{-1}h) = a_0 + a_1(q^{-1}h)^{\gamma_1} + a_2(q^{-1}h)^{\gamma_2} + O(h^{\gamma_3}) \tag{4}$$

and

$$F(q^{-2}h) = a_0 + a_1(q^{-2}h)^{\gamma_1} + a_2(q^{-2}h)^{\gamma_2} + O(h^{\gamma_3}). \tag{5}$$

Solving the simultaneous equations (3), (4), and (5), we obtain the following equation

$$F(0) = F(h) + \frac{A}{C}\left(F(h) - F(q^{-1}h)\right) - \frac{B}{C}\left(F(q^{-1}h) - F(q^{-2}h)\right), \tag{6}$$

where

$$\begin{aligned} A &= q^{-2\gamma_2} - q^{2\gamma_1} + q^{-\gamma_1} - q^{\gamma_2}, \\ B &= q^{-\gamma_2} - q^{\gamma_1}, \\ C &= q^{-2\gamma_2}\left(q^{-\gamma_1} - 1\right) - q^{-2\gamma_1}\left(q^{-\gamma_2} - 1\right) + q^{-\gamma_2} - q^{\gamma_1}. \end{aligned}$$

From the above derivation, we define $P(1) = F(h)$, $P(2) = F(q^{-1}h)$, and $P(4) = F(q^{-2}h)$, where h equals to the time to maturity of the option, T. Moreover, to use geometric time steps, we set $q = 2$. If we expand $F(h)$ in a Taylor series around $F(h)$ and drop the third order or higher terms, we then have $\gamma_1 = 1$, $\gamma_2 = 2$. Substituting $q = 2$, $\gamma_1 = 1$, and $\gamma_2 = 2$ into equation (6), we obtain the modified Geske-Johnson approximation formula as follows:

$$P(1,2,4) = P(4) + \frac{5}{3}(P(4) - P(2)) - \frac{1}{3}(P(2) - P(1)), \tag{7}$$

where $P(1,2,4)$ denotes the value of the approximated American option using the values of options with 1, 2, and 4 exercise points.

In equation (7), $P(4)$ is the value of option exercisable at time steps $T/4, 2T/4, 3T/4$, and T. Because we use a geometric time step, we can ensure that $P(4) \geq P(2) \geq P(1)$ always holds. The reason for this is that the exercise points of $P(4)$ include all the exercise points of $P(2)$, while the exercise points of $P(2)$ include all the exercise points of $P(1)$. Thus, the modified Geske-Johnson formula is able to overcome the shortcomings of non-uniform convergence encountered in the original Geske-Johnson formula.

4 The Repeated Richardson Extrapolation Technique for Predicting the Intervals of American Option Values

We now turn to the question of how to predict the interval of true option values when the Breen and modified Breen AB models are used to value American-style options. To derive the predicted interval of the true option values, we have to employ a numerical method, the repeated Richardson extrapolation. A repeated Richardson extrapolation will get the same results as those of polynomial Richardson extrapolation methods when the same expansion of the truncation error is used.[4] The advantage to using a repeated Richardson extrapolation is that we can predict the interval of the true option values.

As stated earlier, the Breen AB model uses arithmetic time steps to get the approximation formula with polynomial Richardson extrapolation, while the modified Breen AB model uses geometric time steps. In this section we establish an algorithm that can be used by repeated Richardson extrapolation no matter what kind of time steps are used.

Often in numerical analysis, an unknown quantity, a_0,(the same concept as the value of American options in our case), is approximated by a calculable function, $F(h)$, depending on a parameter $h > 0$, such that $F(0) = \lim_{h\to 0} F(h) = a_0$. If we know the complete expansion of the truncation error about function $F(h)$, then we can perform the repeated Richardson extrapolation. Assume that

$$F(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + \ldots + a_k h^{\gamma_k}, O(h^{\gamma_k}) \tag{8}$$

with known exponents $\gamma_1, \gamma_2, \gamma_3, ...$ and $\gamma_1 < \gamma_2 < \gamma_3 ...$, but unknown a_1, a_2, a_3, etc. According to Schmidt (1968), we can establish the following algorithm when $\gamma_j = \gamma j$, $j = 1...k$.

Algorithm:

For $i = 1, 2, 3, ...$, set $A_{i,0} = F(h_i)$, and compute for $m = 1, 2, 3, .., k-1$.

$$\begin{aligned} A_{i,0} &= F(h_i) \\ A_{i,m} &= A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{(h_i/h_{i+m})^{\gamma} - 1}, \end{aligned} \tag{9}$$

[4]For proof of this statement, refer to any textbook of numerical analysis, such as K.E. Atkinson (1989).

where m is the repeated times when the repeated Richardson extrapolation is used and $0 < m \leq k-1$.

The computations can be conveniently set up in the following scheme

h_i	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$\dots$
h_1	$A_{1,0}$	$A_{1,1}$	$A_{1,2}$	$A_{1,3}$	
h_2	$A_{2,0}$	$A_{2,1}$	$A_{2,2}$		
h_3	$A_{3,0}$	$A_{3,1}$			
h_4	$A_{4,0}$				
$\vdots$					

If we use geometric time steps employed in the modified Geske-Johnson formula, then we can set the time steps as follows: $h_1 = h$, $h_2 = h/2$, $h_3 = h/4$, $h_4 = h/8$,...., where h equals to maturity of option, T. We then define $P(1) = A_{1,0}(h)$, the European option value permitting exercise only at period h, $P(2) = A_{2,0}(h/2)$, the twice exercisable option value permitting exercise only at period h and h/2 only, $P(4) = A_{3,0}(h/4)$, the four-times exercisable option value permitting exercise at period h, $3h/4$, $2h/4$, and $h/4$ only, and $P(8) = A_{4,0}(h/8)$, the eight-times exercisable option value permitting exercise only at period h, $7h/8$, $6h/8$, $5h/8$, $4h/8$, $3h/8$, $2h/8$, and $h/8$ only.

Schmidt proves that $|A_{i,m+1} - F(0)| \leq |A_{i,m+1} - A_{i,m}|$ always holds when i is sufficiently large[5] and m is under the constraint, $0 < m \leq k-1$.

[5]In the literature, mathematicians note that it is very difficult to say how large i must be in order to ensure that $A_{i,m}$ and $U_{i,m}$ ($U_{i,m}$ is defined in Appendix) are the upper or

[6] Here, F(0) is the true American option value when the repeated Richardson extrapolation is used to approximate the American-style option values. Thus, we are able to predict the interval of the allocation of the true option values by setting a desired accurate criterion according to the above inequality. We will show, by simulation, how to predict the interval of the true option values by using the repeated Richardson extrapolation. It is very easy to prove that, in the case of using geometric time steps, the two times repeated Richardson method will yield the same formula as the modified Geske-Johnson formula.[7]

5 Numerical Analysis

5.1 Computational Efficiency and Accuracy

In this section we use the modified Geske-Johnson formula in the context of accelerated binomial computations, termed as the modified Breen AB model, to show the accuracy and efficiency of our method. To compare the computational accuracy of the Breen AB model with that of the modified

lower bound of $F(0)$. However, they suggest that, for practical purpose, the extrapolation should be stopped "if a finite number of $A_{i,m}$ and $U_{i,m}$ decrease or increase monotonically, and if $|A_{i,m} - U_{i,m}|$ is small enough for accuracy." Apart from using the above suggestion to test the data in tables 1 and 2, from Tables 6 and 7 we found out that when i=2 and m=1, m=2, or m=3, there are only a very low percentage violate the inequality. However, the violation of error boundaries is not very significant. Thus, we can ignore them.

[6]The proof of this inequality is presented in the Appendix according to Schmidt (1968).

[7]In the case of geometric time steps, we can also prove that the three points (four points) polynomial Richardson extrapolation will yield the same results as those of two-times (three-times) repeated Richardson extrapolation. The proofs are available on request to the authors.

Breen AB model, we use the CRR model as a benchmark. All the binomial trees are at the time steps of 996.

From Tables 1 and 2 we find that the modified Breen AB model almost always dominates the Breen AB model with only two exceptions. Both the Breen and the modified Breen AB models can approximate the values of American put options to a high degree of accuracy. The pricing errors range from one to two cents far less than the transaction cost. The results are not surprising at all since the modified Breen AB model using the value of $P(4)$ has more exercise information than the Breen's AB model using the value of $P(3)$. This result illustrates that the modified Breen AB model can not only solve the problem of non-uniform convergence, but also improve the computational accuracy of Breen's AB model.

The benefit of using the various AB models to value American-style options is able to reduce the numbers of node calculations. Thus, the AB model can significantly improve the computational efficiency. The total number of node calculations for the CRR binomial model, Breen's AB model, and the modified Breen's AB mode are $(n+1)^2$, $4n+10$, and $4.5n+11$, respectively. Surprising the modified Breen AB model only need a little more node calculation than those of the Breen AB model. The reason for this achievement is that $P(2)$ and $P(4)$ in the modified Breen AB model have common time period $T/2$, whereas $P(2)$ and $P(3)$ in the Breen AB model do not have a common time period. In terms of CPU time for an option's computation, the modified Breen's AB model is as fast as Breen's AB model. However, both the Breen and modified Breen AB models are far more computationally efficient than the CRR binomial model.

5.2 The Accuracy of the BBS Method with Repeated Richardson Extrapolation Techniques

In this subsection we investigate the possibility of combining the BBS method with the repeated Richardson extrapolation technique. We apply the BBS method with a repeated Richardson extrapolation in *number of time steps* to price European options, because the true prices are easily to calculate. The root-mean-squared (hereafter RMS) relative error is used as the measure of accuracy. The RMS error is defined by

$$RMS = \sqrt{\frac{1}{m}\sum_{i=1}^{m} {e_i}^2}, \tag{10}$$

where $e_i = (P^*{}_i - P_i)/P_i$ is the relative error, P_i is the true option price (Black-Scholes), and $P^*{}_i$ is the estimated option price obtained from the BBS Method with repeated Richardson Extrapolation Techniques.

Many points can be drawn from Table 3. First, it is clear from the third column of Table 3 that the pricing error of an N-step BBS model for standard options is at the rate of $O(1/N)$. In contrast, Heston and Zhou (2000) show that the pricing error of an N-step CRR model fluctuates between the rate of $O(1/\sqrt{N})$ and $O(1/N)$. As a result, the BBSR method with geometric time steps produces very accurate prices for European options (see the fourth column of Panel B in Table 3). Second, the pricing errors from geometric time steps are far smaller than that of arithmetic time steps. Third, Table 3 reveals that the repeated Richardson extrapolation in time steps cannot further improve the accuracy. For example, Panel B shows that the pricing error of $A_{4,1}$ (obtained from a two-point Richardson extrapolation of BBS prices with 160 and 320 steps) is actually smaller than that of $A_{3,2}$ (obtained from a three-point Richardson extrapolation of BBS prices with 80, 160, and

320 steps). Therefore, in the rest of the numerical evaluations we apply only two-point Richardson extrapolation of BBS prices with $N/2$ and N steps.

5.3 The Predicted Intervals of the True American Option's values Using Repeated Richardson Extrapolation Techniques

To choose the benchmark method for calculating the true values of American options, we compare the accuracy of the CRR and BBSR models when both methods are applied to price European options. It is clear from Table 4 that the RMS relative error of the BBSR method is far smaller than that of the CRR method. Our result is consistent with the findings of Broadie and Detemple (1996). Therefore, we will use the BBSR method with 10800 steps to calculate benchmark prices of American options.

We now turn to compare the accuracy of the Richardson extrapolation for the number of exercisable points to estimate American option values. Both arithmetic and geometric exercisable times are examined. In Table 5 the true values of all options are estimated by the BBSR method with $10,800$ steps. The results indicate that the pricing errors of geometric exercisable times are smaller than that of arithmetic exercisable times. This finding supports the finding that a Richardson extrapolation with geometric exercisable times can avoid the problem of non-uniform convergence. Moreover, the repeated Richardson extrapolation technique can further reduce the pricing errors. In other words, an $(n+1)$-point Richardson extrapolation generally produces more accurate prices than an n-point Richardson extrapolation. For example, Panel B shows that the RMS relative errors of $A_{1,2}$ (obtained from a three-point Richardson extrapolation of $P(1)$, $P(2)$, and $P(4)$) is 0.346 %, which is smaller than that (0.427 %) of $A_{2,1}$ (obtained from a two-point Richardson extrapolation of $P(2)$ and $P(4)$).

One specific advantage of the repeated Richardson extrapolation is that it allows us to specify the accuracy of an approximation to the unknown true option price. That is, the Schmidt inequality can be used to predict tight bounds (with desired tolerable errors) of the true option values. We test the validity of the Schmidt inequality over 243 options for both arithmetic and geometric exercisable times in Tables 6 and 7. The denominator represents the number of option price estimates that match $|A_{i,m+1} - F(0)| <$ the desired errors, and the numerator is the number of option price estimates that match $|A_{i,m+1} - F(0)| <$ the desired errors and $|A_{i,m+1} - A_{i,m}| <$ the desired errors.

The results in Tables 6 and 7 indicate that increasing i or m will increase the number of price estimates with errors less than the desired accuracy. It is also clear that the Schmidt inequality is seldom violated especially when i or m is large ($i = 3, 4$ and $m = 2, 3$). For example, when i= m=2, 228 out of 243 option price estimates have errors smaller than 0.2% of the European option value, and 225 out of 228 option price estimates satisfy Schmidt inequality. Moreover, the findings support that the repeated Richardson extrapolation with geometric exercisable times works better than with arithmetic exercisable times. This confirms the previous result that a Richardson extrapolation with geometric exercisable times can avoid the problem of non-uniform convergence.

6 Conclusion and Suggestion

In this paper we re-examine the original Geske-Johnson formula. We then extend the analysis by deriving a modified Geske-Johnson formula which is able to overcome the possibility of non-uniform convergence encountered in the original Geske-Johnson formula. Another contribution of this paper is that we propose a numerical method which can estimate the predicted intervals of the true option values when the accelerated binomial option pricing models are used to value the American-style options.

The findings are summarized as follows: (i) The modified Breen AB model is as fast as the Breen AB model, whereas both the Breen and modified Breen AB model are faster than the CRR binomial model. However, on average the modified Breen AB model performs better than the Breen model for valuing American-style put options with a range of exercise prices. More importantly the modified Breen AB model can overcome the drawback from possible non-uniform convergence in the Breen AB model. Therefore, we recommend replacing the Breen AB model with the modified Breen AB model for the valuation of American-style put options. (ii) The Richardson extrapolation approach can improve the computational accuracy for the BBS method proposed by Broadie and Detemple (1996), while the repeated Richardson extrapolation technique cannot. (iii) Using Schmidt's inequality, we are able to obtain the intervals of the true American option values. This helps to specify the accuracy of an approximation to the unknown true option price and to determine the smallest value of n that can solely be used in an option price approximation. This research article is the first to discuss how to get the predicted intervals of the true option values in the literature of finance. We believe that the repeated Richardson method will be very useful for practitioners to predict the intervals of the true option values.

The advantage in using the accelerated binomial option pricing model is its ability to reduce tremendous amounts of node calculations. It is especially useful to calculate the prices of options where n is relatively large, or where more than one variable affects the option's value. We will leave the above-mentioned potential applications to future research.

Appendix: The Proof of Schmidt's Inequality

In this appendix we prove that $|A_{i,m+1} - F(0)| \leq |A_{i,m+1} - A_{i,m}|$ is always true when i is sufficiently large and m is under the constraint, $0 < m \leq k-1$, where k is the order of powers of the expansion of truncation errors. Let $F(h)$ be the appropriate solution gained through discretization for a problem. We assume that $F(h)$ can be developed for the parameter $h > 0$

$$F(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + \ldots + a_k h^{\gamma_k} O(h^{\gamma_k}), \tag{11}$$

where $\gamma_1 < \gamma_2 < \gamma_3 < .... < \gamma_k$. The solution of the original problem is $F(0) = \lim_{h\to 0} = a_0$ and the convergence speed is determined by h^{γ}.

Schmidt (1968) shows that, when $\gamma_k = \gamma k + \delta$ and $h_{i+1}/h_i \leq \rho \leq 1$ (ρ is a constant and $0 \leq \rho \leq 1$), iterative extrapolation can be carried out according to following procedure

$$\begin{aligned}
A_{i,0} &= F(h_i) \\
H_{i,0} &= {h_i}^{-\delta} \\
A_{i,m} &= A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{D_{i,m-1}} \\
H_{i,m} &= H_{i+1,m-1} + \frac{H_{i+1,m-1} - H_{i,m-1}}{[h_i/h_{i+m}]^{\gamma} - 1},
\end{aligned} \tag{12}$$

where

$$D_{i,m} = \frac{{h_i}^{\gamma} H_{i+1,m-1}}{h_{i+m}^{\gamma} H_{i,m-1}}$$

and $0 < m \leq k-1$.

If δ is equal to zero (i.e. $\gamma_k = \gamma k$), then $H_{i,m}$ is equal to one. Thus, equation (11) can be reduced to the following equation

$$\begin{aligned}
A_{i,0} &= F(h_i) \\
A_{i,m} &= A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{D_{i,m-1}},
\end{aligned} \tag{13}$$

where

$$D_{i,m} = [h_i/h_{i+m}]^\gamma.$$

Schmidt defined $U_{i,m}$ as the following

$$U_{i,m} = (1+\beta)A_{i+1,m} - \beta A_{i,m}, \tag{14}$$

where

$$\beta = 1 + \frac{2}{[h_i/h_{i+m+1}]^\gamma - 1} = 1 + \frac{2}{D_{i,m+1} - 1}.$$

According to the proof of theorem 2 in Schmidt's paper, we can get equation (14) when i is sufficiently large, m is under the constraint, $0 < m \le k-1$ and a_{m+1} $(m = 1, ..k-1)$ is not equal to zero.

$$\begin{aligned} A_{i,m} \le F(0) \le U_{i,m} \\ U_{i,m} \le F(0) \le A_{i,m}. \end{aligned} \tag{15}$$

This is equivalent to

$$|[A_{i,m} + U_{i,m}]/2 - F(0)| \le \frac{1}{2}|U_{i,m} - A_{i,m}|. \tag{16}$$

Rearranging the definition of $U_{i,m}$ in equation (13), we obtain the following equation

$$\frac{1}{2}(A_{i,m} + U_{i,m}) = \frac{1}{2}(1+\beta)A_{i+1,m} + \frac{1}{2}(1-\beta)A_{i,m} \tag{17}$$

Furthermore, from the definition of β in equation (13), we are able to get the following relationship

$$\begin{aligned} 1+\beta &= 2\left(1 + \frac{1}{D_{i,m+1} - 1}\right), \\ 1-\beta &= \frac{-2}{D_{i,m+1} - 1}. \end{aligned} \tag{18}$$

Substituting equation (17) into equation (16) and referring to equation (12), we obtain

$$\frac{1}{2}(A_{i,m} + U_{i,m}) = A_{i,m+1}. \tag{19}$$

Similarly, we also can acquire the following relationship

$$\frac{1}{2}(U_{i,m} - A_{i,m}) = A_{i,m+1} - A_{i,m}. \tag{20}$$

Finally, substituting equations (18) and (19) into equation (15), we obtain Schmidt's inequality

$$|A_{i,m+1} - F(0)| \leq |A_{i,m+1} - A_{i,m}|. \tag{21}$$

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