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The Random–Time Binomial Model

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Abstract

In this paper we study Binomial Models with random time steps. We explain, how calculating values for European and American Call and Put options is straightforward for the Random–Time Binomial Model. We present the conditions to ensure weak–convergence to the Black–Scholes setup and convergence of the values for European *and* American put options. Differently to the CRR–model the convergence behaviour is extremely smooth in our model. By using extrapolation we therefore achieve order of convergence two. This way it is an efficient tool for pricing purposes in the Black–Scholes setup, since the CRR model and its extrapolations typically achieve order one. Moreover our model allows in a straightforward manner to construct approximations to jump–diffusions. The simple valuation approaches and the convergence properties carry immediatly over from the Black–Scholes case.

Keywords

binomial model, order of convergence, smoothing, extrapolation, jump–diffusion

JEL Classification

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1. Introduction

In a continuous setup, where stock evolution is modeled by geometric Brownian Motion Black and Scholes[73] derived a closed–form solution for the value of the European–style option. Later Harrison and Kreps[79] and Harrison and Pliska[81] developed the concept of equivalent martingale measure, which gave an elegant technique to express and solve pricing problems.

Corresponding to the original Black–Scholes framework Cox, Ross and Rubinstein[79] (henceforth CRR) and Rendleman and Bartter[79] independently presented the binomial model. This is a discrete reflection of the continuous process since it merges into the continuous model in the limit. These models are an easy way to explain how continuous trade takes place and infinitely many states are spanned. Besides these didactical advantages it turned out that binomial models can serve in an easy way to give approximations for option values where no closed form solution is available as for example for the American put option. Moreover they are an elegant alternative to PDE–methods for pricing purposes.

The idea to approximate the Black–Scholes setup by a Binomial Model with random time–steps appears already in the works of Föllmer and Sondermann[86] and Sondermann[87] calculating risk–minimizing strategies for a two–sided compound jump–processes. Sondermann[87] suggested this as a way to approximate the process resulting from hedging only when the stock–process reaches certain prespecified discrete levels. It is only recently that Binomial Models with random time–steps came back into consideration. Dengler and Jarrow[96] used it as an incomplete markets approximation justifing the use of Delta and Gamma for hedging by market participants. Recently Rogers and Stapelton[97] reconsidered the approach of hedging when crossing prespecified levels to price efficiently barrier options. The idea is that incorporting a discrete barrier can be done easily by adjusting the corresponding probabilities. Valuation is simply a mixture of CRR–trees. Unfortunately the distribution on the number of jumps can only be approximated using the work of Petrov[95]. It is easy to see that the simple valuation approach works for any renewal process, yet being limited to time–homogeneous options. We will therefore immediatly suppose that a poisson–process is driving the jumps, which makes valuation much more accessible. Another major contribution is the extension to the valuation of American put options. In all these cases we achieve a very smooth convergence behaviour by choosing the S–spacing suitably, which we can significantly improve through extrapolation in the lines of Leisen[96]. This makes it a very competitive valuation tool.

It is well known that the volatilities implicit in market prices exhibits a strong dependence of volatility on the strike price. According to market participants this is due to the inherent fear of sudden strong price changes ("crashes"). The assumption of continuous sample paths may also be criticized from empirical studies (see for example Jarrow and Rosenfeld[84], Ball and Torous[85] and Jorion[88]).

Already Merton[76] proposed a model in which he superimposed on the Black–Scholes setup a compound jump process ("jump–diffusions"). Whereas geometric Brownian motion describes the arrival of "normal" information, the jump–part models large (discontinuous) price changes due to the arrival of rare "information shocks".

A framework for the valuation using Binomial Models was given by Amin[93] using multinomial models with suitably chosen factors and probabilities. Weak convergence of the processes and using the results of Kushner and DiMasi[78] for the American put was proven. Mercurio and Runggaldier[93] relaxed the assumption of constant intensity by assuming that the jump–part has time–dependent intensity. The value of European Options is now the multidimensional summation of infinite series. Thus it is at least difficult to evaluate. Mulinacci[96] extended the study to american put options and proved convergence of the algorithm to the continuous solution.

The remainder of the paper is organised as follows. In section 2 we will briefly review the properties of the Binomial Model and conditions for weak–convergence in the Skorohod–topology, as well as the extension of Amin[93]. Section 3 presents the model and necessary and sufficient conditions for weak–convergence to geometric Brownian Motion. Section 4 discusses the valuation of European and American Options. Section 5 discusses jump–diffusions.

2. The Binomial Model

On a probability space $(\Omega, \mathcal{F}, P)$ we suppose the stock price process to be described by

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2.1}$$

$$\iff \quad S_t = \exp\{\mu t + \sigma W_t\} \tag{2.2}$$

where $\mu := r - \frac{\sigma^2}{2}$ and the interest rate r as well as the volatility σ are supposed to be constant. $(W_t)_t$ is a standard Wiener–process on the probability space. It is more convenient to work on the logarithm:

$$X_t := \ln S_t \tag{2.3}$$

$$= \mu t + \sigma W_t \tag{2.4}$$

Now suppose we are given a refinement n, a set $\mathcal{T}^n = \{0 = t_{n,0} < t_{n,1} \ldots < t_{n,n} = T\}$ of equidistant trading dates: $t_{n,i+1} - t_{n,i} = \Delta t_n := \frac{T}{n}$ and a sequence $(\kappa_{n,i})_{i=0,\ldots,n} \subset$

$\mathbb{R}$. Then define the independent random variables:

$$\overline{R}_{n,i} \sim \begin{cases} \kappa_{n,i} + v_{n,i} & ; p_{n,i} \\ \kappa_{n,i} - v_{n,i} & ; 1 - p_{n,i} \equiv q_{n,i} \end{cases} \tag{2.5}$$

Denote:

$$\begin{aligned} \overline{X}_t^{(n)} &:= \sum_{i=1}^{N_t} \overline{R}_{n,i} \\ \overline{S}_t^{(n)} &:= \exp \overline{X}_t^{(n)} \\ \text{where} \quad N_t &:= \left\lfloor \frac{t}{\Delta t_n} \right\rfloor \end{aligned}$$

The process $\overline{S}_t^{(n)}$ is called a *Binomial Model* with refinement n. In the sequel we restrict ourselves to Binomial Models, where $\overline{R}_{n,1}, \ldots, \overline{R}_{n,n}$ are identically distributed. Thus we assume that $\kappa_{n,i} = \kappa_n$, $v_{n,i} = v_n$ and call $(u_n, d_n, p_n) = (\exp(\kappa_n + v_n), \exp(\kappa_n - v_n), p_n)$ its *characteristics.*

Please note that differently to the common literature where the discrete process are defined only at dates $t_{n,i} \in \mathcal{T}^n$ we define it on the whole interval $[0, T]$ as a càdlàg process. This is for technical convenience. Yet, as long as we assume that trading takes place only at the discrete dates this makes no difference.
We will further suppose that the space $\mathcal{D}$ of càdlàg processes is equipped with the Skorohod topology and denote by $\stackrel{d}{\Longrightarrow}$ the weak–convergence in distribution on $\mathcal{D}$. We are then interested in $\overline{X}^{(n)} \stackrel{d}{\Longrightarrow} X$ as processes.
Obviously necessary conditions are

$$E\left[\frac{\overline{R}_{n,1}}{\Delta t_n}\right] \stackrel{n}{\longrightarrow} r - \frac{\sigma^2}{2} \tag{2.6}$$

$$\frac{\operatorname{Var}\left(\overline{R}_{n,1}\right)}{\Delta t_n} \stackrel{n}{\longrightarrow} \sigma^2 \tag{2.7}$$

Theorem 2.1:
Suppose that conditions (2.6) and (2.7) are fulfilled.
Then:

$$\overline{X}^{(n)} \stackrel{d}{\Longrightarrow} X \tag{2.8}$$

$$\overline{S}^{(n)} \stackrel{d}{\Longrightarrow} S \tag{2.9}$$

Proof. For (2.8) we refer to Donsker's theorem (see corollary VII.3.11 in Jacod and Shiryaev[87]) and for (2.9) we note that the function exp is continuous. □

The martingale measure condition is $E[\exp\{\overline{R}_{n,1}\}] = \exp\{r\Delta t_n\}$. Similarly to the transfer from (2.2) to (2.4) it can be shown by applying the Itô – formula that condition (2.6) corresponds to this. We therefore require to hold with equality: $E\left[\overline{R}_{n,i}\right] = \mu\Delta t_n$. Easy calculations reveal immediatly:

$$p_n = \frac{\mu\Delta t_n - \kappa_n + v_n}{2v_n} \tag{2.10}$$

It is easy to see that (2.7) requires:

$$\frac{|v_n|}{\sqrt{\Delta t_n}} \xrightarrow{n} \sigma \tag{2.11}$$

To fulfill this property we set $v_n := \sigma\sqrt{\Delta t_n}$. The models proposed in the literature differ in the specification of κ_n:

1. Jarrow and Rudd[83]:
   Setting $\forall n : \kappa_n := \left(r - \frac{\sigma^2}{2}\right)\Delta t_n$, we get $p_n = \frac{1}{2}$
2. Cox, Ross and Rubinstein[79]:
   Setting $\forall n : \kappa_n := 0$, we get $p_n = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma\sqrt{\Delta t_n}}$

Similarly to Merton[76] we will now suppose that our stock–process can described by

$$dS_t = (r+\pi)S_t dt + \sigma S_t dW_t + S_t dJ_t \tag{2.12}$$

$$\text{where} \quad J_t = \sum_{i=1}^{N_t} U_i \tag{2.13}$$

for a sequence of iid random variables $(U_i)_{i\in\mathbb{N}}$ with $U_i \subset]-1,\infty[$ and a poisson–process $(N_t)_t$ with intensity λ. π is called the risk–premium.

Then setting again $X_t = \ln S_t$ and $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i] + \pi$ an easy application of Itô's formula yields:

$$X_t = \mu dt + \sigma dW_t + d\overline{J}_t \tag{2.14}$$

$$\text{where} \quad \overline{J}_t = \sum_{i=1}^{N_t} V_i \tag{2.15}$$

$$\text{with} \quad V_i := \ln(1+U_i) \tag{2.16}$$

Since we are working on the logarithm, equation (2.16) makes sense only for $U_i > -1$. This is the reason we excluded -1 in the assumption for U_i. Yet, this is of only technical nature.

By assuming that jumps occur, markets are no longer complete. Thus there is no longer a *unique* equivalent martingale measure, describing the stock process, which precludes arbitrage. The choice of a martingale measure was performed implicitly in Merton[76] by assuming that jump–risk could be fully diversified and was therefore not priced ($\pi = 0$). Assuming that markets require an exogenously fixed risk–premium for bearing risk requires this to the above formula. Another approach was suggested by Föllmer and Sondermann[86] who specified the measure endogenously by adopting those which minimizes the writers risk in some sense.

Merton[76] obtained the call option price as a mixture of Black–Scholes prices:

$$e^{-\lambda T}\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}E[c(S_0X_ne^{-\lambda kT},T,K,\sigma^2,r)] \tag{2.17}$$

where $c(S_0, T, K, \sigma^2, r)$ is the Black–Scholes formula and $X_n \overset{d}{=} U_1 + \ldots + U_n$.

In the case where there is a probability of immediate ruin, i.e. $U_i = -1$ this formula simplifies to:

$$e^{-\lambda T}c(S_0e^{\lambda T},T,K,\sigma^2,r) \tag{2.18}$$

which can be further simplified to the standard Black–Scholes formula where the interest rate is replaced by $r + \lambda$.

N being a poisson–process with intensity λ, on a discrete interval Δt_n the probability of one jump equals $\lambda\Delta t_n$. The probability of more than one jump is small in comparison to this. Therefore Amin[93] assumed that between two dates exactly one jump occurs. Using a Binomial Model with grid $v_{n,i} := \sigma\sqrt{\Delta t_n}$, $\kappa_{n,i} = \mu\Delta t_n$ he superimposed the jump–process by assuming that jumps occur only into points of the grid. Amin[93] explains how to approximate U_i suitably by $U_{n,i}$ on the grid–points. The return $\overline{R}_{n,i}$ is now modeled by

$$\overline{R}_{n,i} \sim \begin{cases} \kappa_{n,i} + v_{n,i} & ; (1-\lambda\Delta t_n)p_{n,i} \\ \kappa_{n,i} - v_{n,i} & ; (1-\lambda\Delta t_n)(1-p_{n,i}) \\ U_{n,i} & ; \lambda\Delta t_n \end{cases} \tag{2.19}$$

Choosing its probability $p_{n,i}$ suitably, i.e. by setting it according to (2.10) with $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i]$, Amin[93] presents weak–convergence results and convergence results for American Options.

In figure 2.1 we present calculations for the value of a European call option with the following selection $S = 100$, $K = 90$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $\lambda = 0.1$ of parameters and iterating refinement $n = 10, \ldots, 110$. The left hand part corresponds to the Black–Scholes setup using the CRR Binomial Model. We observe very wavy patterns and that prices converging very erratically to the continuous time solution. Prices which overestimate are followed by others which underestimate. Moreover we need very high refinements to achieve sufficiently high accuracy. For example to ensure "penny–accuracy" in the example of figure 2.1 we need at least a refinement of $n = 200$.

On the right hand in figure 2.1 we present calculations using the model of Amin[93] as an approximation to a model allowing immediate ruin ($U_i = -1$, $\lambda = 0.1$). This model inherits the poor convergence properties from the Binomial Model. Yet, this

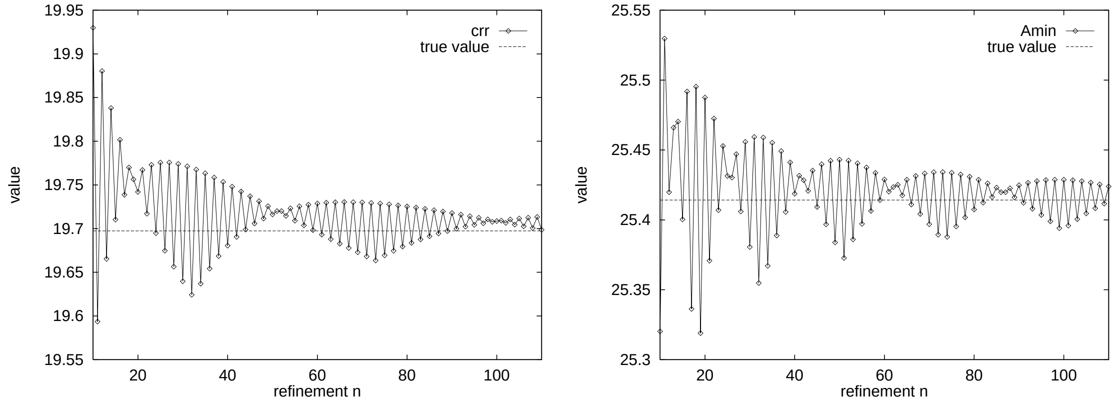

FIGURE 2.1. typical pattern resulting from European Call option price calculations in the BS–setup using the CRR Model resp. in Merton's setup using the Amin Model

is not the only drawback. Superimposing jumps in Amin's way corresponds more to a numerical caluation scheme than to the original idea of the Binomial Model, which presented a didactically simple analogue to the continuous–time model and explained how continuous trading may take place and infinitely many states may be spanned.

3. Randomization of the Binomial Model

In this section we will extend the approach of the previous section by allowing random time steps between two trading dates. We explain, how to construct the process suitably, in order to ensure weak–convergence to the Black–Scholes setup. In the next section we will address the valuation task.

To be as general as possible, we suppose to be given a sequence of renewal processes $N^{(m)} = (N_t^{(m)})_{t\geq 0}$. If we denote the i–th interarrival time by $\tau_{m,i}$ this means that

1. $(\tau_{m,i})_i$ are iid non–negative random variables
2. $N_t^{(m)} = \max\{n \,|\sum_{i=1}^n \tau_{m,i} \leq t\}$

We will denote $\mu_m(t) = E[N_t^{(m)}]$ the renewal function.
The simplest example of a renewal process is a poisson process with parameter λ_m. It has the renewal function $\mu_m(t) = \lambda_m t$.

In section 2 we approximated the process X between two trading dates $t_{n,i}, t_{n,i+1} \in \mathcal{T}^n$ by iid random variables $\overline{R}_{n,i}$. Similarly here we will now approximate it by iid random variables $\overline{R}_{m,i}$ between two interarrival times $\tau_{m,i}, \tau_{m,i+1}$, i.e.

$$\overline{R}_{m,i} \sim \begin{cases} \Delta x_m & ; p_m \\ -\Delta x_m & ; 1 - p_m \end{cases} \tag{3.1}$$

Under this assumption we have set the drift $\kappa_n \equiv 0$. This is concenvient, because the grid becomes stationary, i.e. the process will evolve in the grid $\Delta x_m \cdot \mathbb{Z}$, independent of any i.

This yields the approximation:

$$\overline{X}_t^{(m)} = \sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i}$$

We will moreover assume that $N^{(m)}$ is independent of $(\overline{R}_{m,i})_i$. Then we have the following

Theorem 3.1:
Necessary conditions for $\overline{X}^{(m)} \overset{d}{\Longrightarrow} X$ *are that for all* $t \in \mathcal{T}$*:*

$$E\left[\overline{R}_{m,i}\right] \frac{\mu_m(t)}{t} \overset{n}{\longrightarrow} r - \frac{\sigma^2}{2} \tag{3.2}$$

$$Var\left[\overline{R}_{m,i}\right] \frac{\mu_m(t)}{t} \overset{n}{\longrightarrow} \sigma^2 \tag{3.3}$$

Proof. For the first moment we have, using the Wald equality:

$$\begin{aligned} E\left[\overline{X}_t^{(m)}\right] &= E\left[\sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i}\right] \\ &= E\left[N_t^{(m)}\right] E\left[\overline{R}_{m,i}\right] \\ &= \mu_m(t) E\left[\overline{R}_{m,i}\right] \end{aligned}$$

Since $E\left[X_t\right] = \left(r - \frac{\sigma^2}{2}\right) t$, condition (3.2) follows.
For the second moment we have, using the Wald equality again:

$$\begin{aligned} & E\left[\overline{X}_t^{(m)} - \left(r - \frac{\sigma^2}{2}\right) t\right] \\ = \;& E\left[\left(\overline{X}_t^{(m)}\right)^2\right] \\ = \;& E\left[\left(\sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i}\right)^2\right] - \left(r - \frac{\sigma^2}{2}\right)^2 t^2 \\ = \;& E\left[\sum_{i=1}^{N_t^{(m)}} \left(\overline{R}_{m,i}\right)^2 + \sum_{i=1}^{N_t^{(m)}} \sum_{j=1, j\neq i}^{N_t^{(m)}} \overline{R}_{m,i} \, \overline{R}_j^{(m)}\right] - \left(r - \frac{\sigma^2}{2}\right)^2 t^2 \\ = \;& E[N_t^{(m)}] E\left[\left(\overline{R}_{m,i}\right)^2\right] + E\left[N_t^{(m)}\right]^2 E\left[\left(\overline{R}_{m,i}\right)^2\right] - \left(r - \frac{\sigma^2}{2}\right)^2 t^2 \\ = \;& \mu_m(t) E\left[\left(\overline{R}_{m,i}\right)^2\right] \end{aligned}$$

Since $E\left[X_t^2 - E\left[X_t\right]^2\right] = \sigma^2 t$, condition (3.3) follows. □

If we model the return as in (2.5) then the counterpart to theorem 2.1 becomes:

Theorem 3.2:
Suppose that conditions (3.2) and (3.3) are fulfilled.
Then:

$$\overline{X}^{(m)} \stackrel{d}{\Longrightarrow} X \tag{3.4}$$

$$\overline{S}^{(m)} \stackrel{d}{\Longrightarrow} S \tag{3.5}$$

Proof. From (3.2) and (3.3) follows that $\overline{X}_t^{(m)} - \overline{X}_s^{(m)} \stackrel{d}{\Longrightarrow} X_t - X_s$. Since $\overline{X}^{(m)}$ has independent increments, according to Lemma VII.1.3 in Jacod and Shiryaev[87] this is sufficient to deduce (3.4).
(3.5) follows from (3.4), since the function exp is continuous. □

If λ is sufficiently great, jumps will occur almost always. Thus in the limit $\lambda_m \stackrel{m}{\longrightarrow} \infty$ we can expect that with suitably adjusted return $\exp \overline{R}_{m,i}$ we obtain geometric Brownian Motion, which is exactly what theorem 3.2 tells us.

To fulfill (3.2) or (3.3) we need that $\frac{\mu_m(t)}{t}$ is "almost" constant. For simplicity we will subsequently assume that

$$\exists \lambda_m \, \forall t \, : \, \frac{\mu_m(t)}{t} = \lambda_m \tag{3.6}$$

This means that here we are assuming that $N^{(m)}$ is a poisson process with parameter λ_m.

Then condition (3.2) in theorem (3.1) can be resolved by setting

$$p_m = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{\lambda_m \Delta x_m} \tag{3.7}$$

An important question is now how to choose the jump–intensity λ_m. From condition (3.3) follows immediatly:

Lemma 3.1:
Asymptotically we have:

$$\lambda_m \sim \left(\frac{\sigma}{\Delta x_m}\right)^2$$

Allowing the Binomial Model to jump at Random Times the market becomes incomplete. Whereas in the original Binomial Model framework of the previous section there was a unique equivalent martingale measure, represented by $p_{m,i}$, here we are loosing this property. Instead we have a whole set of equivalent martingale measures, all compatible with the assumption of absence of arbitrage opportunities. We can index the possible martingale measures by the choice of the jump–intensity

λ_m. For valuation purposes we need to choose one measure among all these measures. Different approaches from utility to risk–minimization have been studied. Yet, according to theorem 3.2 in the limit this problem cancels out. The easiest way to meet the conditions of this theorem is according to Lemma 3.1 obtained by setting

$$\lambda_m = \left(\frac{\sigma}{\Delta x_m}\right)^2 \tag{3.8}$$

which we will adopt in the sequel. We will describe the probability measure by $\mathbb{Q}_m$ and its expectation by E_m.

Under this choice we deduce from (3.7) that

$$p_m = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma^2}\Delta x_m \tag{3.9}$$

Since in the CRR model we have $\Delta x_m = \sigma\sqrt{\Delta t_n}$ the only difference with the model of CRR obviously consists in replacing Δt_n by random times $(\tau_{m,i})_i$. However all formulas hold with the expected time $\Delta t_n = E[\tau_{m,1}] = \frac{1}{\lambda_m}$.

Rogers and Stapleton[96] presented another approach to obtain $\overline{R}_{m,i}$. For some Δx is any real number they obtain the interarrival times by stopping X at the grid $\Delta x \cdot \mathbb{Z}$. Thus with $\tau_0 = 0$:

$$\tau_{i+1} = \inf\{t \geq \tau_i |\ |X_t - X_{\tau_i}| = \Delta x\}$$

We can view the process $\overline{X}^{(m)}$ also as the sum of two poisson–processes. If we denote by N^+ (resp. N^-) a poisson–process with intensity $\lambda_m^+ = \frac{\lambda_m}{2} + \frac{r-\frac{\sigma^2}{2}}{2\Delta x_m}$ (resp. $\lambda_m^- = \frac{\lambda_m}{2} - \frac{r-\frac{\sigma^2}{2}}{2\Delta x_m}$) then it follows easily that the process $(X_t^{(m)})$ is equal in distribution to the difference between the two jump–processes when the amplitude is Δx_m, i.e.

$$X^{(m)} \stackrel{d}{=} \Delta x_m (N^+ - N^-)$$

Taking $\Delta x_m = \frac{\sigma}{\sqrt{2n}}$ and the jump–intensities $\lambda_n^+ = n(1 + \frac{\mu}{\sigma\sqrt{2n}})$, $\lambda_n^- = n(1 - \frac{\mu}{\sigma\sqrt{2n}})$ in this form the Random–Time Binomial Model was studied by Dengler and Jarrow[96] to explain, why market participants use Delta *and* Gamma for Hedging purposes. In this context they derived Lemma 3.1 using the Brownian scaling relation.

4. Valuation

In the previous section we introduced the Random–Time Binomial Model and motivated our choice of the intensity λ_m.

In this section we will present a valuation algorithm, which resolves the further randomness in an easy and straightforward manner for European and American call and put options and we will give convergence results.

A European option is completely described by its payoff function f. For example for a European call with strike K this is $f : x \mapsto (x-K)^+$. Due to the independence of $N^{(m)}$ and the random variables in the sequence $\overline{R}_m$, we can derive the Merton[76] formula (equation (2.17)) by conditioning first on the $N_T^{(m)}$ and then averaging over all possible values. More specifically under our choice of the equivalent martingale measure its value is equal to

$$
\begin{aligned}
&(4.1) \qquad v_m^e(0,S_0) &:=& \; e^{-rT} E_m\left[f(\overline{S}_T)\right] \\
&(4.2) \qquad &=& \; e^{-rT} E_m\left[E_m[f(\overline{S}_T)|N_T^{(m)}]\right] \\
&(4.3) \qquad &=& \; e^{-rT} \sum_{n=0}^{\infty} E_m\left[f(\overline{S}_T)|N_T^{(m)}=n\right] \cdot P\left[N_T^{(m)}=n\right]
\end{aligned}
$$

$E_m\left[f(\overline{S}_T)|N_T^{(m)}=n\right]$ is the value calculated by backward–induction in an n–step tree grid with characteristics (u_m, d_m, p_m), if we *do not* perform discounting. If we denote this value by

$$
\Phi_n^m(0,S_0) := \sum_{i=1}^{n} \binom{n}{j} p_m^i (1-p_m)^{n-i} f(u_m^i d_m^{n-i} S_0) \tag{4.4}
$$

then we have:

$$
v_m^e(0,S_0) = e^{-(r+\lambda_m)T} \sum_{n=0}^{\infty} \frac{(\lambda_m T)^n}{n!} \Phi_n^m(0,S_0) \tag{4.5}
$$

This holds for European put as well as for European call options. Whereas convergence of put prices follows immediatly from weak–convergence of the respective processes, convergence of call prices follows via put–call-parity.

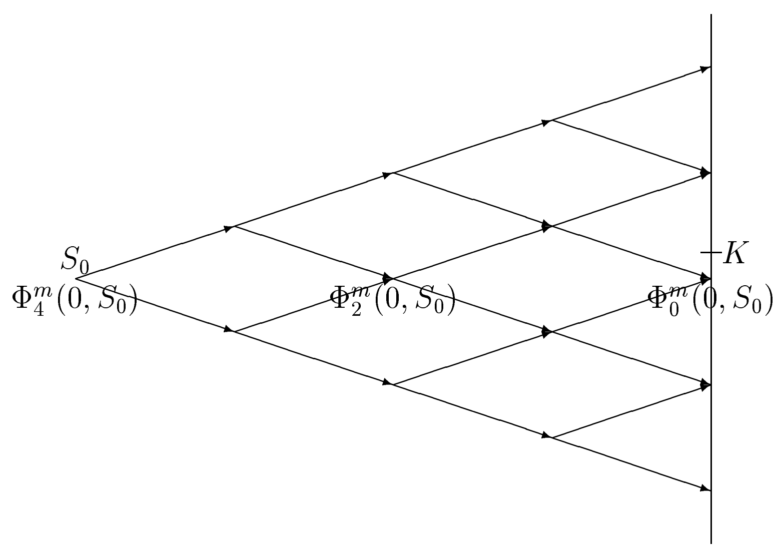

FIGURE 4.1. Example grids for n even ($n = 0, 2, 4$)

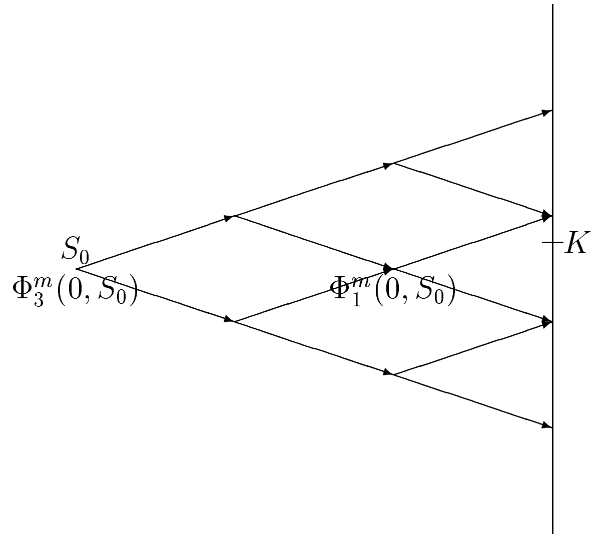

FIGURE 4.2. Example grids for n odd ($n = 1, 3$)

Please note that we need different grids for odd and even n. However $\Phi^m_{n+2}(0, S_0) = p_m^2 \cdot \Phi_n^m(0, u_m^2 S_0) + p_m(1 - p_m) \cdot \Phi_n^m(0, S_0) + (1 - p_m)^2 \cdot \Phi_n^m(0, d_m^2 S_0)$ (see figures 4.1 and 4.2). Thus we can calculate prices as intermediate calculations in an $2\lfloor \lambda_m \rfloor$ step resp. in an $2\lfloor \lambda_m \rfloor - 1$ step tree with parameters (u_m, d_m, p_m).

This works for any renewal process $N_T^{(m)}$. Yet, it remains to calculate $P[N_T^{(m)} = n]$ for any $n \in \mathbb{N}$. Unfortunately for the "stopping–approach" proposed by Rogers and Stapelton[96] the distribution of the the stopping times is unknown. We only know its Laplace transform (see Karatzas and Shreve[]):

$$\begin{aligned} \varphi(\lambda) &= E\left[\exp\{-\lambda \tau_1\}\right] \\ &= \frac{\cosh \mu \sigma^{-2} \Delta x}{\cosh \sqrt{\mu^2 + 2\lambda\sigma^2} / \sigma^2 \Delta x} \end{aligned}$$

This allows for a calculation of the mean and variance of τ_1. Rogers and Stapelton[96] now calculate the probabilities through the application of a Limit theorem in Petrov[95]. This procedure is quite complicated and does in our eyes not add a survalue. Therefore we restrict ourselves to a jump–process in the sequel and prefer a suitable choice of Δx_m.

Of course in a first step we need simplify the infinite series in (4.5) to a finite one. From the Central Limit Theorem for renewals we deduce:

$$\frac{N_T^{(m)} - T\lambda_m}{\sqrt{T\lambda_m}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0, 1)$$

Thus:

$$\lim_{m \to \infty} P[N_T^{(m)} \in \{0, \ldots, 2\lfloor \lambda_m \rfloor\}] = 1$$

Therefore it is not necessary to calculate the infinite series in (4.5) and we will subsequently use the approximation:

$$v_m^e(0, S_0) \approx e^{-(r+\lambda_m)T} \sum_{n=0}^{2\lfloor\lambda_m\rfloor} \frac{(\lambda_m T)^n}{n!} \Phi_n^m(0, S_0) \tag{4.6}$$

In figures 4.3 and 4.4 we present the results of calculating European Put Option prices with the CRR Binomial Model and our Randomized Binomial Model.

For a given Call or Put option with strike K and a refinement m we will in the sequel always take $\Delta x_m := \frac{\ln S/K}{m}$, yielding the grid $\mathcal{G}_m := \Delta x_m \cdot \mathbb{Z}$. The same grid results in a CRR Binomial Model if we take a refinement of $n = \lfloor(\frac{\sigma}{\Delta x_m})^2\rfloor$. Since we can calculate the values $\Phi_n^m(0, S_0)$ as intermediate calculations in an in an $2\lfloor\lambda_m\rfloor$ step resp. in an $2\lfloor\lambda_m\rfloor - 1$ step tree with parameters (u_m, d_m, p_m), in order to compare both approaches properly depending on its complexity, we index calculations in the Random–Time Binomial Model by $2\lfloor\lambda_m\rfloor$ and calculate the corresponding CRR prices taking this as refinement. Figure 4.3 contains calculations for an out–of–the–

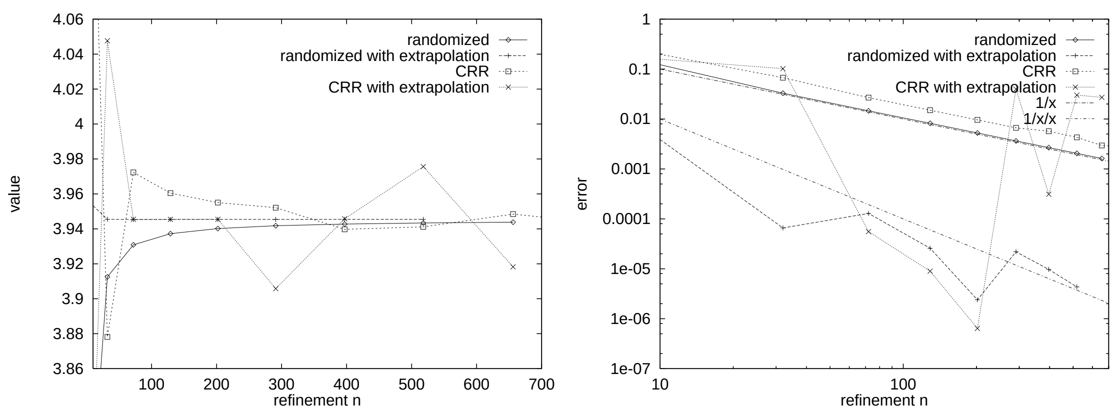

FIGURE 4.3. typical pattern resulting from european put option price calculations using the randomized model and its extrapolation with the following selection of parameters: $S = 100$, $K = 110$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 700$

money Option, whereas figure 4.4 presents calculations for in–the–money Option. Left hand we display the values according to the refinement which shows us the convergence behaviour. As we are interested in the convergence, we make use of the fact that we know the true (continuous time) price from the Black–Scholes formulae. Thus the right hand part contains the absolute difference. We have chosen a log–log–scale in the error pricture for the following reason: On this scale the function $\log \frac{\kappa}{n^\rho} = \log \kappa - \rho \cdot \log n$ becomes a straight line with slope ρ. Such an upper–bounding function gives us the order of convergence ρ.

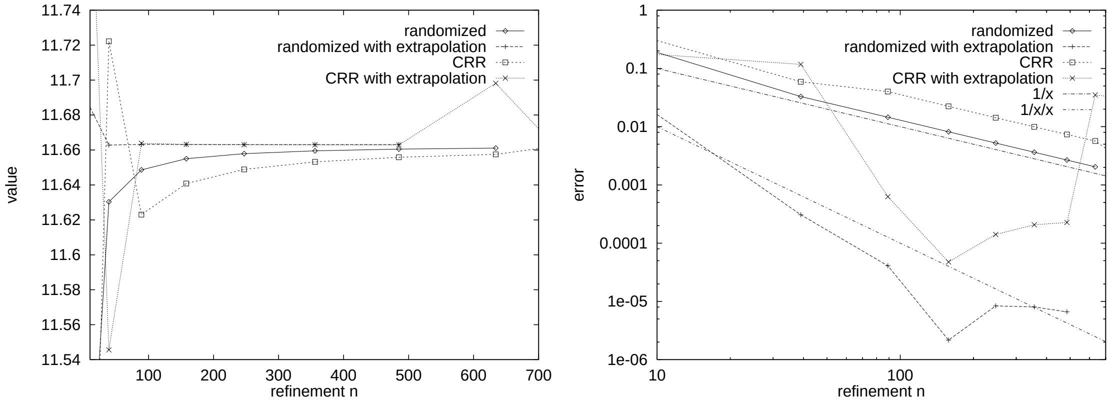

FIGURE 4.4. typical pattern resulting from european put option price calculations using the randomized model and its extrapolation with the following selection of parameters: $S = 100$, $K = 90$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 700$

We observe in the figures that due to our judicious choice of the refinement n the CRR model shows a much smoother structure than in the figures in section 2. Yet convergence is not completely monotonical, since there is under– and overestimation remaining (left hand figure). In contrast to this, the Random–Time Binomial Model exhibits an extremely smooth structure. Moreover it has slightly lower initial error than the CRR Model.

According to Leisen[96] the smooth convergence structure of the randomized model allows us to make use of extrapolation. To apply this properly we need to know the order of convergence. We see that the error is bounded by the line $1/x$ for the CRR as well as the Random–Time Binomial Model. Thus we deduce that both models converge with order one. It was proven by Leisen and Reimer[96] that the CRR model converges with order one. The following theorem establishes the corresponding result for the Random–Time Binomial Model:

Theorem 4.1:
For European Call and Put Option prices, if we denote by $v^e(0, S_0)$ the Black–Scholes formula, and by $v^e_m(0, S_0)$ the value of equation (4.5), then:

$$|v^e(0, S_0) - v^e_m(0, S_0)| \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Proof. We have according to theorem 2 in Leisen and Reimer[96]:

$$
\begin{aligned}
|p_m - p| &\leq \sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \left| e^{-rT} \Phi_n^m(0, S_0) - c(0, S_0) \right| \\
&\leq \sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \kappa \cdot n \cdot \left(\mathfrak{m}_n^1 + \mathfrak{m}_n^2 + \mathfrak{m}_n^3 + \mathfrak{p}_n + \frac{1}{n^2} \right)
\end{aligned}
$$

where $\mathfrak{m}_n^1 := \left| \overline{E} \left[\overline{R}_{m,1} \right] - e^{r\frac{T}{n}} \right|$, $\mathfrak{m}_n^2 := \left| \overline{E} \left[\left(\overline{R}_{m,1} \right)^2 \right] - e^{(2r+\sigma^2)\frac{T}{n}} \right|$,
$\mathfrak{m}_n^3 := \left| \overline{E} \left[\left(\overline{R}_{m,1} \right)^3 \right] - e^{(3r+3\sigma^2)\frac{T}{n}} \right|$ and $\mathfrak{p}_n := \left| \overline{E} \left[\left(\ln \overline{R}_{m,1} \right) \left(\overline{R}_{m,1} - 1 \right)^3 \right] \right|$

We have

$$
\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \mathfrak{m}_n^1 = \mathcal{O}\left(\frac{1}{m^2} \right)
$$

since

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{P[N_T^{(m)} = n]}{n} &= e^{-\lambda_m T} \sum_{n=0}^{\infty} \underbrace{\frac{(\lambda_m T)^n}{n!\, n}}_{\leq 2\frac{(\lambda_m T)^n}{(n+1)!}} \\
&\leq 2e^{-\lambda_m T} \sum_{n=0}^{\infty} \frac{(\lambda_m T)^n}{(n+1)!} \\
&= 2e^{-\lambda_m T} \frac{1}{\lambda_m T} \sum_{n=1}^{\infty} \frac{(\lambda_m T)^{n+1}}{(n+1)!} \\
&\leq 2e^{-\lambda_m T} \frac{1}{\lambda_m T} \sum_{n=1}^{\infty} \frac{(\lambda_m T)^n}{n!} \\
&\leq \frac{2}{\lambda_m T} = \mathcal{O}\left(\frac{1}{\lambda_m} \right) \\
&= \mathcal{O}\left(\frac{1}{m^2} \right)
\end{aligned}
$$

From Leisen and Reimer[96] follows that $\mathfrak{p}_n = \mathcal{O}\left(\frac{1}{m^2}\right)$. We will prove now that: $\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \mathfrak{m}_n^1 = \mathcal{O}\left(\frac{1}{m^2}\right)$ Since $\mathfrak{m}_n^1 = \left| \overline{E} \left[\overline{R}_{m,1} \right] - e^{r\frac{T}{n}} \right| = \left| e^{r\frac{T}{\lambda_m}} - e^{r\frac{T}{n}} \right| =$

$e^{r\frac{T}{\lambda_m}}|1 - e^{rT(\frac{1}{n}-\frac{1}{\lambda_m})}|$, we will study:

$$\begin{aligned}
&\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \left|1 - e^{rT(\frac{1}{n}-\frac{1}{\lambda_m})}\right| \\
= &\sum_{n\geq\lambda_m} P[N_T^{(m)} = n] \underbrace{(1 - \underbrace{e^{rT(\frac{1}{n}-\frac{1}{\lambda_m})}}_{\geq -e^{r\frac{T}{\lambda_m}}})}_{\leq 1-e^{-r\frac{T}{\lambda_m}} \approx r\frac{T}{\lambda_m}} + \sum_{n\leq\lambda_m} P[N_T^{(m)} = n] \underbrace{(\underbrace{e^{rT(\frac{1}{n}-\frac{1}{\lambda_m})}}_{\leq e^{r\frac{T}{n}}} - 1)}_{e^{r\frac{T}{n}} \approx r\frac{T}{n}} \\
\leq &\underbrace{\sum_{n\geq\lambda_m} P[N_T^{(m)} = n]\, r\frac{T}{\lambda_m}}_{=\mathcal{O}(\frac{1}{\lambda})=\mathcal{O}\left(\frac{1}{m^2}\right)} + \underbrace{\sum_{n\geq\lambda_m} P[N_T^{(m)} = n]\, r\frac{T}{n}}_{=\mathcal{O}\left(\frac{1}{m^2}\right)}
\end{aligned}$$

With the methods in Appendix B of Leisen and Reimer[96] it is easy to derive from this, that: $\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \mathfrak{m}_n^2 = \mathcal{O}\left(\frac{1}{m^2}\right)$, $\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \mathfrak{m}_n^3 = \mathcal{O}\left(\frac{1}{m^2}\right)$ □

We can derive from this result the following extrapolation rule (see Leisen[96]) for both models, where $(m_1, m_2) \in \mathbb{N}^2$:

$$p_{(m_1,m_2)} = \frac{m_2^2 p_{m_2} - m_1^2 p_{m_1}}{m_2^2 - m_1^2}$$

In figures 4.3 and 4.4 we applied it with the values calculated earlier, i.e. $(m, m+1)$. We applied it to the CRR model, too. Extrapolated prices in the CRR Model behave much more like those in the original CRR Model. The wavy patterns seem to be even enforced. Although there are very accurate results we are forced to rely on the upper error bound, which is not better than in the original CRR Model.

Differently, extrapolating the Random–Time Binomial Model yields very impressive results. We have very small initial errors which give us almost immediatly "penny–accuarcy". Moreover extrapolated prices seem to converge with the higher order of two. We would like to remark that this result is in line with those of Leisen[96] who predicts this for models with sufficiently smooth convergence structure.

Next we come to American Options. Unfortunately it is not straightforward to generalize equation (4.5) to American Options, since discounting will typically introduce path–dependency. Thus valuation of American Options is slightly more difficult.

Proposition 4.1:
For fixed m, the value of the American Put Option is

$$v_m^a(0, S_0) = e^{-\lambda_m T} \sum_{n=0}^{\infty} \frac{(\lambda_m T)^n}{n!} \Phi_n^m(0, S_0)$$

where Φ_n^m is its price in an n–step tree with characteristics (u_m, d_m, p_m) performing discounting in each time–step as in the original CRR-model.

Proof. Fix some $n \in \mathbb{N}$ and $t \in [0, T]$.
On $A^{n,t} := \{N_T^{(m)} - N_t^{(m)} = n\}$ we have a sequence of exactly n jumps $(\tau_1, \ldots, \tau_n)$ on $[t, T]$.
Any such sequence denotes a tree with varying time step sizes $\tau_{m,i} - \tau_{m,i-1}$, where we can easily calculate using the usual backward–induction argument the "value" of the american put

$$\overline{J}^{n,t}_{(\tau_1,\ldots,\tau_n)}$$

Denote the maximum over all such sequences by

$$\tilde{J}_t^n := \max_{(\tau_1,\ldots,\tau_n)} \overline{J}^{n,t}_{(\tau_1,\ldots,\tau_n)}$$

and

$$\overline{J}_t := E\left[\tilde{J}_t^{N_T(\omega)-N_t} \,\middle|\, \overline{S}_t, N_t\right]$$

Then a little thought immediatly reveals that $(\overline{J}_t)_t$ is the smallest supermartingale which majorizes $f(\overline{S}_t)$.
Since $(\overline{S}_t)_t$ is càdlàg, $\overline{J}_t$ is the value of the above american put option (see El Karoui[79]).
A case study of deviations to the left resp. right reveals that for any n, t and any sequence $(\tau_1, \ldots, \tau_n)$ and $\tau_i' \neq \frac{\tau_{i-1}+\tau_{i+1}}{2}$ we have:

$$\overline{J}^{n,t}_{(\tau_1,\ldots,\tau_i',\ldots,\tau_n)} \leq \overline{J}^{n,t}_{(\tau_1,\ldots,\frac{\tau_{i-1}+\tau_{i+1}}{2},\ldots,\tau_n)}$$

Thus:

$$\begin{aligned} \tilde{J}_t^n &= \overline{J}^{n,t}_{(\Delta t_n, 2\Delta t_n,\ldots,T)} \\ \text{where} \qquad \Delta t_n &:= \frac{T-t}{n} \end{aligned}$$

This completes the proof. □

Theorem 4.2:

$$v_m^a(0, S_0) \xrightarrow{m} v^a(0, S_0)$$

Proof. Lamberton and Pagès[90] by checking condition (H) using the sufficient conditions of Mulinacci and Pratelli[96]. □

Thus we can calculate continuous–time prices via the above algorithm.

In figures 4.5 and 4.6 we present American Put Option Price calculations. True values are calculated using the CRR Binomial Model with a refinement of $n = 50000$.

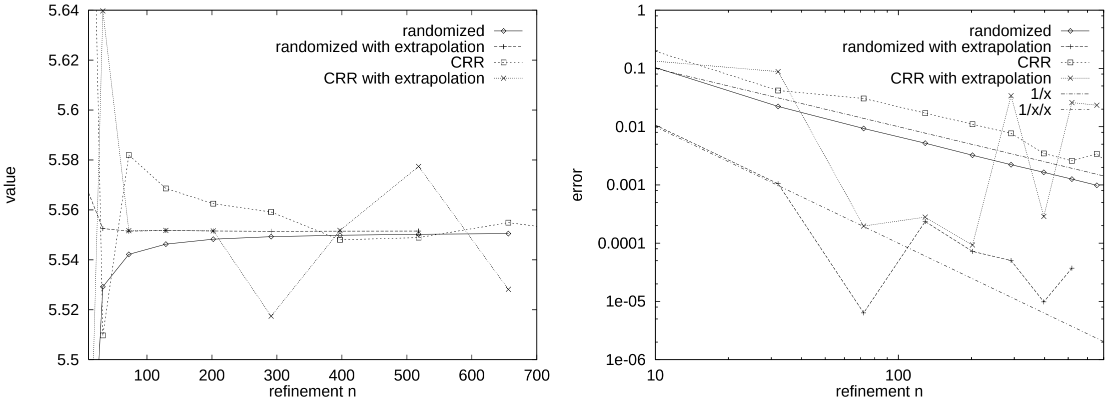

Figure 4.5. typical pattern resulting from American put option price calculations using the randomized model and its extrapolation with the following selection of parameters: $S = 100$, $K = 110$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 700$

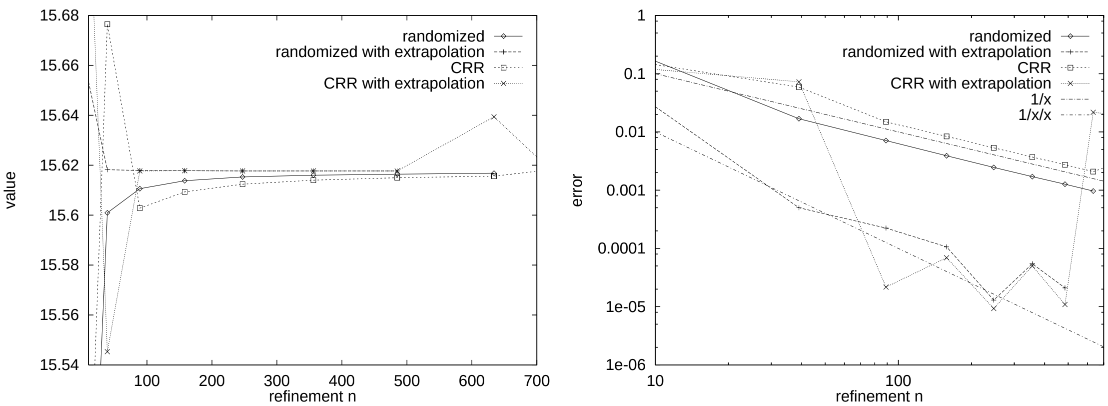

Figure 4.6. typical pattern resulting from American put option price calculations using the randomized model and its extrapolation with the following selection of parameters: $S = 100$, $K = 90$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $n = 10, \ldots, 700$

Besides this the figures correspond to figures 4.3 resp. 4.4. Moreover the analysis and the conlusions immediatly carry over to this case.

Through extrapolation the complication resulting from randomization is more than offset. Although randomization complicates valuation we observe that through extrapolation it gives a very competitive pricing tool.

5. Approximating Jump–Diffusions

In the approach of Amin the jump–process was simply put on top of the Binomial Model. Thus it inherited its poor convergence properties from this. Moreover though the model of Amin[96] is computationally correct, it does not match the original idea of jump–diffusions of a rare event at random time.

Using the randomized Binomial Model it is straightforward to add the jump–part: The sum of two jump–processes is simply a jump–process whose intensity is the sum of the two parts. The tools developed in section 4 carry immediatly over to this case. They can be very simple and competitive for valuation purposes.

Suppose we are in the jump–diffusion framework presented at the end of section 2 and suppose we are given a sequence $(N^m, \overline{R}^m)_m$, independent from N and converging weakly to the following process:

$$X_t := \mu t + \sigma W_t \tag{5.1}$$

where here $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i] + \pi$. Such a sequence can be constructed easily with the results of section 3 by applying equations (3.1) and (3.9) using this μ. The process $\overline{N}^m := N + N^m$ is a poisson with intensity $\lambda + \lambda_m$. Now define the sequence of random variables $(Z_i^m)_i$ by

$$Z_i^m \sim \begin{cases} V_i & ; \frac{\lambda}{\lambda+\lambda_m} \\ \overline{R}_i^m & ; \frac{\lambda_m}{\lambda+\lambda_m} \end{cases} \tag{5.2}$$

and the process $\overline{X}^m$, $\overline{S}^m$ by

$$\overline{X}_t^m := \sum_{i=1}^{\overline{N}_t^m} Z_i^m \tag{5.3}$$

$$\overline{S}_t^m := \exp \overline{X}_t^m \tag{5.4}$$

This approximation is very easy to perform. We believe also that due to the fact that it is straightforward, it represents much better the didactical advantages of the original CRR Binomial Model.

We have the following:

Theorem 5.1:

$$\overline{X}^m \overset{d}{\Longrightarrow} X \tag{5.5}$$

$$\overline{S}^m \overset{d}{\Longrightarrow} S \tag{5.6}$$

Proof. Denote by h the function $h : x \mapsto x + \sum_{i=1}^{N} U_i$ and by $\mathcal{C}$ where it is continuous. We have

$$\sum_{i=1}^{N^m} \overline{R}^m \overset{d}{\Longrightarrow} \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t$$

Since this is continuous, we have according to proposition VI.1.23 in Jacod and Shiryaev[87] that

$$P\left[\sum_{i=1}^{N^m} \in \mathcal{C}\right] = 1$$

Thus according to VI.3.8 (ii) in Jacod and Shiryaev[87]:

$$\overline{X}^m = \sum_{i=1}^{N^m} \overline{R}^m + \sum_{i=1}^{N} U_i \stackrel{d}{\Longrightarrow} X$$

□

The put Payoff–function is continuous and bounded. This and the above theorem immediatly gives us:

Proposition 5.1:
For European Put Options we have:

$$v_m^e(0, S_0) \stackrel{m}{\longrightarrow} v^e(0, S_0)$$

From Put–Call Parity follows then:

Proposition 5.2:
For European Call Options we have:

$$v_m^e(0, S_0) \stackrel{m}{\longrightarrow} v^e(0, S_0)$$

This means that we have convergence for european calls, especially in the case of Merton to the formula (2.17).

The methods and proofs presented in the previous section carry immediatly over:

Proposition 5.3:
For European Options we can calculate its value by

$$v_m^e(0, S_0) = e^{-(r+\lambda+\lambda_m)T} \sum_{n=0}^{\infty} \frac{((\lambda + \lambda_m)T)^n}{n!} \Phi_n^m(0, S_0)$$

where $\Phi_n^m(0, S_0)$ *can be calculated through intermediate calculations in backward–induction as in section 4.*

For American Options we can calculate its value by

$$v_m^a(0, S_0) = e^{-(\lambda+\lambda_m)T} \sum_{n=0}^{\infty} \frac{((\lambda + \lambda_m)T)^n}{n!} \Phi_n^m(0, S_0)$$

where $\Phi_n^m(0, S_0)$ *is its price in an* n*–step tree with characteristics* (u_m, d_m, p_m)*.*

Theorem 5.2:
For American Put Options we have:

$$v_m^a(0, S_0) \stackrel{m}{\longrightarrow} v^a(0, S_0)$$

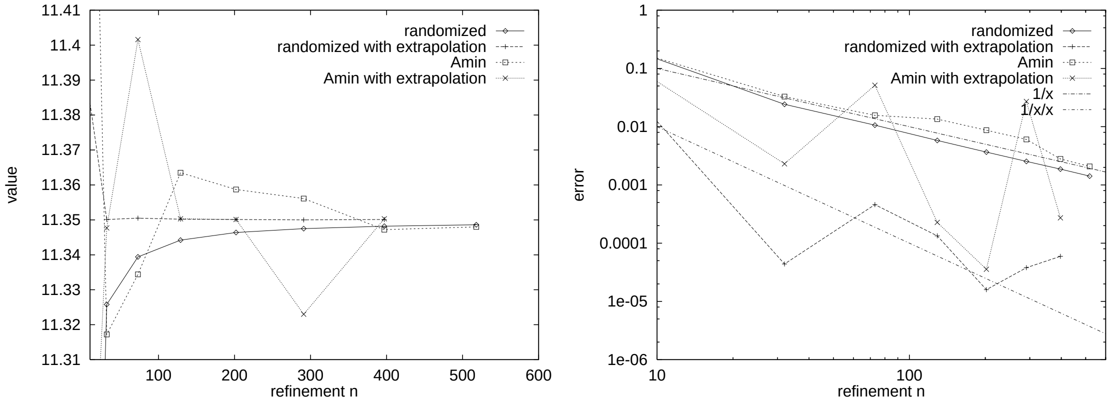

FIGURE 5.1. typical pattern resulting from American put option price calculations in a model allowing immediate ruin using the randomized model and the Amin Model resp. their extrapolations with the following selection of parameters: $S = 100$, $K = 110$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $\lambda = 0.1$, $n = 10, \ldots, 600$

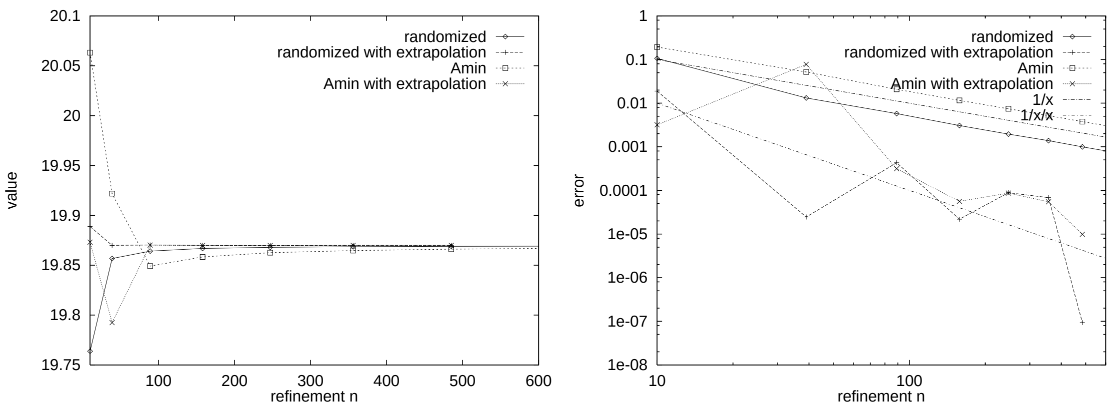

FIGURE 5.2. typical pattern resulting from American put option price calculations in a model allowing immediate ruin using the randomized model and the Amin Model resp. their extrapolations with the following selection of parameters: $S = 100$, $K = 90$, $T = 1$, $r = 0.05$, $\sigma = 0.3$, $\lambda = 0.1$, $n = 10, \ldots, 600$

For the problem of approximating a jump–diffusion allowing immediate ruin in the stock, we presented in section 2 the formula of Merton[76] (equation (2.18)). According to this formula the value for a European Call Option can be calculated via the Black–Scholes formula with adjusted interested rate $r+\lambda$. The argumentation of Merton[73] tells us that in the original Black–Scholes framework the right to exercise an American Call Option is worthless, i.e. both prices coincide. Adjusting it to this case we deduce that in the framework of Merton[76], where there is a probability of immediate ruin, the problem of pricing European and American Call Options is exactly equal to those in the Black–Scholes framework. Since we discussed this detailed in section 4 and explained its superiour performance we will now restrict ourselves to American Put Options.

Figures 5.1 and 5.1 are organised as figures 4.5 and 4.6: Figure 5.1 contains out–of–the–money options, whereas 5.2 contains in–the–money options. Left–hand figures contain the values; right–hand figures the error to the true value. The true value is calculated using the Amin model with refinement $n = 50000$.

We observe in the figures that in contrast to the Amin model, the Random–Time Binomial Model exhibits an extremely smooth structure. Moreover the upper bounding error line has slightly lower initial error. Extrapolated prices in the Amin Model behave much more like those in the original CRR Model. The wavy patterns seem to be even enforced. Although there are some accurate results we are forced to rely on the upper error bound, which is not better than in the original CRR Model. Differently, extrapolating the Random–Time Binomial Model yields very small initial errors which give us almost immediatly "penny–accuracy". Moreover extrapolated prices seem to converge with the higher order of two. Thus the remarkable convergence properties of the Random–Time Binomial Model carry over to the approximation of jump–diffusions.

6. Conclusion

In this paper we studied a Binomial Model with Random Time steps. We showed how price approximations for European and American Put and Call Options can be calculated easily in the Black–Scholes setup. We prove convergence to the continuous–time solution. For European Put Options we proved order–of–convergence one. Extrapolation improved the results impressively. Thus this Model can serve as an efficient tool in the Black–Scholes setup. A second important contribution is that the Random–Time Binomial Model gives intuitive and straightforward approximations to jump–diffusions.

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