An Introduction to Computational Finance

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"Men wanted for hazardous journey, small wages, bitter cold, long months of complete darkness, constant dangers, safe return doubtful. Honour and recognition in case of success." Advertisement placed by Earnest Shackleton in 1914. He received 5000 replies. An example of extreme risk-seeking behaviour. Hedging with options is used to mitigate risk, and would not appeal to members of Shackleton's expedition.

1 The First Option Trade

Many people think that options and futures are recent inventions. However, options have a long history, going back to ancient Greece.

As recorded by Aristotle in *Politics*, the fifth century BC philosopher Thales of Miletus took part in a sophisticated trading strategy. The main point of this trade was to confirm that philosophers could become rich if they so chose. This is perhaps the first rejoinder to the famous question *"If you are so smart, why aren't you rich?"* which has dogged academics throughout the ages.

Thales observed that the weather was very favourable to a good olive crop, which would result in a bumper harvest of olives. If there was an established Athens Board of Olives Exchange, Thales could have simply sold olive futures short (a surplus of olives would cause the price of olives to go down). Since the exchange did not exist, Thales put a deposit on all the olive presses surrounding Miletus. When the olive crop was harvested, demand for olive presses reached enormous proportions (olives were not a storable commodity). Thales then sublet the presses for a profit. Note that by placing a deposit on the presses, Thales was actually manufacturing an option on the olive crop, i.e. the most he could lose was his deposit. If had sold short olive futures, he would have been liable to an unlimited loss, in the event that the olive crop turned out bad, and the price of olives went up. In other words, he had an option on a future of a non-storable commodity.

2 The Black-Scholes Equation

This is the basic PDE used in option pricing. We will derive this PDE for a simple case below. Things get much more complicated for real contracts.

2.1 Background

Over the past few years derivative securities (options, futures, and forward contracts) have become essential tools for corporations and investors alike. Derivatives facilitate the transfer of financial risks. As such, they may be used to hedge risk exposures or to assume risks in the anticipation of profits. To take a simple yet instructive example, a gold mining firm is exposed to fluctuations in the price of gold. The firm could use a forward contract to fix the price of its future sales. This would protect the firm against a fall in the price of gold, but it would also sacrifice the upside potential from a gold price increase. This could be preserved by using options instead of a forward contract.

Individual investors can also use derivatives as part of their investment strategies. This can be done through direct trading on financial exchanges. In addition, it is quite common for financial products to include some form of embedded derivative. Any insurance contract can be viewed as a *put* option. Consequently, any investment which provides some kind of protection actually includes an option feature. Standard examples include deposit insurance guarantees on savings accounts as well as the provision of being able to redeem a savings bond at par at any time. These types of embedded options are becoming increasingly common and increasingly complex. A prominent current example are investment guarantees being offered by insurance companies ("segregated funds") and mutual funds. In such contracts, the initial investment is guaranteed, and gains can be locked-in (reset) a fixed number of times per year at the option of the contract holder. This is actually a very complex put option, known as a shout option. How much should an investor be willing to pay for this insurance? Determining the fair market value of these sorts of contracts is a problem in *option pricing*.

2.2 Definitions

Let's consider some simple European put/call options. At some time T in the future (the expiry or exercise date) the holder has the right, but not the obligation, to

- Buy an asset at a prescribed price K (the exercise or strike price). This is a call option.
- Sell the asset at a prescribed price K (the exercise or strike price). This is a put option.

At expiry time T, we know with certainty what the value of the option is, in terms of the price of the underlying asset S,

$$\begin{aligned} \text{Payoff} &= \max(S-K,0) \text{ for a call} \\ \text{Payoff} &= \max(K-S,0) \text{ for a put} \end{aligned} \tag{2.1}$$

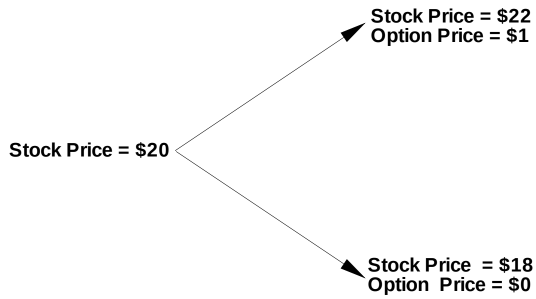
Note that the payoff from an option is always non-negative, since the holder has a right but not an obligation. This contrasts with a forward contract, where the holder *must* buy or sell at a prescribed price.

2.3 A Simple Example: The Two State Tree

This example is taken from *Options, futures, and other derivatives*, by John Hull. Suppose the value of a stock is currently \$20. It is known that at the end of three months, the stock price will be either \$22 or \$18. We assume that the stock pays no dividends, and we would like to value a European call option to buy the stock in three months for \$21. This option can have only two possible values in three months: if the stock price is \$22, the option is worth \$1, if the stock price is \$18, the option is worth zero. This is illustrated in Figure 1.

In order to price this option, we can set up an imaginary portfolio consisting of the option and the stock, in such a way that there is no uncertainty about the value of the portfolio at the end of three months. Since the portfolio has no risk, the return earned by this portfolio must be the risk-free rate.

Figure 1: A simple case where the stock value can either be \$22 or \$18, with a European call option, $K = \$21$.



Consider a portfolio consisting of a long (positive) position of δ shares of stock, and short (negative) one call option. We will compute δ so that the portfolio is riskless. If the stock moves up to \$22 or goes down to \$18, then the value of the portfolio is

$$\begin{aligned} \text{Value if stock goes up} &= \$22\delta - 1 \\ \text{Value if stock goes down} &= \$18\delta - 0 \end{aligned} \tag{2.2}$$

So, if we choose $\delta = .25$, then the value of the portfolio is

$$\begin{aligned} \text{Value if stock goes up} &= \$22\delta - 1 = \$4.50 \\ \text{Value if stock goes down} &= \$18\delta - 0 = \$4.50 \end{aligned} \tag{2.3}$$

So, regardless of whether the stock moves up or down, the value of the portfolio is \$4.50. A risk-free portfolio must earn the risk free rate. Suppose the current risk-free rate is 12%, then the value of the portfolio today must be the present value of \$4.50, or

$$4.50 \times e^{-.12 \times .25} = 4.367$$

The value of the stock today is \$20. Let the value of the option be V. The value of the portfolio is

$$\begin{aligned} 20 \times .25 - V &= 4.367 \\ \rightarrow \quad & V = .633 \end{aligned}$$

2.4 A hedging strategy

So, if we sell the above option (we hold a short position in the option), then we can hedge this position in the following way. Today, we sell the option for \$.633, borrow \$4.367 from the bank at the risk free rate (this means that we have to pay the bank back \$4.50 in three months), which gives us \$5.00 in cash. Then, we buy .25 shares at \$20.00 (the current price of the stock). In three months time, one of two things happens

- The stock goes up to \$22, our stock holding is now worth \$5.50, we pay the option holder \$1.00, which leaves us with \$4.50, just enough to pay off the bank loan.

- The stock goes down to \$18.00. The call option is worthless. The value of the stock holding is now \$4.50, which is just enough to pay off the bank loan.

Consequently, in this simple situation, we see that the *theoretical* price of the option is the cost for the seller to set up portfolio, which will precisely pay off the option holder and any bank loans required to set up the hedge, at the expiry of the option. In other words, this is price which a hedger requires to ensure that there is always just enough money at the end to net out at zero gain or loss. If the market price of the option was higher than this value, the seller could sell at the higher price and lock in an instantaneous risk-free gain. Alternatively, if the market price of the option was lower than the theoretical, or *fair market* value, it would be possible to lock in a risk-free gain by selling the portfolio short. Any such arbitrage opportunities are rapidly exploited in the market, so that for most investors, we can assume that such opportunities are not possible (the *no arbitrage condition*), and therefore that the market price of the option should be the theoretical price.

Note that this hedge works regardless of whether or not the stock goes up or down. Once we set up this hedge, we don't have a care in the world. The value of the option is also independent of the probability that the stock goes up to \$22 or down to \$18. This is somewhat counterintuitive.

2.5 Brownian Motion

Before we consider a model for stock price movements, let's consider the idea of Brownian motion with drift. Suppose X is a random variable, and in time $t \rightarrow t + dt$, $X \rightarrow X + dX$, where

$$dX = \alpha dt + \sigma dZ \tag{2.4}$$

where αdt is the drift term, σ is the volatility, and dZ is a random term. The dZ term has the form

$$dZ = \phi\sqrt{dt} \tag{2.5}$$

where ϕ is a random variable drawn from a normal distribution with mean zero and variance one (ϕ is $N(0,1)$).

If E is the expectation operator, then

$$E(\phi) = 0 \qquad E(\phi^2) = 1 \; . \tag{2.6}$$

Now in a time interval dt, we have

$$\begin{aligned} E(dX) &= E(\alpha dt) + E(\sigma dZ) \\ &= \alpha dt \; , \end{aligned} \tag{2.7}$$

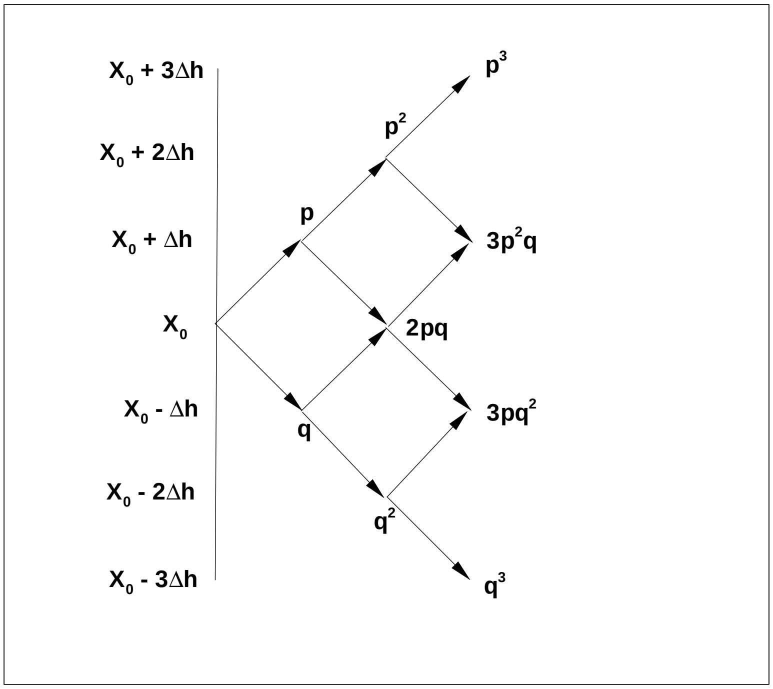


Figure 2: Probabilities of reaching the discrete lattice points for the first three moves.

and the variance of dX, denoted by $Var(dX)$ is

$$\begin{aligned} Var(dX) &= E([dX - E(dX)]^2) \\ &= E([\sigma dZ]^2) \\ &= \sigma^2 dt \ . \end{aligned} \tag{2.8}$$

Let's look at a discrete model to understand this process more completely. Suppose that we have a discrete lattice of points. Let $X = X_0$ at $t = 0$. Suppose that at $t = \Delta t$,

$$\begin{aligned} X_0 &\rightarrow X_0 + \Delta h \ ; \quad \text{with probability } p \\ X_0 &\rightarrow X_0 - \Delta h \ ; \quad \text{with probability } q \end{aligned} \tag{2.9}$$

where $p + q = 1$. Assume that

- X follows a Markov process, i.e. the probability distribution in the future depends only on where it is now.
- The probability of an up or down move is independent of what happened in the past.
- X can move only up or down Δh.

At any lattice point $X_0 + i\Delta h$, the probability of an up move is p, and the probability of a down move is q. The probabilities of reaching any particular lattice point for the first three moves are shown in Figure 2. Each move takes place in the time interval $t \rightarrow t + \Delta t$.

Let ΔX be the change in X over the interval $t \rightarrow t + \Delta t$. Then

$$\begin{aligned} E(\Delta X) &= (p-q)\Delta h \\ E([\Delta X]^2) &= p(\Delta h)^2 + q(-\Delta h)^2 \\ &= (\Delta h)^2, \end{aligned} \tag{2.10}$$

so that the variance of ΔX is

$$\begin{aligned} Var(\Delta X) &= E([\Delta X]^2) - [E(\Delta X)]^2 \\ &= (\Delta h)^2 - (p-q)^2(\Delta h)^2 \\ &= 4pq(\Delta h)^2 \ . \end{aligned} \tag{2.11}$$

Now, suppose we consider the distribution of X after n moves, so that $t = n\Delta t$. The probability of j up moves, and $(n-j)$ down moves ($P(n,j)$) is

$$P(n,j) = \frac{n!}{j!(n-j)!} p^j q^{n-j} \tag{2.12}$$

which is just a binomial distribution. Now, if X_n is the value of X after n steps on the lattice, then

$$\begin{aligned} E(X_n - X_0) &= nE(\Delta X) \\ Var(X_n - X_0) &= nVar(\Delta X) \ , \end{aligned} \tag{2.13}$$

which follows from the properties of a binomial distribution, (each up or down move is independent of previous moves). Consequently, from equations (2.10, 2.11, 2.13) we obtain

$$\begin{aligned} E(X_n - X_0) &= n(p-q)\Delta h \\ &= \frac{t}{\Delta t}(p-q)\Delta h \\ Var(X_n - X_0) &= n4pq(\Delta h)^2 \\ &= \frac{t}{\Delta t} 4pq(\Delta h)^2 \end{aligned} \tag{2.14}$$

Now, we would like to take the limit at $\Delta t \rightarrow 0$ in such a way that the mean and variance of X, after a finite time t is independent of Δt, and we would like to recover

$$\begin{aligned} dX &= \alpha dt + \sigma dZ \\ E(dX) &= \alpha dt \\ Var(dX) &= \sigma^2 dt \end{aligned} \tag{2.15}$$

as $\Delta t \rightarrow 0$. Now, since $0 \leq p, q \leq 1$, we need to choose $\Delta h = Const \sqrt{\Delta t}$. Otherwise, from equation (2.14) we get that $Var(X_n - X_0)$ is either 0 or infinite after a finite time. (Stock variances do not have either of these properties, so this is obviously not a very interesting case).

Let's choose $\Delta h = \sigma\sqrt{\Delta t}$, which gives (from equation (2.14))

$$\begin{aligned} E(X_n - X_0) &= (p-q)\frac{\sigma t}{\sqrt{\Delta t}} \\ Var(X_n - X_0) &= t4pq\sigma^2 \end{aligned} \tag{2.16}$$

Now, for $E(X_n - X_0)$ to be independent of Δt as $\Delta t \to 0$, we must have

$$(p-q) = \ Const.\ \sqrt{\Delta t} \tag{2.17}$$

If we choose

$$p - q = \frac{\alpha}{\sigma}\sqrt{\Delta t} \tag{2.18}$$

we get

$$\begin{aligned} p &= \frac{1}{2}[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}] \\ q &= \frac{1}{2}[1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}] \end{aligned} \tag{2.19}$$

Now, putting together equations (2.16-2.19) gives

$$\begin{aligned} E(X_n - X_0) &= \alpha t \\ Var(X_n - X_0) &= t\sigma^2(1 - \frac{\alpha^2}{\sigma^2}\Delta t) \\ &= t\sigma^2 \ ; \ \ \Delta t \to 0 \end{aligned} \tag{2.20}$$

so that, in the limit as $\Delta t \to 0$, we can interpret the random walk for X on the lattice (with these parameters) as the solution to the stochastic differential equation (SDE)

$$\begin{aligned} dX &= \alpha\ dt + \sigma\ dZ \\ & dZ = \phi\sqrt{dt}. \end{aligned} \tag{2.21}$$

For future reference, if $\alpha = 0, \sigma = 1$, so that $dX = X_i - X_{i-1} = dZ$, note that (from equation (2.19))

$$\begin{aligned} E(X_n - X_0) &= 0 \\ Var(X_n - X_0) &= t \end{aligned} \tag{2.22}$$

so that we can write

$$\begin{aligned} \int_0^t dX &= \int_0^t dZ \\ &= (X_n - X_0) \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} (X_n - X_0) &= N(0,t) \\ &= \int_0^t dZ\ . \end{aligned} \tag{2.24}$$

In other words, after a finite time t, $\int_0^t dZ$ is normally distributed with mean zero and variance t (the limit of a binomial distribution is a normal distribution).

In the case $\alpha = 0$, $\sigma = 1$, we have that $X_i - X_{i-1} = \sqrt{\Delta t}$ with probability p and $X_i - X_{i-1} = -\sqrt{\Delta t}$ with probability q. Note that $(X_i - X_{i-1})^2 = \Delta t$, with certainty, so that we can write

$$(X_i - X_{i-1})^2 = (dZ)^2 = \Delta t \ . \tag{2.25}$$

To summarize

- We can interpret the SDE

$$\begin{aligned} dX &= \alpha\ dt + \sigma\ dZ \\ & dZ = \phi\sqrt{dt}. \end{aligned} \tag{2.26}$$

  as the limit of a discrete random walk on a lattice as the timestep tends to zero.

- $Var(dZ) = dt$, otherwise, after any finite time, the $Var(X_n - X_0)$ is either zero or infinite.

- We can *integrate* the term dZ to obtain

$$\begin{aligned} \int_0^t dZ &= Z(t) - Z(0) \\ &= N(0,t) \ . \end{aligned} \tag{2.27}$$

Going back to our lattice example, note that the total distance traveled over any finite interval of time becomes infinite,

$$E(|\Delta X|) = \Delta h \tag{2.28}$$

so that the the total distance traveled in n steps is

$$\begin{aligned} n\Delta h &= \frac{t}{\Delta t}\Delta h \\ &= \frac{t\sigma}{\sqrt{\Delta t}} \end{aligned} \tag{2.29}$$

which goes to infinity as $\Delta t \to 0$. Similarly,

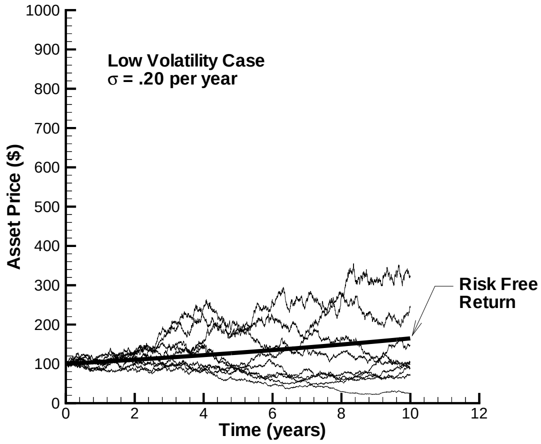
$$\frac{\Delta x}{\Delta t} = \pm\infty \ . \tag{2.30}$$

Consequently, Brownian motion is very jagged at every timescale. These paths are not differentiable, i.e. $\frac{dx}{dt}$ does not exist, so we cannot speak of

$$E(\frac{dx}{dt}) \tag{2.31}$$

but we can possibly define

$$\frac{E(dx)}{dt} \ . \tag{2.32}$$



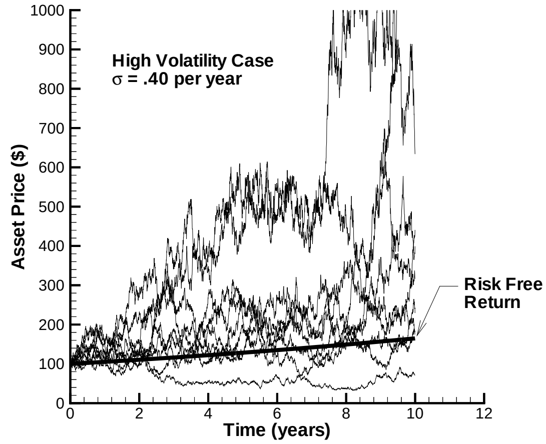


Figure 3: Realizations of asset price following geometric Brownian motion. Left: low volatility case; right: high volatility case. Risk-free rate of return $r = .05$.

2.6 Geometric Brownian motion with drift

Of course, the actual path followed by stock is more complex than the simple situation described above. More realistically, we assume that the relative changes in stock prices (the returns) follow Brownian motion with drift. We suppose that in an infinitesimal time dt, the stock price S changes to $S + dS$, where

$$\frac{dS}{S} = \mu dt + \sigma dZ \tag{2.33}$$

where μ is the drift rate, σ is the volatility, and dZ is the increment of a Wiener process,

$$dZ = \phi\sqrt{dt} \tag{2.34}$$

where ϕ is $N(0, 1)$. Equations (2.33) and (2.34) are called geometric Brownian motion with drift. So, superimposed on the upward (relative) drift is a (relative) random walk. The degree of randomness is is given by the volatility σ. Figure 3 gives an illustration of ten realizations of this random process for two different values of the volatility. In this case, we assume that the drift rate μ equals the risk free rate.

Note that

$$\begin{aligned} E(dS) &= E(\sigma S dZ + \mu S dt) \\ &= \mu S dt \\ &\quad \text{since } E(dZ) = 0 \end{aligned} \tag{2.35}$$

and that the variance of dS is

$$\begin{aligned} Var[dS] &= E(dS^2) - [E(dS)]^2 \\ &= E(\sigma^2 S^2 dZ^2) \\ &= \sigma^2 S^2 dt \end{aligned} \tag{2.36}$$

so that σ is a measure of the degree of randomness of the stock price movement.

Equation (2.33) is a *stochastic differential equation*. The normal rules of calculus don't apply, since for example

$$\begin{aligned} \frac{dZ}{dt} &= \phi \frac{1}{\sqrt{dt}} \\ &\rightarrow \infty \text{ as } dt \rightarrow 0 \quad . \end{aligned}$$

The study of these sorts of equations uses results from stochastic calculus. However, for our purposes, we need only one result, which is Ito's Lemma (see *Derivatives: the theory and practice of financial engineering*, by P. Wilmott). Suppose we have some function $G = G(S,t)$, where S follows the stochastic process equation (2.33), then, in small time increment dt, $G \rightarrow G + dG$, where

$$dG = \left(\mu S \frac{\partial G}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) dt + \sigma S \frac{\partial G}{\partial S} dZ \tag{2.37}$$

An informal derivation of this result is given in the following section.

2.6.1 Ito's Lemma

We give an informal derivation of Ito's lemma (2.37). Suppose we have a variable S which follows

$$dS = a(S,t)dt + b(S,t)dZ \tag{2.38}$$

where dZ is the increment of a Weiner process.

Now since

$$dZ^2 = \phi^2 dt \tag{2.39}$$

where ϕ is a random variable drawn from a normal distribution with mean zero and unit variance, we have that, if E is the expectation operator, then

$$E(\phi) = 0 \qquad E(\phi^2) = 1 \tag{2.40}$$

so that the expected value of dZ^2 is

$$E(dZ^2) = dt \tag{2.41}$$

Now, it can be shown that

$$E((dZ^2 - dt)^2) = O(dt^2) \tag{2.42}$$

so that, in the limit as $dt \rightarrow 0$, we have that $\phi^2 dt$ becomes non-stochastic, so that with probability one

$$dZ^2 \rightarrow dt \quad \text{as } dt \rightarrow 0 \tag{2.43}$$

Now, suppose we have some function $G = G(S,t)$, then

$$dG = G_S dS + G_t dt + G_{SS} \frac{dS^2}{2} + ... \tag{2.44}$$

Now (from (2.38))

$$\begin{aligned} (dS)^2 &= (adt + b\ dZ)^2 \\ &= a^2dt^2 + ab\ dZdt + b^2dZ^2 \end{aligned} \tag{2.45}$$

Since $dZ = O(\sqrt{dt})$ and $dZ^2 \to dt$, equation (2.45) becomes

$$(dS)^2 = b^2dZ^2 + O((dt)^{3/2}) \tag{2.46}$$

or

$$(dS)^2 \to b^2dt \quad \text{as } dt \to 0 \tag{2.47}$$

Now, equations(2.38,2.44,2.47) give

$$\begin{aligned} dG &= G_S dS + G_t dt + G_{SS}\frac{dS^2}{2} + ... \\ &= G_S(a\ dt + b\ dZ) + dt(G_t + G_{SS}\frac{b^2}{2}) \\ &= G_S b\ dZ + (aG_S + G_{SS}\frac{b^2}{2} + G_t)dt \end{aligned} \tag{2.48}$$

So, we have the result that if

$$dS = a(S,t)dt + b(S,t)dZ \tag{2.49}$$

and if $G = G(S,t)$, then

$$dG = G_S b\ dZ + (a\ G_S + G_{SS}\frac{b^2}{2} + G_t)dt \tag{2.50}$$

Equation (2.37) can be deduced by setting $a = \mu S$ and $b = \sigma S$ in equation (2.50).

2.7 The Black-Scholes Analysis

Assume

- The stock price follows geometric Brownian motion, equation (2.33).
- The risk-free rate of return is a constant r.
- There are no arbitrage opportunities, i.e. all risk-free portfolios must earn the risk-free rate of return.
- Short selling is permitted (i.e. we can own negative quantities of an asset).

Suppose that we have an option whose value is given by $V = V(S,t)$. Construct an imaginary portfolio, consisting of one option, and a number of $(-(\alpha^h))$ of the underlying asset. (If $(\alpha^h) > 0$, then we have sold the asset short, i.e. we have borrowed an asset, sold it, and are obligated to give it back at some future date).

The value of this portfolio P is

$$P = V - (\alpha^h)S \tag{2.51}$$

In a small time dt, $P \rightarrow P + dP$,

$$dP = dV - (\alpha^h)dS \tag{2.52}$$

Note that in equation (2.52) we not included a term $(\alpha^h)_S S$. This is actually a rather subtle point, since we shall see (later on) that (α^h) actually depends on S. However, if we think of a real situation, at any instant in time, we must choose (α^h), and then we hold the portfolio while the asset moves randomly. So, equation (2.52) is actually the change in the value of the portfolio, not a differential. If we were taking a true differential then equation (2.52) would be

$$dP = dV - (\alpha^h)dS - Sd(\alpha^h)$$

but we have to remember that (α^h) does not change over a small time interval, since we pick (α^h), and then S changes randomly. We are not allowed to *peek into the future*, (otherwise, we could get rich without risk, which is not permitted by the no-arbitrage condition) and hence (α^h) is not allowed to contain any information about future asset price movements. The principle of *no peeking into the future* is why Ito stochastic calculus is used. Other forms of stochastic calculus are used in Physics applications (i.e. turbulent flow).

Substituting equations (2.33) and (2.37) into equation (2.52) gives

$$dP = \sigma S \left(V_S - (\alpha^h)\right) dZ + \left(\mu S V_S + \frac{\sigma^2 S^2}{2} V_{SS} + V_t - \mu(\alpha^h)S\right) dt \tag{2.53}$$

We can make this portfolio riskless over the time interval dt, by choosing $(\alpha^h) = V_S$ in equation (2.53). This eliminates the dZ term in equation (2.53). (This is the analogue of our choice of the amount of stock in the riskless portfolio for the two state tree model.) So, letting

$$(\alpha^h) = V_S \tag{2.54}$$

then substituting equation (2.54) into equation (2.53) gives

$$dP = \left(V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right) dt \tag{2.55}$$

Since P is now risk-free in the interval $t \rightarrow t + dt$, then no-arbitrage says that

$$dP = rPdt \tag{2.56}$$

Therefore, equations (2.55) and (2.56) give

$$rPdt = \left(V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right) dt \tag{2.57}$$

Since

$$P = V - (\alpha^h)S = V - V_S S \tag{2.58}$$

then substituting equation (2.58) into equation (2.57) gives

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV = 0 \tag{2.59}$$

which is the Black-Scholes equation. Note the rather remarkable fact that equation (2.59) is independent of the drift rate μ.

Equation (2.59) is solved backwards in time from the option expiry time $t = T$ to the present $t = 0$.

2.8 Hedging in Continuous Time

We can construct a hedging strategy based on the solution to the above equation. Suppose we sell an option at price V at $t = 0$. Then we carry out the following

- We sell one option worth V. (This gives us V in cash initially).
- We borrow $(S\frac{\partial V}{\partial S} - V)$ from the bank.
- We buy $\frac{\partial V}{\partial S}$ shares at price S.

At every instant in time, we adjust the amount of stock we own so that we always have $\frac{\partial V}{\partial S}$ shares. Note that this is a *dynamic* hedge, since we have to continually rebalance the portfolio. Cash will flow into and out of the bank account, in response to changes in S. If the amount in the bank is positive, we receive the risk free rate of return. If negative, then we borrow at the risk free rate.

So, our hedging portfolio will be

- Short one option worth V.
- Long $\frac{\partial V}{\partial S}$ shares at price S.
- $V - S\frac{\partial V}{\partial S}$ cash in the bank account.

At any instant in time (including the terminal time), this portfolio can be liquidated and any obligations implied by the short position in the option can be covered, at *zero gain or loss*, regardless of the value of S. Note that given the receipt of the cash for the option, this strategy is *self-financing*.

2.9 The option price

So, we can see that the price of the option valued by the Black-Scholes equation is the market price of the option at any time. If the price was higher then the Black-Scholes price, we could construct the hedging portfolio, dynamically adjust the hedge, and end up with a positive amount at the end. Similarly, if the price was lower than the Black-Scholes price, we could short the hedging portfolio, and end up with a positive gain. By the *no-arbitrage* condition, this should not be possible.

Note that we are *not* trying to predict the price movements of the underlying asset, which is a random process. The value of the option is based on a hedging strategy which is *dynamic*, and must be continuously rebalanced. The price is the cost of setting up the hedging portfolio. The Black-Scholes price is *not* the expected payoff.

The price given by the Black-Scholes price is *not* the value of the option to a speculator, who buys and holds the option. A speculator is making bets about the underlying drift rate of the stock (note that the drift rate does not appear in the Black-Scholes equation). For a speculator, the value of the option is given by an equation similar to the Black-Scholes equation, except that the drift rate appears. In this case, the price can be interpreted as the *expected payoff* based on the guess for the drift rate. But this is art, not science!

2.10 American early exercise

Actually, most options traded are *American* options, which have the feature that they can be exercised at any time. Consequently, an investor acting optimally, will always exercise the option if the value falls below the payoff or exercise value. So, the value of an American option is given by the solution to equation (2.59) with the additional constraint

$$V(S,t) \geq \begin{cases} \max(S-K,0) & \text{for a call} \\ \max(K-S,0) & \text{for a put} \end{cases} \tag{2.60}$$

Note that since we are working backwards in time, we *know* what the option is worth in future, and therefore we can determine the optimal course of action.

In order to write equation (2.59) in more conventional form, define $\tau = T - t$, so that equation (2.59) becomes

$$\begin{aligned} V_\tau &= \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV \\ V(S,\tau=0) &= \begin{cases} \max(S-K,0) & \text{for a call} \\ \max(K-S,0) & \text{for a put} \end{cases} \\ V(0,\tau) &\rightarrow V_\tau = -rV \\ V(S=\infty,\tau) &\rightarrow \begin{cases} \simeq S & \text{for a call} \\ \simeq 0 & \text{for a put} \end{cases} \end{aligned} \tag{2.61}$$

If the option is American, then we also have the additional constraints

$$V(S,\tau) \geq \begin{cases} \max(S-K,0) & \text{for a call} \\ \max(K-S,0) & \text{for a put} \end{cases} \tag{2.62}$$

Define the operator

$$LV \equiv V_\tau - \left(\frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV\right) \tag{2.63}$$

and let $V(S,0) = V^*$. More formally, the American option pricing problem can be stated as

$$\begin{aligned} LV &\geq 0 \\ V - V^* &\geq 0 \\ (V - V^*)LV &= 0 \end{aligned} \tag{2.64}$$

3 Further Reading

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