A Monte-Carlo Method for Optimal Portfolios.

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Abstract

This paper provides (i) new results on the structure of optimal portfolios, (ii) economic insights on the behavior of the hedging components and (iii) simulation-based methods for numerical implementation of allocation rules. The core of our approach relies on closed-form solutions for functionals of diffusion processes which simplify their numerical simulation and facilitate the computation and simulation of the hedging components of optimal portfolios. One of our procedures relies on a variance-stabilizing transformation of the underlying process which eliminates stochastic integrals from the representation of random variables in hedging terms and ensures the existence of an exact weak approximation scheme. This improves the performance of Monte-Carlo methods in the numerical implementation of portfolio rules derived on the basis of probabilistic arguments. Our approach is flexible and can be used even when the dimensionality of the set of underlying state variables is large. We implement the procedure for a class of bivariate and trivariate models in which the uncertainty is described by diffusion processes for the market price of risk (MPR), the interest rate (IR) and other relevant factors. After calibrating the models to the data we document the behavior of the portfolio demand and the hedging components relative to the parameters of the model such as risk aversion, investment horizon, speeds of mean-reversion, IR and MPR levels and volatilities. We show that the hedging terms are important and cannot be ignored for asset allocation purposes. Risk aversion and investment horizon emerge as the most relevant factors: they have a substantial impact on the size of the optimal portfolio and on its economic properties for realistic values of the models' parameters.

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1 Introduction.

Asset allocation models have received extensive attention in the past three decades. Prompted by the seminal work of Merton (1969, 1971) researchers have explored various aspects of the problem in the context of financial markets with diffusion price processes (e.g. Richard (1975)). Numerical methods based on the dynamic programming approach employed in this literature have also been used to examine the properties of optimal portfolios (Brennan, Schwarz and Lagnado (1997)). Numerical schemes based on PDEs, however, become increasingly difficult to implement when the number of underlying state variables increases. More recent contributions by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) have proposed an alternative resolution method based on martingale techniques. This approach yields a closed form solution for optimal consumption when markets are complete even when asset prices follow Ito processes with history-dependent coefficients. The optimal portfolio was derived by Ocone and Karatzas (1991) using the Clark-Ocone formula. This representation formula expresses the portfolio in terms of expectations of random variables which involve "abstract" Malliavin derivatives of the coefficients of the model, namely the interest rate (IR) and the market price of risk (MPR).

But while theoretical formulas for optimal portfolios are available in general contexts little is known about the structure and properties of the hedging components. Even if we restrict attention to diffusion models, realistic specifications with stochastic IR and MPR give rise to complex hedging terms which depend on multiple state variables and are often difficult to evaluate numerically. As a result attention has been devoted to (i) state variable specifications for which closed form solutions are available (Kim and Omberg (1996), Liu (1999), Wachter (1999)) or (ii) specifications which are computationally tractable based on dynamic programming techniques (Brennan, Schwarz and Lagnado (1997)), or (iii) discrete time models based on approximated Euler equations (Campbell and Viceira (1999)).

This paper provides three main contributions. First we exploit the diffusion nature of the opportunity set to provide explicit expressions for the Malliavin derivatives arising in the hedging components of the optimal portfolio. Hedging demands are expressed as conditional expectations of random variables which depend on the drift and variance of the relevant state variables. These formulas are valid for any structure of the underlying processes and of the utility function and reduce the computation of hedging demands to the computation of expectations, as in traditional option pricing. Our approach can therefore be seen as a translation of the dynamic asset allocation problem into an option pricing problem for which Monte Carlo methods, as summarized in Boyle, Broadie and Glasserman (1997), have long been successfully applied by practitioners. Furthermore, the formulas enable us to establish new theoretical results about the hedging behavior.

Second we derive an alternative representation of Malliavin derivatives of diffusion processes which simplifies their evaluation. Our formula relies on a variance-stabilizing transformation of the underlying process and eliminates stochastic integrals from their representation. Aside from its theoretical interest this new expression has interesting computational benefits. Indeed, the absence of stochastic integrals ensures the existence of an exact weak approximation scheme for the martingale part of the Malliavin derivatives and this improves the rate of convergence of approximations of Malliavin derivatives to their true values. The scheme also increases the speed of convergence of simulated trajectories of hedging terms and of any statistic (such as confidence intervals) of simulated hedging terms. Finally it may also help to reduce the second-order bias and therefore the size distortion of asymptotic confidence intervals of the Monte Carlo estimator of the hedging demands and portfolios given the realization of the state variables.

Third we provide new results on the economic properties of optimal portfolios. We examine bivariate and trivariate IR and MPR models in a setting with constant relative risk aversion. In our benchmark bivariate model the IR process is mean-reverting with square-root volatility (MRSR) and the MPR process is Gaussian with either mean-reversion (MRG) or with mean-reversion and interest rate dependence in the drift (MRGID). More elaborate trivariate models with stochastic dividend yield or volatility, and with multiple assets are also considered. In these settings we document the magnitude of the hedging terms and their behavior relative to the parameters of the model such as risk aversion, investment horizon or IR and MPR values. All our results are based on a portfolio formula which evolves from the Ocone-Karatzas representation. This modified formula which emphasizes the role of relative risk aversion and wealth sheds further light on the portfolio/hedging behavior. It can be viewed as a minor contribution of the paper.

Some of the lessons drawn from our simulations can be summarized as follows:

1. Our methodology involving the combination of Monte-Carlo simulation and our variance-stabilizing transformation produces very reasonable values for the shares of wealth invested in the stock. Unlike some earlier studies of optimal portfolios interior solutions are obtained and portfolio shares are stable in simulation exercises such as market timing experiments.

2. Hedging components are important for asset allocation purposes. For long horizons the adjustment to mean-variance demands can represent up to 80% of the stock demand. Hedging demands also exhibit low volatility and are therefore very stable over time.

3. Critical factors in optimal asset allocation are the risk aversion and the investment horizon of the investor. For instance, in our basic bivariate model, investors with short (long) horizons and whose risk aversion exceeds 1 want to reduce (increase) their stock demand relative to the logarithmic investor in order to hedge against MPR (IR) fluctuations. The effects documented in the paper rationalize the marketing of investment products tailored to different categories of investors classified according to those criteria.

4. Allocation rules are remarkably stable relative to the other parameters of the model. Variations of the order of 2 standard deviations around estimated parameter values have little impact on the magnitude of investment shares.

5. The global behavior of the optimal portfolio in the multiasset case parallels the behavior displayed with a single risky asset. Hedging terms exhibit strong patterns with respect to correlations when asset returns are highly correlated. Correlations between returns and among state variables emerge as additional factors driving the size of hedging demands.

The portfolio choice problem is stated next. Section 3 presents a closed-form solution and discusses its structure. Section 4 develops an alternative formula for Malliavin derivatives of diffusion processes. Numerical implementation is discussed in section 5. Our basic bivariate model with MRSR interest rate and MRG/MRGID market price of risk is analyzed in sections 6 and 7. Sections 8 and 9 provide trivariate extensions to stochastic dividends and stochastic, imperfectly correlated volatility. A multiasset model is analyzed in section 10. Proofs are in appendix A; appendix B extends the procedure to multivariate diffusions; appendix C contains results for the MRGID model; appendix D reports asymptotic properties of state variable estimators.

2 The portfolio choice problem.

We consider a portfolio choice problem in an economy with d state variables $Y_{jt}, j = 1, ..., d$, and d sources of Brownian uncertainty $W_{it}, i = 1, ..., d$.[1] State variables follow the vector diffusion process

$$dY_t = \mu^Y(t, Y_t)dt + \sigma^Y(t, Y_t)dW_t \tag{1}$$

where the coefficients satisfy appropriate Growth and Lipschitz conditions for the existence of a unique strong solution.[2] The investor allocates his wealth between d risky securities and one riskless asset (a money market account) with instantaneous riskless rate of return $r_t = r(t, Y_t)$. The security prices $S_i, i = 1, ..., d$, satisfy the stochastic differential equations

$$dS_{it} = S_{it}[(\mu_i(t, Y_t) - \delta_i(t, Y_t))dt + \sigma_i(t, Y_t)dW_t]; \quad 1 \leq i \leq d \tag{2}$$

where μ_i is the expected return, δ_i the dividend rate and σ_i the vector of volatility coefficients of security i. We assume that $r(t, Y_t), \mu_i(t, Y_t), \delta_i(t, Y_t)$ are integrable $(P - a.s.)$ and that $\sigma_i(t, Y_t)$ is square-integrable $(P - a.s.)$. Let σ denote the $d \times d$-dimensional volatility matrix whose rows are $\sigma_i, i = 1, ..., d$. Suppose that σ is nonsingular almost everywhere and that the market price of risk

$$\theta_t = \theta(t, Y_t) = \sigma(t, Y_t)^{-1}(\mu(t, Y_t) - r(t, Y_t)\mathbf{1}),$$

where $\mathbf{1}$ is the unit vector, is continuously differentiable and satisfies the Novikov condition $E \exp\left(\frac{1}{2}\int_0^T \theta_t'\theta_t dt\right) < \infty$. Under this condition the risk neutral measure is well defined and given by $dQ = \eta_T dP$ where

$$\eta_t = \exp\left[-\int_0^t \theta_t' dW_t - \frac{1}{2}\int_0^t \theta_t'\theta_t dt\right].$$

The state price density is $\xi_t \equiv B_t^{-1}\eta_t$ where $B_t \equiv \exp[\int_0^t r_s ds]$ is the date t-value of a dollar investment in the money market account. Relative state prices are written $\xi_{t,v} \equiv \xi_v/\xi_t$. Under Q the process $W_t^{\mathbf{Q}} = W_t + \int_0^t \theta_v dv$ is a Brownian motion.

Suppose that an investor seeks to maximize the expected utility of his terminal wealth by selecting a dynamic portfolio policy composed of the d risky assets and the riskless asset

$$\max_{\pi} U(X_T) \equiv E[u(T, X_T)] \quad s.t. \tag{3}$$

$$\begin{cases} dX_t = r_t X_t dt + \pi_t'[(\mu_t - r_t\mathbf{1})dt + \sigma_t dW_t], & X_0 = x \\ X_t \geq 0 \text{ for all } t \in [0, T]. \end{cases} \tag{4}$$

Here X_t represents the investor's wealth at date t, x is his initial wealth and π_t the amounts invested in the risky assets at date t. The nonnegativity constraint is a typical no-bankruptcy condition. The utility function is strictly increasing and concave with limiting values $\lim_{X\to\infty} \partial_2 u(T, x) = 0$ and $\lim_{X\to 0} \partial_2 u(T, x) = \infty$ for all $T < \infty$. (For any function $f(t, X)$ we write $\partial_i f$ for the first derivative relative to i, $i = 1, 2$ and $\partial_{ij} f$ the second derivative, $i, j = 1, 2$; when the second argument is a vector $\partial_2 f$ is the gradient and $\partial_{22} f$ the hessian of second derivatives).

[1] It is straightforward to consider $k \neq d$ state variables. To simplify notation, in particular for the expressions of the Malliavin derivatives, we assume that $k = d$.

[2] Note that the d state variables are joint solutions of the system (1), i.e. they influence each other. Remark 1 considers the special case of an autonomous system in which each state variable is determined independently.

3 The optimal portfolio: the investor's hedging behavior.

The portfolio choice problem described above can be resolved by using a martingale approach (Karatzas, Lehoczky and Shreve (1987), Cox-Huang (1989)) to identify optimal terminal wealth in explicit form and then applying the Clark-Ocone formula on the representation of Brownian functionals to obtain the financing portfolio. This approach was adopted by Ocone and Karatzas (1991) who provide formulas in the form of conditional expectations of random variables involving Malliavin derivatives. Due to the generality of their model in which asset prices follow Ito processes (with unspecified coefficients) these Malliavin derivatives are abstract quantities without an explicit structure. In this section we exploit the diffusion specification of the financial market to derive explicit expressions for the Malliavin derivatives and hence for the optimal portfolio.

3.1 The optimal portfolio policy.

Let $V(x)$ denote the value function in the optimization problem (3)-(4), $I(T,y)$ the inverse marginal utility, $\widehat{y}$ the marginal value of initial wealth and $\hat{X}$ the optimal wealth. Our first result identifies the general structure of the optimal portfolio and of its hedging components.

Theorem 1 *If $V(x) < \infty$ and $\xi_T I(T,\widehat{y}\xi_T) \in \mathbb{D}^{1,2}$ we have that*[3]

$$\begin{aligned}\hat{\pi}_t &= \hat{X}_t \frac{1}{R(t,\hat{X}_t)}(\sigma(t,Y_t)')^{-1}\theta(t,Y_t)c(t,Y_t) \\ &\quad + \hat{X}_t(\frac{1}{R(t,\hat{X}_t)} - 1)(\sigma(t,Y_t)')^{-1}a(t,Y_t) \\ &\quad + \hat{X}_t(\frac{1}{R(t,\hat{X}_t)} - 1)(\sigma(t,Y_t)')^{-1}b(t,Y_t)\end{aligned} \tag{5}$$

where $R(t,x) := \frac{-\partial_{22}u(t,x)x}{\partial_2 u(t,x)}$ denotes the Arrow-Pratt measure of relative risk aversion, and

$$a(t,Y_t)' \equiv \mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\left(\frac{1-1/R(T,\hat{X}_T)}{1-1/R(t,\hat{X}_t)}\right)\int_t^T \mathcal{D}_t r_s ds\right] \tag{6}$$

$$b(t,Y_t)' \equiv \mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\left(\frac{1-1/R(T,\hat{X}_T)}{1-1/R(t,\hat{X}_t)}\right)\int_t^T (dW_s^{\mathbf{Q}})'\mathcal{D}_t\theta_s\right] \tag{7}$$

$$c(t,Y_t) \equiv \mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\frac{R(t,\hat{X}_t)}{R(T,\hat{X}_T)}\right]. \tag{8}$$

In these expressions optimal wealth equals $\hat{X}_t = \mathbf{E}_t[\xi_{t,T}I(T,\widehat{y}\xi_T)]$. The Malliavin derivatives in (6)-(7) are given in explicit form by $\mathcal{D}_t\theta_s' = \partial_2\theta(s,Y_s)'\mathcal{D}_tY_s$ and $\mathcal{D}_tr_s = \partial_2 r(s,Y_s)\mathcal{D}_tY_s$ where

$$\mathcal{D}_tY_s = \sigma^Y(t,Y_t)\exp\left\{\int_t^s dL_v\right\}, \tag{9}$$

[3] $\mathbb{D}^{1,2}$ is the domain of the Malliavin derivative. See Nualart (1995) for exact definitions.

with the $d \times d$ random variable dL_v defined by[4]

$$dL_v \equiv \left[\partial_2 \mu^Y(v, Y_v) - \frac{1}{2} \sum_{j=1}^{d} \partial_2 \sigma^Y_{\cdot j}(v, Y_v) (\partial_2 \sigma^Y_{\cdot j}(v, Y_v))' \right] dv + \sum_{j=1}^{d} \partial_2 \sigma^Y_{\cdot j}(v, Y_v) dW_{jv} \qquad (10)$$

where $\sigma^Y_{\cdot j}$ denotes the j^{th} column of the matrix σ^Y.

Note that the first component of the optimal portfolio (5) is a mean-variance component while the two other components are intertemporal hedging terms (see Merton (1971)).[5] In this general formula the mean-variance term varies with optimal wealth since the coefficient of relative risk aversion is allowed to change with wealth. Hedging arises since the investor seeks insurance against fluctuations in the interest rate (second component of (5)) and in the market prices of risk (third component of (5)). That the second term is motivated by the desire to hedge interest rate risk is evidenced by the presence of the Malliavin derivative $\mathcal{D}_t r_s$ which captures the interest rate's sensitivity to the underlying risk factors, i.e. the Brownian motion processes W_i. In accordance we call this term an IR-hedge.[6] Similarly the third term is seen to emerge when the market prices of risk are sensitive to the W_i (i.e. when $\mathcal{D}_t \theta_s \neq 0$) and is called an MPR-hedge. When (r, θ) are constant or deterministic all these hedging terms are null since in the Malliavin derivatives $\mathcal{D}_t r_s$ and $\mathcal{D}_t \theta_s$, the partial derivatives $\partial_2 r(s, Y_s)$ and $\partial_2 \theta(s, Y_s)$ are zero.

Before discussing the behavior embedded in the hedging components it is also of interest to point out that formula (5) expresses the hedging components in explicit form: hedging demands are conditional expectations of random variables which depend entirely on the exogenous coefficients of the model and the utility function. The key to these explicit expressions is the derivation of closed-form solutions for the Malliavin derivatives $\mathcal{D}_t r_s$ and $\mathcal{D}_t \theta_s$ which are obtained due the diffusion structure of the uncertainty. As mentioned above these results complement Ocone and Karatzas (1991) who express the optimal portfolio in terms of abstract Malliavin derivatives.

3.2 The intertemporal hedging behavior.

Let us now focus on the hedging behavior of the investor. First, it should be noted that a myopic individual ($R(t, \widehat{X}_t) = 1$) does not hedge[7]. The signs of the hedging terms will otherwise depend on the signs of the conditional expectations $a(t, Y_t)$ and $b(t, Y_t)$. For example, when these are positive, an individual who is more (less) risk tolerant than the logarithmic investor will over- (under) invest in the risky assets. For the IR-hedge simple sufficient conditions ensure an unambiguous behavior.

[4]The exponential in (9) should be interpreted as the exponential of a matrix, i.e. (9) is short hand notation for the solution of $d\mathcal{D}_t Y_s = \left(dL_s + \frac{1}{2} d[L]_s\right) \mathcal{D}_t Y_s$ subject to the boundary condition $\mathcal{D}_t Y_t = \sigma^Y(t, Y_t)$, where $[L]$ is the quadratic variation process.

[5]The optimal portfolio formula extends easily to the case of intermediate consumption. It also extends to settings with infinite horizon provided that the Novikov condition is satisfied.

[6]Expression (9) shows that the Malliavin derivative, in a Markovian model, corresponds to the derivative of the stochastic flow of the SDE of state variables with respect to the initial position of the state variables (Colwell, Elliott and Kopp (1991)).

[7]When $R(t, \widehat{X}_t)$ and $R(T, \widehat{X}_T)$ tend to one, the ratio inside the conditional expectations (6)-(7) tends to one as seen by applying l'Hôpital's rule.

Proposition 2 *Fix $t \in [0,T]$. Suppose that the conditions*

(i) $(\sigma(t,Y_t)')^{-1}(\mathcal{D}_t r_s)' \leq 0$ for all $s \geq t$,(P-a.s)

(ii) $R(t,\hat{X}_t) \geq 1$ and $R(T,\hat{X}_T) \geq 1$ (P-a.s).

hold. Then, intertemporal hedging of interest rate risk raises the demand for stocks (i.e. the IR-hedge is nonnegative). If (i)-(ii) hold for all $t \in [0,T]$ the IR-hedge boosts the proportion of wealth invested in stocks at all times.

Conditions (i)-(ii) are very general. The first condition holds in a variety of special cases that are of interest for empirical or theoretical reasons. For instance it holds if state variables are autonomous (see remark 1 below) and

$$(\sigma(t,Y_t)')^{-1}\left(\partial_2 r(t,Y_t)\sigma^Y(t,Y_t)\right)' \leq 0. \tag{11}$$

In the single risky asset case this simply boils down to negative correlation between the interest rate and the risky asset price, which is empirically verified if the risky asset is interpreted as the SP500 index. Condition (11) also holds with multiple risky assets that are independent and negatively correlated with the interest rate. In all these cases the particular structure of the coefficients of the state variables processes (whether they are increasing, decreasing, convex or concave functions) does not matter for the sign of the hedging term: the only aspect of relevance is whether (11) is verified.

The second condition applies even to models in which relative risk aversion varies with optimal wealth. As long as an investor displays more risk aversion than a myopic investor at date t and for all possible realizations of optimal terminal wealth the condition will hold.

When we combine both conditions we obtain, for instance, the intuitive proposition that individuals that are more risk averse than the log investor ($R(t,\hat{X}_t) \geq 1, R(T,\hat{X}_T) \geq 1$) will increase their demand for the market portfolio of risky assets when the interest rate covaries negatively with the portfolio return (single risky asset model) in order to hedge interest rate risk.

We conclude this discussion with a description of hedging demands and Malliavin derivatives for the case of autonomous state variables.

Remark 1 *When the system of stochastic differential equations (1) is composed of d autonomous equations $dY_{it} = \mu_i^Y(t,Y_{it})dt + \sigma_i^Y(t,Y_{it})dW_t$ for $i = 1,...,d$, we can write*

$$\mathcal{D}_t Y_{is} = \sigma_i^Y(t,Y_{it}) \exp\left\{\int_t^s dL_v^i\right\} \tag{12}$$

$$dL_v^i \equiv [\partial_2\mu_i^Y(v,Y_{iv}) - \frac{1}{2}\partial_2\sigma_i^Y(v,Y_{iv})(\partial_2\sigma_i^Y(v,Y_{iv}))']dv + \partial_2\sigma_i^Y(v,Y_{iv})dW_v \tag{13}$$

In this instance the sign of the Malliavin derivative (12) is positive (negative) when $\sigma_i^Y(t,Y_{it})$ is positive (negative). The results discussed above follow immediately from this property.

3.3 Constant relative risk aversion.

Since our numerical results in later sections assume constant relative risk aversion we specialize the formulas of theorem 1 to that case.

Theorem 3 *(CRRA) When the utility function exhibits constant relative risk aversion R the optimal portfolio is*

$$\hat{\pi}_t = \hat{X}_t(\sigma(t,Y_t)')^{-1}\left[\frac{1}{R}\theta(t,Y_t) + (\frac{1}{R}-1)a(t,Y_t) + (\frac{1}{R}-1)b(t,Y_t)\right] \tag{14}$$

where

$$a(t,Y_t)' \equiv \mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\int_t^T \mathcal{D}_t r_s ds\right] = \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\int_t^T \mathcal{D}_t r_s ds\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \tag{15}$$

$$b(t,Y_t)' \equiv \mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\int_t^T \mathcal{D}_t \theta_s' dW_s^{\mathbf{Q}}\right] = \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\int_t^T (dW_s^{\mathbf{Q}})'\mathcal{D}_t\theta_s\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \tag{16}$$

and $\xi_{t,T} = \exp(-\int_t^T (r_v + \frac{1}{2}\theta_v^2)dv - \int_t^T \theta_v dW_v)$. *The Malliavin derivatives in a and b are given in theorem 1.*

In this formula two expressions are provided for the coefficients a, b in the hedging components. The first is simply the specialization of the previous result to the case under consideration. The second formula uses the relation between optimal wealth and state prices in order to express a, b in terms of the relative state price density $\xi_{t,T}$ between periods t and T. This formula clearly demonstrates that the functions a, b depend only on the state variables Y.

The formulas described in theorems 1 and 3 provide useful information about the qualitative behavior of the investor. In order to assess the magnitude of the various components, and hence their relevance for asset allocation purposes, it is nevertheless necessary to get quantitative estimates. Practical implementations require the computation of the conditional expectations appearing in the portfolio formulas. Clearly Monte Carlo simulation appears to be an appealing way to proceed. In the next section we pursue this avenue and suggest a further transformation which facilitates the computation of Malliavin derivatives and may also help in the estimation of the hedging demands.

4 An alternative formula for Malliavin derivatives of diffusions.

The key to our simplification is a change of variables which transforms a stochastic differential equation into an ordinary differential equation. In effect this (variance-stabilizing) transformation removes stochastic integrals from expressions such as $a(t, Y_t)$ and $b(t, Y_t)$. Changes of variables of this type are used by Doss (1977) to prove that an SDE can be solved pathwise, since it can be reduced to an ordinary differential equation.[8] Appendix A shows how Doss' arguments can be used to derive alternative expressions for Malliavin derivatives of solutions of one-dimensional SDEs. In this section we state the result and discuss its implications.

[8] This result also plays an important role in the approximation of solutions of SDEs (e.g. Talay and Pardoux (1985)). In this context it can be used to conclude that convergence of the underlying Wiener process implies the convergence of the solution of an SDE.

4.1 The main result.

Consider a process Y which satisfies the one-dimensional SDE

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t; Y_0 = y.$$

The Malliavin derivative of Y has the following alternative representation.

Proposition 4 *If the following conditions hold*[9]
(i) differentiability of drift: $\mu \in \mathcal{C}^1([0,T] \times \mathbb{R})$
(ii) differentiability of volatility: $\sigma \in \mathcal{C}^2([0,T] \times \mathbb{R})$
(iii) growth condition: $\mu(t,0)$ *and* $\sigma(t,0)$ *are bounded for all* $t \in [0,T]$,
then we have for $t \leq s$ *that*

$$\mathcal{D}_t Y_s = \sigma(s, Y_s) \exp\left[\int_t^s [\partial_2\mu - \frac{\mu\partial_2\sigma}{\sigma} - \frac{1}{2}(\partial_{22}\sigma)\sigma - \frac{\partial_1\sigma}{\sigma}](v, Y_v)dv\right] \tag{17}$$

Note that (17) expresses the Malliavin derivatives entirely in terms of Riemann-Stieltjes integrals of first and second derivatives of the coefficients of Y. Thus the stochastic integrals which appeared in the earlier formulas ((10) and (13)) have been entirely eliminated. Formula (17) is therefore easily computed using standard methods to approximate the Riemann integrals involved. With the variance stabilizing transformation the numerical calculation of the Malliavin derivatives is therefore of the same complexity as the numerical solution of an ODE. A second difference with the earlier expressions is that the leading term is the future volatility of the process at date s instead of the current volatility at t. This implies that this leading term cannot be factored out of conditional expectations at date t as was the case in (5) or (9). Randomness of the leading term however does not increase the computational difficulty involved in evaluating the Malliavin derivative.

With this numerically appealing expression for the Malliavin derivative we obtain a formula for the IR-hedge which does not involve stochastic integrals any longer. To achieve the same result for the MPR-hedge we introduce a second transformation which enables us to write the SPD and, as a consequence, also the MPR-hedge without any stochastic integral. We illustrate the idea in the univariate case.

Proposition 5 *Let* $d = 1$. *If the following conditions hold*
(i) differentiability of MPR: $\theta \in \mathcal{C}^2([0,T] \times \mathbb{R})$
(ii) differentiability of volatility: $\sigma \in \mathcal{C}^2([0,T] \times \mathbb{R})$
then the SPD can be written as

$$\xi_t = \exp\left[-\int_0^t [r + \frac{1}{2}\theta^2 - \frac{\theta}{\sigma}\mu - \partial_1\psi - \frac{1}{2}(\partial_2\theta\sigma - \theta\partial_2\sigma)](s, Y_s)ds - \psi(t, Y_t) + \psi(0, Y_0)\right] \tag{18}$$

where the function $\psi \in \mathcal{C}^1([0,T] \times \mathbb{R})$ *solves* $\partial_2\psi\sigma = \theta$. *Consequently, we obtain*

$$\int_t^T \mathcal{D}_t\theta_s[dW_s + \theta_s ds] = \frac{\theta}{\sigma}(T, Y_T)\mathcal{D}_t Y_T - \theta(t, Y_t) - \int_t^T (g_1(s, Y_s) + g_2(s, Y_s))\, \mathcal{D}_t Y_s ds \tag{19}$$

[9]The space $\mathcal{C}^i([0,T] \times \mathbb{R})$ is the space of i times continuously differentiable functions on the domain $[0,T] \times \mathbb{R}$.

where

$$g_1(s,Y_s) \equiv \left[\frac{\partial_1\theta}{\sigma} - \frac{\theta}{\sigma}\frac{\partial_1\sigma}{\sigma}\right](s,Y_s)$$

$$g_2(s,Y_s) \equiv \left[\frac{1}{2}(\partial_{22}\theta\sigma - \theta\partial_{22}\sigma) + \frac{\partial_2\theta}{\sigma}\mu - \mu\frac{\theta}{\sigma}\frac{\partial_2\sigma}{\sigma} + \frac{\theta}{\sigma}\partial_2\mu - \theta\partial_2\theta\right](s,Y_s)$$

4.2 A bivariate state variable example.

To illustrate the formulas above consider the model with CRRA of theorem 3 and suppose that the state variables are given by the pair (r,θ) which satisfies[10]

$$dr_t = \kappa_r(\overline{r} - r_t)dt + \sigma_r r_t^{\gamma_r} dW_t, \qquad r_0 \text{ given} \tag{20}$$

$$d\theta_t = \kappa_\theta(\overline{\theta} - \theta_t)dt + \sigma_\theta \theta_t^{\gamma_\theta} dW_t, \qquad \theta_0 \text{ given} \tag{21}$$

where $(\kappa_r, \overline{r}, \sigma_r, \gamma_r, \kappa_\theta, \overline{\theta}, \gamma_\theta)$ are nonnegative constants, $(\sigma_r, \sigma_\theta)$ are constants (possibly negative) and $(\gamma_r, \gamma_\theta) \in [0,1]$. The Brownian motion W is unidimensional. This model nests standard formulations as special cases. The class of interest rate processes (20) is used in another context in Chan, Karolyi, Longstaff and Sanders (1992). The class of models (21) for the MPR has not been explored in the literature yet. We also assume that the stock volatility is stochastic and equal to $\sigma(r_t,\theta_t)$. This financial market is then described by two state variables (r,θ).

The transition from the general model with state variables Y to the model (20)-(21) with state variables (r,θ) is immediate since the Malliavin derivative $\mathbf{D}_t\theta_v$ can now be computed directly from the process (21). In order to state the result define the process

$$h_{t,v}(\gamma,\kappa,\sigma,\overline{x};x) = -(1-\gamma)\int_t^v \left(\kappa(1+\overline{x}\frac{\gamma}{1-\gamma}\frac{1}{x_u}) - \frac{1}{2}\sigma^2\gamma(\frac{1}{x_u})^{2(1-\gamma)}\right)du$$

for a quadruple of constants $(\gamma,\kappa,\sigma,\overline{x})$ and some process x. Taking account of the specific structure (20)-(21) then leads to

Corollary 6 *In the financial market (20)-(21) the optimal portfolio for CRRA utility is given by (14)-(16) where*

$$\mathbf{D}_t r_v = r_v^{\gamma_r}\sigma_r \exp\left[h_{t,v}(\gamma_r,\kappa_r,\sigma_r,\overline{r};r)\right]$$
$$\mathbf{D}_t\theta_v = \theta_v^{\gamma_\theta}\sigma_\theta \exp\left[h_{t,v}(\gamma_\theta,\kappa_\theta,\sigma_\theta,\overline{\theta};\theta)\right].$$

and

$$\xi_{t,T} = \exp\left[-\int_t^T r_s ds - \frac{1}{2}\int_t^T [\theta_s^2 + \sigma_\theta(1-\gamma_\theta)\theta_s^{\gamma_\theta} + 2\frac{\kappa_\theta}{\sigma_\theta}\theta_s^{1-\gamma_\theta}(\bar{\theta}-\theta_s)]ds - \phi(\theta_T) + \phi(\theta_t)\right]$$

with $\phi(x) = \frac{1}{\sigma_\theta(2-\gamma_\theta)}x^{2-\gamma_\theta}$. *The stochastic integral in the MPR-hedge (16) can also be written*

$$\int_t^T \mathcal{D}_t\theta_s[dW_s + \theta_s ds] = \theta_T \exp\left[h_{t,v}(\gamma_\theta,\kappa_\theta,\sigma_\theta,\overline{\theta};\theta)\right] - \theta_t - \int_t^T g_2(s,Y_s)\mathcal{D}_t\theta_s ds$$

with $g_2(s,Y_s) = \frac{1}{2}\sigma_\theta\gamma_\theta(1-\gamma_\theta)\theta_s^{\gamma_\theta-1} + \frac{\kappa_\theta}{\sigma_\theta}\left((1-\gamma_\theta)\bar{\theta}\theta_s^{-\gamma_\theta} - (2-\gamma_\theta)\theta_s^{1-\gamma_\theta}\right)$.

[10] This is equivalent to a model with two state variables $Y=(Y_1,Y_2)$ in which the equations ($r_t = r(t,Y_1), \theta_t = \theta$ (t,Y_2)) can be inverted and the state variables can be expressed as $Y_t = (f_1(r_t), f_2(\theta_t))$.

When $\gamma_r, \gamma_\theta = 0$ (Ornstein-Uhlenbeck IR and MPR processes) the formulas above simplify even further.

Corollary 7 *Suppose that $u \in CRRA$. When the interest rate and the market price of risk follow Ornstein-Uhlenbeck processes ($\gamma_r, \gamma_\theta = 0$) the optimal portfolio is given by (14)-(16) where*

$$a(t, r_t) = \frac{\sigma_r}{\kappa_r}\left(1 - \exp[-\kappa_r(T-t)]\right) \tag{22}$$

$$b(t, \theta_t) = \sigma_\theta \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\left(\int_t^T e^{-\kappa_\theta(v-t)} W_v^{\mathbf{Q}}\right)\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}. \tag{23}$$

The analytical expression for the IR-hedge in (22) clarifies the influence of the parameters of the interest rate process and the time horizon. Given the expressions provided in corollary 6 the MPR-hedge in (23) also has an analytical expression, albeit more complicated than the interest rate expression.

5 Numerical implementation.

It follows from the results in the prior sections that the problem of finding the optimal portfolio for power utility function reduces to the identification of the functions a and b. When closed form expressions for a, b are not available, one must resort to a numerical scheme to estimate their values. As explained before Monte-Carlo simulation is naturally suggested by the structure of the problem and this is the approach that we adopt. In our context the simulation procedure involves two sources of error. First, since the joint law of the SPD and the Malliavin derivatives involved in the IR- and MPR- hedge terms are generally unknown we have to use a discretization scheme to approximate these random variables. It is well-known that such a discretization procedure produces a bias. Second, since we do not know how to calculate analytically the conditional expectation we rely on a law of large numbers to evaluate the expectation using independent replications of the random variables which enter in the hedging terms. This Monte-Carlo estimation of the conditional expectation also introduces an error.

In the discussion which follows we shall restrict attention to the model with CRRA utility. In this context, we estimate the functions a and b with M replications and N discretization points for the investment horizon by

$$a^{N,M}(T-t, y) = \frac{\sum_{i=1}^{M}(\xi_{T-t}^{N,i}(Y^N(y)))^{1-1/R} H_{T-t}^{a,N,i}(Y^N(y))}{\sum_{i=1}^{M}(\xi_{T-t}^{N,i}(Y^N(y)))^{1-1/R}}. \tag{24}$$

$$b^{N,M}(T-t, y) = \frac{\sum_{i=1}^{M}(\xi_{T-t}^{N,i}(Y^N(x))^{1-1/R} H_{T-t}^{b,N,i}(Y^N(y))}{\sum_{i=1}^{M}(\xi_{T-t}^{N,i}(Y^N(x))^{1-1/R}} \tag{25}$$

where $H_{T-t}^{a,N,i}(Y^N(y))$ and $H_{T-t}^{b,N,i}(Y^N(y))$ are estimators of $\int_t^T \mathcal{D}_t r_s ds$ and $\int_t^T \mathcal{D}_t \theta_s [dW_s + \theta_s ds]$ respectively. In these expressions we have emphasized that these quantities are functionals of the approximated state variables starting at $Y_0^N = y$.

Since the state variables, the SPD and the Malliavin derivatives of the state variables are all given as solutions of SDEs, the simplest approach for estimation is to use the Euler scheme. It has been shown by Kurtz and Protter (1991) that the order of convergence for this scheme is $1/\sqrt{N}$[11] due to the discretization error in the martingale parts of the SDEs. In Detemple, Garcia and Rindisbacher (DGR) (2000) we show that our variance-stabilizing transformation eliminates discretization errors in the martingale part of the SDE of the transformed state variables and therefore attains a rate of convergence of order $1/N$, which is the same convergence rate as for the Euler scheme of an ODE (see Appendix D).[12] In order to illustrate this difference in performance between the two schemes we estimate the respective absolute computational errors in the Malliavin derivative of the IR for different discretizations N of the time interval $[0, T]$. We estimate errors by the strong criterion

$$\widehat{\epsilon}^{N,M} = \hat{E}^M \left| \mathcal{D}_0^N r_T - \mathcal{D}_0 r_T \right| = \frac{1}{M} \sum_{i=1}^{M} \left| \mathcal{D}_0^{N,i} r_T - \mathcal{D}_0^i r_T \right|$$

where $\mathcal{D}_0 r_T$ denotes the true value of the derivative and $\mathcal{D}_0^N r_T$ its approximation based on N discretization points using M independent replications. We also compute the respective errors with and without transformation for the state variable r_T. Since the computation of this statistic requires the true distribution of the Malliavin derivative we assume that the IR follows the MRSR process ((20) with $\gamma_r = \frac{1}{2}$) with parameters $T = 1,\ \kappa_r = .004, \overline{r} = .06,\ \sigma_r = .0309839,\ r_0 = .06$.[13] To compute the expectation above we take 20 batches of $1,000$ simulations each. For each batch an absolute error is estimated. Estimated absolute errors are then averaged over the batches. Table 1 below reports the results. Columns 2 and 4 show that the speed of convergence of the Euler scheme is roughly of order $1/\sqrt{N}$. Columns 3 and 5 illustrate the increase in the speed of convergence to $1/N$ when the scheme with transformation is used.

[Insert Table 1 here].

However, to compute hedging terms, we evaluate expectations of functionals of the state variables. In DGR (2000), we show that the increased speed of convergence obtained with the transformation for the numerical solution of SDEs of state variables fails to increase the speed of convergence of expectations of functionals of the state variables. This extends a result of Talay and Tubaro (1991). They have shown that, for the Euler scheme, $\mathbf{E}[f(Y_T^N) - f(Y_T)]$ is of order $\frac{1}{N}$ for functions f and diffusion coefficients μ and σ satisfying certain boundedness assumptions. Even though our problem is more complicated since we are not evaluating a function of a terminal

[11] That is $\sqrt{N}(Y^N - Y) \Rightarrow U^Y$ where convergence is in the weak sense and the error process in non-trivial $U^Y (\neq 0)$.

[12] That is $N(G(Z^N) - Y) \Rightarrow V^Y$, where $G(Z^N)$ is an estimator of the state variables Y and Z^N is obtained using the Euler scheme for the transformed state variables.

[13] Since $\sigma_r = 2\sqrt{\kappa_r \overline{r}}$ the interest rate r is the square of an Ornstein-Uhlenbeck process $Y_t = \sqrt{r_t}$. The true value can then be calculated by using the exact simulation of the transformed state variables

$$Y_{t+\Delta} = Y_t e^{\alpha\Delta} + \beta(\sigma_r e^{\alpha\Delta}\sqrt{\Delta}(W_{t+\Delta} - W_t) + \sqrt{|s_{22}|}Z)$$

where Z is a Gaussian variate independent of W, $\alpha = -\frac{\kappa_r}{2}$, $\beta = \sigma_r/2, \Delta = \frac{T}{N}$ and $s_{22} = e^{2\alpha\Delta}(\frac{1}{2\alpha} - \Delta) + 2(\Delta - \frac{1}{\alpha}) + \frac{3}{2\alpha}$. This choice of coefficients ensures that Y has the correct variance and covariance with the increment of the Brownian motion $W_{t+\Delta} - W_t$.

point of a numerical solution to a SDE but a functional which depends on the whole trajectory of the solution, the same result holds. Nevertheless, as we will discuss next, the transformation may still be useful as it may reduce the asymptotic second order bias.

Denoting estimators without our transformation (by direct application of the Euler scheme) by $\tilde{\cdot}$ and estimators with the transformation by $\hat{\cdot}$, we obtain under certain integrability conditions (see DGR (2000) for details) for the $a(\cdot)$ function

$$\sqrt{M}(\tilde{a}^{N,M}(T-t,y) - a(T-t,y)) \Rightarrow \epsilon \tilde{K}^{a(y)}_{T-t} + M^{a(y)}_{T-t} \tag{26}$$

$$\sqrt{M}(\hat{a}^{N,M}(T-t,y) - a(T-t,y)) \Rightarrow \epsilon \hat{K}^{a(y)}_{T-t} + M^{a(y)}_{T-t} \tag{27}$$

where $\epsilon = \frac{\sqrt{M}}{N}$ is fixed for all M, N. Corresponding limit laws are also obtained for the $b(\cdot)$ function. The vector processes $\tilde{K}^{a(y)}$ and $\hat{K}^{a(y)}_{T-t}$ (resp. $\tilde{K}^{b(y)}$ and $\hat{K}^{b(y)}_{T-t}$) are deterministic whereas $M^{a(y)}$ (resp. $M^{b(y)}$) is a Gaussian martingale. As indicated both types of processes depend on the initial position of the state variables, y.

In these expressions, the deterministic processes $K^{\cdot}$ correspond to the discretization error resulting from the approximation scheme and therefore depend on the approximation method used. Ideally, they should be zero. Using our transformation this is indeed the case if the underlying state variables are given by an invertible, twice continuously differentiable function of lognormal processes. It happens in this case that the approximation using the transformation is also exact for the part of the SDE involving Riemann integrals. But in general $\hat{K}^{\cdot}$ will be different from zero. Therefore, although the estimators are consistent, a smaller $\hat{K}^{\cdot}$ reduces the second order bias. If in the construction of confidence intervals we do not correct for this second order bias the size distortion[14] will be smaller with the transformation whenever $\hat{K}^{\cdot} < \tilde{K}^{\cdot}$. Consequently, a reduced second order bias will also improve the validity of statistical tests based on the law of $M^{\cdot}$ only. Furthermore, a small second order bias is potentially important for a good performance of the estimators given a finite number of replications and discretization points.

The processes $M^{\cdot}$ are for both approximation methods the same. They result from the Monte Carlo estimation of the conditional expectation and would not vanish even if we could sample from the true joint law of $H^{\cdot}$ and the SPD ξ. The expressions for both processes K and M are obtained in explicit form and described in detail in DGR (2000) and can therefore be used to implement error corrections and variance reductions.

All the results discussed above are conditional on the knowledge of the state variables at a given moment in time. If we are only interested at point estimators of the optimal composition of our portfolio given a certain state the estimators of a and b are all we have to calculate. But for other purposes, such as risk management, we may well be interested in testing a given portfolio strategy against a specific benchmark. Since this type of exercise requires the probabilistic structure of the optimal portfolio strategy, we need the distribution of conditional estimators of the mean-variance component, the IR-hedge and the MPR-hedge. Since we cannot sample from the true law of the state variables it follows that we have to rely on an approximation of their dynamic evolution described by the SDE. As we show in DGR (2000) the conditional estimators converge weakly with order $\frac{1}{N}$ with transformation and order $\frac{1}{\sqrt{N}}$ without. The limit laws of these conditional

[14] Size distortion refers to the fact that the actual coverage probability is different from the prescribed level.

estimators are non-Gaussian [15] but known and therefore can be used to construct asymptotically valid confidence intervals or statistical tests.

6 Calibration of the model.

In order to examine the economic properties of optimal portfolios we need to specify and calibrate our model of the financial market. We will focus on the class of bivariate processes for (r, θ) described in the section above. Specifically we estimate the following IR-MPR model

$$dr_t = \kappa_r(\overline{r} - r_t)dt - \sigma_r r_t^{1/2} dW_t, \qquad r_0 \text{ given} \tag{28}$$

$$d\theta_t = \kappa_\theta(\overline{\theta} - \theta_t)dt + \sigma_\theta dW_t, \qquad \theta_0 \text{ given} \tag{29}$$

where $(\kappa_r, \overline{r}, \sigma_r, \gamma_r, \kappa_\theta, \overline{\theta}, \sigma_\theta, \gamma_\theta)$ are constants.

We assume that the approximate discrete-time process is the true time-series model.[16] The econometric procedure described in this section is based on the maximization of the loglikelihood of the following discrete-time model:

$$r_{t_{n+1}}^{(h)} = r_{t_n}^{(h)} + \kappa_r(\overline{r}_h - r_{t_n}^{(h)}) + \sigma_{r,h}\sqrt{r_{t_n}^{(h)}}\varepsilon, \qquad r_0 \text{ given} \tag{30}$$

$$\theta_{t_{n+1}} = \theta_{t_n} + \kappa_\theta(\overline{\theta} - \theta_{t_n}) + \sigma_\theta v, \qquad \theta_0 \text{ given.} \tag{31}$$

where $\overline{r}_h = \overline{r}h$ and $\sigma_{r,h} = \sigma_r\sqrt{h}$ and $\{t_n : n = 0, ..., N\}$ is a partition of $[0, T]$. In our estimations, we consider a monthly frequency with $h = \frac{1}{12}$.

Since the MPR, $\theta_t = \sigma_t^{-1}(\mu_t - r_t)$, is unobservable it must be filtered from the data. We take two approaches.[17] First we assume that the stock volatility σ is constant. In other words, we estimate the MPR from the conditional mean μ_t of the stock return series (taken as the SP500 index), assuming a simple AR(1) process for the conditional mean. The estimation period is January 1965-June 1996.

In the continuous-time model the same Brownian motion applies to r and θ, but with a perfect negative correlation. We therefore produce two sets of estimates, one with the correlation coefficient between ε_{t+1} and v_{t+1} left unconstrained, another one with a negative correlation of -0.9.[18] The results are presented in Tables 2 and 3 respectively. The estimates obtained for the parameters of the interest rate CIR process are comparable to the values obtained by Broze, Scaillet and Zakoïan (1995) and Chan et al. (1992). The process slowly reverts to an annualized mean of about 6% with a yearly volatility of about 1.76% for the unconstrained model

[15] The reader is referred to DGR (2000) for the exact expressions.

[16] Estimating the parameters of a continuous-time diffusion model based on a discrete-time approximation of the likelihood function leads to a discretization bias (Lo (1988)). However, for the monthly estimation of interest rate processes, Broze, Scaillet and Zakoian (1995) use an indirect estimation to correct for the bias and find that the bias is small for the mean-reversion κ_r, the mean $\overline{r}$ and the variance σ_r. We therefore follow the simpler approach to calibrate the parameters. We also investigate the sensitivity of the results to changes in the parameters.

[17] This filtering approach is in the spirit of Nelson and Foster (1994), although we do not claim any optimality property for the GARCH(1,1) process we use.

[18] Since at a correlation of -1, the variance-covariance matrix would be singular, we chose the closest approximation that did not create numerical problems.

and around 3.6% for the constrained estimate. The estimation results for the MPR Orstein-Uhlenbeck process show that the market price of risk reverts rather quickly to its mean. The mean is about 8%, which is low compared with the standard estimates of the market price of risk. The MPR volatility is about ten times the volatility of the interest rate process in both the unconstrained and constrained estimations; almost perfect negative correlation between the interest rate and the MPR forces upwards the volatilities of the two processes by a factor of two. Given the low value of the MPR, we also investigate a specification where the interest rate enters in the drift of the market price of risk, since excess returns are known to be predictable by the interest rate. Equation (32) replaces (31)

$$\theta_{t_{n+1}} = \theta_{t_n} + \kappa_\theta(\overline{\theta} - \theta_{t_n}) + \delta r_{t_n}^{(h)} + \sigma_\theta v, \qquad \theta_0 \text{ given.} \tag{32}$$

The estimation results, reported in Table 4, are quite similar to the previous specification, except for the mean level of the MPR, which is more in line with the usual estimate of 0.3. The expected negative coefficient of the interest rate δ comes out quite significantly different from zero. As we will see, this specification will only change the absolute magnitude of the stock position and the hedging terms, but not the relative importance of the later with respect to the former.

To assess the robustness of the results obtained with a constant σ,we use a GARCH (1,1) model for the stock returns to construct the series for the market price of risk θ_t. We keep as before an AR(1) specification for the conditional mean of the stock returns. The results are reported in Tables 5 and 6, where as before we estimate two versions of the model, with correlation coefficient $\rho_{r\theta}$ left unconstrained (Table 5) and constrained to a value of -0.9 (Table 6). The most notable differences are a moderate increase (decrease) in the interest rate (MPR) speed of mean-reversion by about 10%, an increase in the long run mean of the MPR by about 5% and a decrease in the MPR volatility by about 7%. The estimates obtained for the other parameters are roughly the same as before. Overall these differences should not exert much influence on the magnitude of the hedging terms and will not be considered in our numerical computation of optimal portfolios.

7 Economic properties of optimal portfolios.

We now implement our numerical procedure for the model with (i) constant relative risk aversion, (ii) a single risky stock with constant volatility, (iii) an MRSR (mean reverting - square root) process for the interest rate and (iv) a MRG (mean-reverting Gaussian) process or a MRGID (mean-reverting with interest rate dependence in the drift) process for the MPR**.** The uncertainty is thus captured by a bivariate system of state variables (r, θ). For this specification of preferences and uncertainty we recall that the stock demand is

$$\hat{\pi}_t = \hat{X}_t \frac{1}{R}\sigma^{-1}\theta_t + \hat{X}_t\left(\frac{1}{R} - 1\right)\sigma^{-1}a(t, r_t) + \hat{X}_t\left(\frac{1}{R} - 1\right)\sigma^{-1}b(t, \theta_t) \tag{33}$$

$$a(t, r_t) := \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\int_t^T \mathcal{D}_t r_s ds\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}, \tag{34}$$

$$b(t, \theta_t) := \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\int_t^T \mathcal{D}_t\theta_s dW_s^{\mathbf{Q}}\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \tag{35}$$

where $\mathcal{D}_t r_s, \mathcal{D}_t \theta_s$ and $\xi_{t,T}$ are provided in corollary 6. For the MRGID model see Appendix C.

Parameter values are set at their estimated values reported in Tables 3 and 4 and at values equal or close to the means for r_0 and θ_0; the volatility of the stock is set at its historical average 0.2. Specifically, in the first model (Table 3), we take $\kappa_r = .0824$, $\overline{r} = .0050 \times 12$, $\gamma_r = .5$, $\sigma_r = .01050 \times \sqrt{12}$ (recall that there is a minus sign in front of σ_r in (28)), $r_0 = .0050 \times 12$, $\sigma = .20$, $\kappa_\theta = .6950$, $\overline{\theta} = .0871$, $\gamma_\theta = 0$, $\sigma_\theta = .21$, $\theta_0 = .10$. In the second model, we take the following values: $\kappa_r = .0005$, $\overline{r} = .0050 \times 12$, $\gamma_r = .5$, $\sigma_r = .01050 \times \sqrt{12}$, $r_0 = .0050 \times 12$, $\sigma = .20$, $\kappa_\theta = .7771$, $\overline{\theta} = .2675$, $\gamma_\theta = 0$, $\sigma_\theta = .205, \delta = -26.29/12$, $\theta_0 = .30$. Simulations are carried out using daily increments and $5,000$ paths with variance reduction by antithetic variables method ($M = 5,000$, $h = 1/365$). Since the results are very similar, except for the difference in the absolute magnitude of the hedging terms as mentioned earlier, we only report in Table 7 summary results for the first model and provide a full-fledged analysis with graphs for the second model.[19]

7.1 Optimal portfolios and hedging components.

Figures 1-3 illustrate the behavior of the optimal portfolio and the hedging components relative to the risk aversion coefficient and the investment horizon. Risk aversion varies from .5 to 5; the investment horizon from 1 year to 5 years. As expected the fraction of wealth invested in the stock decreases as risk aversion increases and increases as the horizon increases. The hedges, however, display strikingly different behavior. The MPR-hedge displays mildly humped decreasing-increasing behavior relative to risk aversion and appears to decrease relative to horizon. The IR-hedge increases relative to both variables. As noted before the signs of the hedges change depending on whether risk aversion exceeds or falls short of 1. This illustrates the standard knife-edge behavior of (myopic) logarithmic utility. For investors that are more risk averse than the Bernoulli investor the negative values of the MPR-hedge stem from the positive correlation between the stock return and the MPR. Such an investor tries to hedge the additional risk away by reducing his/her stock demand. Similarly the IR-hedge tends to boost stock demand since it covaries negatively with the stock return. Note also that the combination of the two hedges is negative for short investment horizons (less than 4 years in the numerical example) and positive for longer holding periods. Thus, hedging behavior reduces (increases) the stock investment for short run (long run) horizons relative to a pure mean-variance investor. In fact, the increase in stock holdings increases with longer investment horizons.

[Insert figures 1-3 here]

Figures 4-6 display the behavior relative to the levels of the IR and the MPR r_0, θ_0 for risk aversion $R = 2$ and investment horizon $T - t = 1$. Again the fraction invested in the stock varies considerably over the range of initial values investigated, from over 90% of wealth to nearly 25%. The hedge components' ranges are much narrower: while the IR-hedge varies between about 1.8% and 2.6%, the MPR-hedge lies between -1.8% and about -6%.

Second note that the fraction invested in the stock is an increasing function of the MPR and is almost insensitive to the interest rate. As θ_0 increases the IR-hedge stays flat (figure 5) while

[19] Of course the expressions for the Malliavin derivatives with respect to the interest rate and the MPR as well as the state price density change compared to our previous expressions in section 5. The new expressions are given in Appendix C.

the MPR-hedge becomes more negative (figure 6). These effects, however, are of second order relative to the increase in the mean-variance component of the stock demand. When r_0 increases the IR-hedge increases moderately and becomes more positive (see figure 6): it tends to increase stock demand. The MPR-hedge also increases but even more moderately. Combining these two effects produces a mildly increasing total stock demand. For typical values of the MPR (between .20 and .40) the sum of the hedging terms is negative and tends to reduce the overall demand for the stock.

[Insert figures 4-6 here].

7.2 Market timing strategies.

In order to assess the importance and stability over time of our hedging demand estimates we perform two market timing experiments. The first consists in drawing trajectories of the underlying state variable processes r, θ and computing the portfolio and hedging demands along these trajectories. The second experiment simulates the optimal portfolio for very long horizons and using actual market data.

Results for the first experiment are reported in figures 7-10. A typical trajectory of the pair (r, θ) is drawn in figures 7 and 8. The interest rate is seen to vary between 3.9% and 5.4%; the MPR takes values between $-.08$ and .30. Figure 9 illustrates the stock demand behavior for an investor with risk aversion of 4 and a fixed horizon of 5 years. For the trajectory drawn the proportion invested in the stock evolves between -3% and 40%. Close inspection of the graph, however, shows that changes superior to 30% in the portfolio share are usually spread over periods of 6 month or more. There are also long stretches of time, of duration larger than a year, over which the stock share varies within at 10% interval.

Figure 10 which shows the respective contributions of the IR-hedge, the MPR-hedge and the sum of the two hedges sheds further light on this issue. First note that the IR-hedge is remarkably stable over time. It experiences very small fluctuations and decreases slowly toward zero due to the maturity effect of the fixed horizon. It also remains positive throughout the period. The MPR-hedge is negative and exhibits stronger volatility, which is not surprising since it is sensitive to the MPR level which is more volatile. Within intervals of a year though the fluctuations rarely exceed 5%. Again a trend toward zero is observed due to the fixed investment horizon. Both hedges work in opposite direction and partly offset each other. The net hedging correction is of the order of $5\% - 10\%$ at the beginning of the investment horizon, thus boosting the stock demand. It then slowly converges toward zero taking negative values along the way, thus reducing stock demand, in the last couple of years of the period. The net hedging correction inherits the stability of its two components: its fluctuations rarely exceeds 5% over periods of a year or longer. Over the whole 5 year period the hedging correction varies between -3% and 10%.

Although not reported in the paper similar properties are recorded when the analysis is performed for rolling horizons of 2 years and 5 years (though hedging terms do not converge to zero in that case) and for risk aversions in the range $2 - 4$.

We conclude from this (representative) experiment that hedging components are remarkably stable over time in the sense that they exhibit low volatility. The variation in the total stock demand which is observed in figure 9 stems primarily from the variation of its mean variance component.

[Insert figures 7-10 here]

Our second experiment examines the actual behavior, based on market data, of the portfolio over time for an investor with long horizon of about 30 years at the beginning of the period. Hedging demands and portfolio positions are computed using our model along the realized trajectory of the IR and the MPR in the last 31.5 years (our estimation sample). Based on these data, we compute each month of the sample the optimal share of the stock in the portfolio with and without hedging for an investor with a relative risk aversion of 4 (computations are performed using 25,000 replications and variance reduction, i. e. 50,000 replications). As figure 11 shows, intertemporal hedging will increase the optimal share to a reasonable level of about 60% at the beginning of the investment horizon to roughly 10% at the end, with an average holding of 44%. This is in sharp contrast with the myopic mean-variance optimal share which varies substantially around an average level of about 10%. Note also that the hedging investor will short the stock by 15% only once during the investment period (during the 1987 crash) and only because the triggering event happened shortly (10 years) before the end of the investment horizon. The observed increase in stock holdings comes mainly from the positive IR-hedge. From this realistic situation we then conclude that intertemporal hedging has a fundamental impact when the investment horizon is long. As in the previous experiment it tends to stabilize the overall stock demand.

[Insert figure 11 here]

8 Stochastic dividends (trivariate model).

Suppose now that the dividend-price ratio (DPR), denoted by p, is a relevant stochastic factor which influences the evolution of the market price of risk. The following trivariate process for (r, θ, p) generalizes the MRGID model by incorporating such an effect

$$dr_t = \kappa_r(\overline{r} - r_t)dt - \sigma_r r_t^{1/2} dW_t, \qquad r_0 \text{ given} \tag{36}$$

$$d\theta_t = [\kappa_\theta(\overline{\theta} - \theta_t) + \delta_r r_t + \delta_p p_t]dt + \sigma_\theta dW_t, \qquad \theta_0 \text{ given} \tag{37}$$

$$dp_t = \kappa_p(\overline{p} - p_t)dt - \sigma_p p_t^{1/2} dW_t, \qquad r_0 \text{ given.} \tag{38}$$

In this specification the DPR follows a mean-reverting square root process and has a linear effect on the drift of the MPR.

The model is estimated as previously: we maximize the loglikelihood of the discretized model using for the MPR the filtered series based on an AR(1) specification and a constant stock volatility. For the sake of brevity, we just report the estimated values of the parameters. These are $\kappa_r = 0.06977$, $\overline{r} = 0.005 \times 12$, $\kappa_\theta = 0.9088$, $\overline{\theta} = 0.1685$, $\delta_r = -23.90/12$, $\delta_p = 17.63/12$, $\kappa_p = 0.0344$, $\overline{p} = 0.003 \times 12$, $\sigma_r = 0.01227\sqrt{12}$, $\sigma_\theta = 0.16127$, $\sigma_p = 0.004578\sqrt{12}$. It should, however, be noted that these estimates, in particular those corresponding to the impact of the IR and the DPR on the drift of the MPR, are statistically different from zero. Other parameters are also seen to be close to the values obtained for the model with two state variables only.

Table 8 shows that optimal behavior changes when stochastic dividends are accounted for. The most notable feature is the reversal in the sign of the MPR-hedge. Inspection of the trivariate process reveals the root of this behavior. Recall that the estimated model displays positive impact of the dividend-price ratio on the drift of the MPR ($\delta_p = 17.63/12$) and negative correlation between stock returns and the dividend-price ratio ($-\sigma\sigma_p = -0.2 \times 0.004578\sqrt{12}$). Under these conditions hedging MPR-risk will involve two components. The first results from the positive association between stock returns and innovations in the MPR. This hedge against direct MPR-risk is negative, as in the earlier models. The second is the consequence of the indirect negative association between the drift of the MPR and innovations in the dividend-price ratio. This hedge, against indirect MPR-risk, is positive. Evidently, the two hedging motives work in opposite direction. As illustrated in the table (see also figure 12) the second effect dominates in the context of our estimated model and results in a positive overall MPR-hedge when risk aversion exceeds unity.

9 Stochastic volatility with imperfect correlation (trivariate model).

Consider now the trivariate state variable model (r, θ, σ) described by

$$dr_t = \kappa_r(\overline{r} - r_t)dt - \sigma_r r_t^{1/2} dW_{1t} \tag{39}$$

$$d\theta_t = [\kappa_\theta(\overline{\theta} - \theta_t) + \delta r_t]dt + \sigma_\theta dW_{1t} \tag{40}$$

$$d\sigma_t = [\kappa_\sigma(\overline{\sigma} - \sigma_t) + \sigma_t\{\delta_{1\theta}\theta_t \mathbf{1}_{\{\theta \geq 0\}} + \delta_{2\theta}\theta_t \mathbf{1}_{\{\theta < 0\}}\}]dt + \sigma_t^{1/2}[\lambda_1 dW_{1t} + \lambda_2 dW_{2t}] \tag{41}$$

where $(r_0, \theta_0, \sigma_0)$ are given, the coefficients $(\kappa_r, \overline{r}, \sigma_r, \kappa_\theta, \overline{\theta}, \delta, \sigma_\theta, \kappa_\sigma, \overline{\sigma}, \delta_{1\theta}, \delta_{2\theta}, \lambda_1, \lambda_2)$ are all constant and W_1, W_2 are independent Brownian motion processes.

The model (39)-(41) contains several innovations relative to the prior MRSR-MRGID model. The most important feature is that volatility is now stochastic. Furthermore, the volatility process is imperfectly correlated with the interest rate and the MPR processes. As a result our basic model is one with (apparently) incomplete markets. The drift of the volatility process also permits an asymmetric dependence on the MPR process, conditioned on positive or negative realizations of the MPR. This structure seeks to capture the notion that volatility is high when the magnitude (absolute value) of the MPR is large. As in the MRGID model the MPR process also involves an interaction in the drift with the rate of interest.

Even though this trivariate model (39)-(41) is driven by two underlying Brownian motions, and hence appears to have incomplete markets since there are only two assets, the portfolio formulas of the previous sections are still valid. The intuition for this seemingly surprising result is that the state price density ξ depends only on (r, θ) which are independent of the risk W_2. Since the investor's marginal utility is proportional to the state price density at the optimum it follows that optimal terminal wealth is independent of W_2. The portfolio that finances optimal wealth, in turn, will be independent of this idiosyncratic volatility risk. It follows that the individual valuation of the risk W_2 is null at the optimum.

Assuming CRRA preferences gives the optimal stock demand

$$\hat{\pi}_t = \hat{X}_t \frac{1}{R} \sigma_t^{-1} \theta_t + \hat{X}_t \left(\frac{1}{R} - 1\right) \sigma_t^{-1} a(t, r_t, \theta_t) + \hat{X}_t \left(\frac{1}{R} - 1\right) \sigma_t^{-1} b(t, r_t, \theta_t) \tag{42}$$

$$a(t, r_t, \theta_t) \equiv \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R} \int_t^T \mathcal{D}_t r_s ds\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}, \tag{43}$$

$$b(t, r_t, \theta_t) \equiv \frac{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R} \int_t^T \mathcal{D}_t \theta_s dW_{1s}^{\mathbf{Q}}\right]}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \tag{44}$$

where $\xi_{t,T}$ is defined in corollary 6 and where $\mathcal{D}_t r_s, \mathcal{D}_t \theta_s$ are given in explicit form in appendix C. The only notable impact of stochastic volatility is that it implies a continuous rescaling of the stock demand as it changes over time: the volatility-scaled portfolio demand $\sigma_t \hat{\pi}_t$ is immune to volatility risk.

The economic properties of the optimal portfolio follow directly from the scaling property. The fraction of each hedging demand relative to total stock demand is insensitive to volatility fluctuations. Since the magnitude of each component is simply rescaled as volatility changes the portfolio components exhibit more volatility. This behavior is illustrated in figure 13.[20]

10 A multiasset-trivariate model: hedging with two mutual funds.

We now consider a financial market with three assets (2 risky and a riskless asset) and a triplet of state variables. Our objectives are to provide a decomposition of the optimal portfolio and to examine the effects of correlations on the hedging terms.

The state variables (r, θ_1, θ_2) evolve according to

$$dr_t = \kappa_r(\overline{r} - r_t)dt - \sigma_r r_t^{1/2} dW_{1t}, \qquad r_0 \text{ given} \tag{45}$$

$$d\theta_{1t} = \left(\kappa_1(\overline{\theta}_1 - \theta_{1t}) + \delta_{1r} r_t\right) dt + \sigma_1^\theta \frac{1}{\sqrt{2\alpha}} \left(\alpha dW_{1t} + \rho_\theta dW_{2t}\right), \quad \theta_0^1 \text{ given} \tag{46}$$

$$d\theta_{2t} = \left(\kappa_2(\overline{\theta}_2 - \theta_{2t}) + \delta_{2r} r_t\right) dt + \sigma_2^\theta \frac{1}{\sqrt{2\alpha}} \left(\rho_\theta dW_{1t} + \alpha dW_{2t}\right), \quad \theta_0^2 \text{ given} \tag{47}$$

where $\alpha = 1 + \sqrt{1 - \rho_\theta^2}$ and $(\kappa_r, \overline{r}, \sigma_r, \kappa_1, \overline{\theta}_1, \delta_{1r}, \sigma_1^\theta, \kappa_2, \overline{\theta}_2, \delta_{2r}, \sigma_2^\theta, \rho_\theta)$ are constants. In this formulation σ_i^θ is the standard deviation of $\theta_i, i = 1, 2$, and ρ_θ represents the correlation coefficient between θ_1 and θ_2. The correlation between the interest rate and the market price of W_1-risk is negative and equals $\rho_{r\theta_1} = -\sqrt{\frac{1}{2}(1 + \sqrt{1 - \rho_\theta^2})}$. The correlation with the market price of W_2-risk, which equals $\rho_{r\theta_2} = -\rho_\theta / \sqrt{2(1 + \sqrt{1 - \rho_\theta^2})}$, is negative (positive) when ρ_θ is positive (negative).

[20] Again for this extension of the basic two-state variable model, we maximize the loglikelihood of the discretized model. The MPR series is now filtered with a GARCH(1,1) model with an AR(1) conditional mean as described in section 6. The estimates of the parameters in (r, θ) are found to be stable relative to those obtained in the earlier bivariate model. For the volatility process we find evidence of different effects for the positive and negative values of the MPR. This confirms the asymmetry reported in the literature. Estimated parameters are $\kappa_r = 0.004575$, $\overline{r} = 0.007 \times 12$, $\kappa_\theta = 0.7772$, $\overline{\theta} = 0.2689$, $\delta = -26.3514/12$, $\kappa_\sigma = 0.0445$, $\overline{\sigma} = 0.0594\sqrt{12}$, $\delta_{1\theta} = -0.2159/12$, $\delta_{2\theta} = 0.1254/12$, $\sigma_r = 0.01045\sqrt{12}$, $\sigma_\theta = 0.185$, $\lambda_1 = 0.00081$, and $\lambda_2 = \sqrt{0.01252^2 + 0.00174^2}$. The numerical simulation is based on these estimates.

When MPRs are positively correlated (ρ_θ positive) an increase in their correlation will increase $\rho_{r\theta_1}$ and decrease $\rho_{r\theta_2}$.

The riskless asset pays interest at the rate r in (45). The price S_i of asset i, $i = 1, 2$, satisfies

$$dS_{it} + \delta_{it} S_{it} dt = S_{it} \left[\mu_{it} dt + \sigma_i \left(\rho_i dW_{1t} + \sqrt{1-\rho_i^2} dW_{2t} \right) \right] \tag{48}$$

where the dividend rate δ_i and the drift μ_i are stochastic. The volatility matrix of asset returns is constant and assumed to be invertible (i.e. $\Delta \equiv \sigma_1\sigma_2(\rho_1\sqrt{1-\rho_2^2} - \rho_2\sqrt{1-\rho_1^2}) \neq 0$). Asset prices induce the (bivariate) MPR process (θ_1, θ_2) whose evolution is described in (46)-(47). The first risky asset can be interpreted as the market portfolio of risky stocks (SP500); the second is a portfolio of assets (mutual fund) whose correlation with the market portfolio is $\rho = \rho_1\rho_2 + \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}$. The correlation coefficients between the portfolio returns and the interest rate are respectively $-\rho_1$ and $-\rho_2$.

In this setting with two assets the optimal portfolio is given by[21]

$$\begin{aligned}
\begin{bmatrix} \hat{\pi}_{1t}/\hat{X}_t \\ \hat{\pi}_{2t}/\hat{X}_t \end{bmatrix} &= \frac{1}{R}\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2}\theta_{1t} - \sigma_2\rho_2\theta_{2t} \\ -\sigma_1\sqrt{1-\rho_1^2}\theta_{1t} + \sigma_1\rho_1\theta_{2t} \end{bmatrix} \\
&+ (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} \\ -\sigma_1\sqrt{1-\rho_1^2} \end{bmatrix} \overline{a}(t, r_t, \theta_t) \\
&+ (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2}\alpha - \rho_\theta\sigma_2\rho_2 \\ -\sigma_1\sqrt{1-\rho_1^2}\alpha + \rho_\theta\sigma_1\rho_1 \end{bmatrix} \sigma_1^\theta \frac{1}{\sqrt{2\alpha}} \overline{b}_1^{dir}(t, r_t, \theta_t) \\
&+ (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} \\ -\sigma_1\sqrt{1-\rho_1^2} \end{bmatrix} \delta_{1r} \overline{b}_1^{ind}(t, r_t, \theta_t) \\
&+ (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \rho_\theta\sigma_2\sqrt{1-\rho_2^2} - \alpha\sigma_2\rho_2 \\ -\rho_\theta\sigma_1\sqrt{1-\rho_1^2} + \alpha\sigma_1\rho_1 \end{bmatrix} \sigma_2^\theta \frac{1}{\sqrt{2\alpha}} \overline{b}_2^{dir}(t, r_t, \theta_t) \\
&+ (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} \\ -\sigma_1\sqrt{1-\rho_1^2} \end{bmatrix} \delta_{2r} \overline{b}_2^{ind}(t, r_t, \theta_t)
\end{aligned} \tag{49}$$

where

$$\overline{a}(t, r_t, \theta_t) = \mathbf{E}_t \left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \int_t^T \mathcal{D}_{1t} r_s ds \right]$$

$$\overline{b}_i^{dir}(t, r_t, \theta_t) = \mathbf{E}_t \left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \int_t^T e^{-\kappa_i(s-t)} dW_{is}^{\mathbf{Q}} \right], \quad i = 1, 2$$

$$\overline{b}_i^{ind}(t, r_t, \theta_t) = \mathbf{E}_t \left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]} \int_t^T \int_t^s e^{-\kappa_i(s-v)} \mathcal{D}_{1t} r_v dv dW_{is}^{\mathbf{Q}} \right], \quad i = 1, 2.$$

[21] The structure of the optimal portfolio remains the same if the volatility coefficients σ_1, σ_2 are stochastic. As in the prior section structure is preserved even when volatility risk can only be partially hedged with traded assets.

and $\mathcal{D}_{1t} r_s = -\sigma_r r_s^{\frac{1}{2}} \exp(-\frac{1}{2}\int_t^s (\kappa_r - \frac{\sigma_r^2}{4})\frac{1}{r_s} ds - \frac{1}{2}\kappa_r (s-t))$ is nonpositive at all times (taking $\sigma_r > 0$). The first line in (49) is the MV-demand, the second the IR-hedge, the third and fourth the MPR(θ_1)-hedge and the last two the MPR(θ_2)-hedge. The function $\overline{a}(t, r_t, \theta_t)$ is the cross-moment between the cost of optimal consumption $\xi_{t,T}^{1-1/R}$ and the sensitivity of the cumulative interest rate to W_1-risk (i.e. $\int_t^T \mathcal{D}_{1t} r_s ds$). Since θ_1 and θ_2 depend on the interest rate (see (46)-(47)), innovations in W_i will have a direct effect on future values of the MPRs as well as an indirect effect through the interest rate. The covariances $\overline{b}_i^{dir}(t, r_t, \theta_t)$ and $\overline{b}_i^{ind}(t, r_t, \theta_t)$ capture, respectively, these two aspects.

Let us focus on the effects of correlation between the two funds. Assume that all the coefficients are positive except for the correlation coefficients $\rho_1, \rho_2, \rho_\theta$ which may take positive or negative values. As it turns out the sign of all the demand components result from the spanning properties of the two traded assets and the risk exposure of the present value of terminal (optimal) consumption (PVC). This follows since the optimal portfolio is selected so as to finance this present value.

In particular, note that the MV components result from the desire to synthesize the vector (θ_1, θ_2) which describes the risk exposure of the state price density and captures the impact of the SPD on the PVC. When (θ_1, θ_2) is a convex combination of the vectors generated by the two funds returns, namely $(\rho_1, \sqrt{1-\rho_1^2})$ and $(\rho_2, \sqrt{1-\rho_2^2})$ then both demands are positive. This is the case when $\rho_1/\sqrt{1-\rho_1^2} > \theta_1/\theta_2 > \rho_2/\sqrt{1-\rho_2^2} > 0$. Otherwise, one fund is held short and the other long, and the MV demands are of opposite signs.

The IR-hedges in the two portfolio components reflect similar considerations. Here it is the risk exposure of the PVC induced by the interest rate that is being synthesized, i.e. the vector

$$(\frac{1}{R} - 1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overline{a}(t, r_t, \theta_t).$$

Since this risk exposure is outside the convex cone generated by asset returns demands will necessarily be of opposite sign. When risk aversion exceeds one interest rate risk has a positive impact on the PVC ($(1/R-1)\overline{a}(t, r_t, \theta_t) > 0$). If $\Delta > 0$ fund one exhibits more sensitivity to W_1-risk and will be held long. The second fund is used to neutralize the exposure to W_2-risk induced by the IR-hedge component of fund 1. Combining the IR-hedging demands of the two funds produces a perfect hedge against the impact of IR-risk on the PVC. The overall IR-hedge achieved is positive when risk aversion exceeds 1. This parallels the results found in prior sections.

MPR-hedges display an interesting structure. Since MPRs respond directly to exogenous shocks as well as indirectly through the interest dependence of their drift these MPR-hedges have two components. The direct hedges correspond to the terms with $(\sigma_i^\theta/\sqrt{2\alpha})\overline{b}_i^{dir}(t, r_t, \theta_t)$; indirect hedges involve $\delta_{ir}\overline{b}_i^{ind}(t, r_t, \theta_t)$. Considerations similar to those above govern the signs of these components. Let us focus on the MPR(θ_1)-hedge. We have:

1. direct hedge: fluctuations in the PVC related to the direct impact of (W_1, W_2) on θ_1 are described by the vector

$$(\frac{1}{R} - 1) \begin{bmatrix} \alpha \\ \rho_\theta \end{bmatrix} \sigma_1^\theta \frac{1}{\sqrt{2\alpha}} \overline{b}_1^{dir}(t, r_t, \theta_t).$$

When $\Delta > 0$, $\rho_\theta < 0$ and $(1/R-1)\overline{b}_1^{dir}(t, r_t, \theta_t)$ is positive this wealth component is financed by a long (short) position in fund 1 (fund 2). Under these conditions fund 1 is used to span W_1-risk. This, however, will result in an overexposure to W_2-risk. Shorting fund 2 in suitable proportion creates a perfect hedge. When one of the two funds provides a perfect hedge (i.e. $\rho_1/\sqrt{1-\rho_1^2} = \alpha/\rho_\theta$ or $\alpha/\rho_\theta = \rho_2/\sqrt{1-\rho_2^2}$) holdings of the other fund are null.

2. indirect hedge: fluctuations in the PVC induced through the interest rate dependence of θ_1 are described by

$$(\frac{1}{R} - 1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta_{1r} \overline{b}_1^{ind}(t, r_t, \theta_t)$$

   Properties of the fund positions which synthesize this risk parallel the properties of the IR-hedges.

We now provide a numerical illustration of the properties described above as well as others. Consider the symmetric case $(\kappa_1, \overline{\theta}_1, \delta_{1r}, \sigma_1) = (\kappa_2, \overline{\theta}_2, \delta_{2r}, \sigma_2)$ and calibrate the model by using the parameter estimates reported in section 6: $\kappa_r = 0.06977$, $\overline{r} = 0.005 \times 12$, $\kappa_1 = \kappa_2 = 0.9088$, $\overline{\theta}_1 = \overline{\theta}_2 = 0.1685$, $\delta_{1r} = \delta_{2r} = -23.90/12$, $\sigma_r = 0.01227\sqrt{12}$, $\sigma_1^\theta = \sigma_2^\theta = 0.16127$. The common volatilities of the two funds are $\sigma_1 = \sigma_2 = 0.2$ and the correlation of the market portfolio with the interest rate is $\rho_1 = 0.1$. In order to simulate the portfolio components we set initial values at $r_0 = .06$, $\theta_{10} = \theta_{20} = .10$. The graphs show results for correlations between the MPRs from $\rho_\theta = -0.9$ to $+0.9$ with increment 0.1. Given that $\rho_1 = 0.1$, the correlation ρ_2 of the second risky fund with the interest rate is chosen such that the implied correlation between the risky assets, $\rho = \rho_1\rho_2 + \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}$ varies between $-.2$ and $+1$. Values for ρ_2 vary from $\rho_2 = -0.99$ to $+.01$ in increments of 0.01. Finally, risk aversion $R = 4$ and the investment period is taken to be 5 years.

Figures 14 and 15 illustrate, respectively, the behaviors of the mean-variance components and of the IR-hedges. In addition to the theoretical effects described above it should be noted that the IR-hedges are nearly insensitive, and the MV components completely insensitive, to the correlation between the MPRs. Both components increase in magnitude as the returns correlation becomes more positive. Figure 16 shows that the hedge against θ_1 embedded in the demand for fund 1 (fund 2) displays concavity (convexity) with respect to ρ_θ and is increasing (decreasing) with respect to ρ. Figures 17 and 18 reveal that concavity (convexity) reflects the structure of both the direct and indirect components. Furthermore the indirect hedge changes sign when ρ_θ is in a neighborhood of ± 1. This follows from a sign reversal of the covariance $\overline{b}_1^{ind}(t, r_t, \theta_t)$ when ρ_θ approaches ± 1.

The hedge against θ_2 displays surprising behavior (figure 19). Its direct component, in the demand for fund 1, exhibits a convex-concave structure (horizontal S-shape) relative to ρ_θ for large values of ρ; a symmetric pattern characterizes the direct component in the demand for fund 2 (figures 20-21). This S-shape is a consequence of the behavior of the covariance $\overline{b}_2^{dir}(t, r_t, \theta_t)$ which is convex with respect to ρ_θ and takes positive values in a neighborhood of $\rho_\theta = \pm 1$. For values of ρ_θ close to -1 direct hedging entails duplicating the vector $\overline{b}_2^{dir}(t, r_t, \theta_t)(\rho_\theta, \alpha)$ and this is achieved by taking a long position in fund 1. As ρ_θ increases the covariance $\overline{b}_2^{dir}(t, r_t, \theta_t)$ becomes negative which implies a short position in fund 1. As ρ_θ increases further the vector $\overline{b}_2^{dir}(t, r_t, \theta_t)(\rho_\theta, \alpha)$ enters the convex cone formed by asset returns. Both funds are then held long. Eventually, as

ρ_θ approaches 1 the covariance $\overline{b}_2^{dir}(t, r_t, \theta_t)$ becomes positive and a short position in fund 1 is required to synthesize $\overline{b}_2^{dir}(t, r_t, \theta_t)(\rho_\theta, \alpha)$.

When combined the total MPR-hedging demand is concave (convex) for fund 1 (fund 2) reflecting the dominance of the hedge against θ_1 (figure 22). Finally figures 23 and 24 show the behavior of the sum of all the hedging components and the overall behavior of the portfolio. The overall hedging demand reflects the reinforcing behaviors of the IR- and MPR-hedges. The overall portfolio structure also exhibits the same pattern. In general hedging implies a significant departure from mean-variance demand behavior.

[Insert figures 14-24 here.]

Numerical values for fund holdings and hedging demands are provided in Table 9 for selected values of the correlation coefficients. They illustrate some of the features discussed above.

[Insert table 9 here.]

11 Conclusions.

In this paper we have developed a comprehensive approach for the calculation of the optimal portfolio in the asset allocation problem. One major benefit of our method which relies on Monte-Carlo simulation is its flexibility. Indeed the approach can be easily adapted to encompass (i) any finite number of state variables, (ii) any process for the state variables which satisfies the conditions described and (iii) any number of risky assets. It is also valid for any preference relation in the von Neumann-Morgenstern class. This flexibility provides a distinct advantage over alternative approaches to the problem.

The paper also derives a number of economic results which can be used as guidelines for sound asset allocation rules. The lessons drawn from our simulations can be summarized in the following observations:

1. Hedging components cannot be ignored for asset allocation purposes. Even for short investment horizons they imply an adjustment to mean-variance demands which may represent up to 20% of the stock demand. For long investment horizons hedging behavior has a major impact: the adjustment to mean-variance demands can represent up to 80% of the stock demand.

2. Hedging corrections are fairly stable over time: market timing experiments show that the volatility of the hedging components is low relative to the variation in the mean-variance component.

3. The most important factors in optimal allocation shares are the risk aversion of the investor and the investment horizon. Of particular interest is the behavior of the optimal stock demand relative to the investment horizon, namely the fact that long (short) investment horizons mandate an increase (decrease) in stock holdings relative to myopic behavior. Although this effect was only recorded in the context of our basic bivariate model, it confirms the interest of tailoring investment products and strategies to different categories of clienteles.

4. Allocation shares are also remarkably stable relative to the other parameters of the model. Variations of the order of 2 standard deviations around estimated parameter values have little impact on the magnitude and the behavior of investment shares.

5. In multiasset models return correlations and correlations between returns and state variables emerge as important factors composition of the optimal portfolio. Even for short horizons asset demands can increase by a factor of 5 when assets returns are highly correlated.

Naturally, the performance of any asset allocation rule will also depend on the soundness of the underlying model of financial markets. Clearly we do not suggest that the models investigated in this paper are adequate in that respect. However, the approach that we have proposed is universal in the sense that it can be used to address the asset allocation problem under complete markets for any realistic specification of the uncertainty structure no matter how complex.

12 Appendix.

12.1 Appendix A: proofs.

Proof of Theorem 1: It follows from Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987) that optimal final wealth must be given by $\hat{X}_T = I(T, \hat{y}\xi_T)$ where $I = [\partial_2 u]^{-1}$ is the inverse marginal utility of consumption and $\hat{y}$ satisfies $\mathbf{E}[\xi_T I(T, \hat{y}\xi_T)] = x$. Since $\xi_t \hat{X}_t = \mathbf{E}_t[\xi_T \hat{X}_T]$ we have for $J(t,y) := yI(t,y)$ that

$$\hat{X}_t = I(t, \hat{y}\xi_t)\mathbf{E}_t[J_{t,T}]$$

where

$$J_{t,T} \equiv \frac{J(T, \hat{y}\xi_T)}{J(t, \hat{y}\xi_t)}.$$

Using the chain rule of Malliavin calculus and the relation $-\partial_2 I(t,y) = \frac{1}{-\partial_{22}u(y, I(t,y))}$ (which follows from the definition $\partial_2 u(t, I(t,y)) = y$) we obtain

$$\frac{\mathcal{D}_s \hat{X}_t}{\hat{X}_t} = -\frac{1}{\xi_t R(t, I(t, \hat{y}\xi_t))}\mathcal{D}_s \xi_t + \frac{\mathcal{D}_s \mathbf{E}_t[J_{t,T}]}{\mathbf{E}_t[J_{t,T}]}$$

where $R(t,x) \equiv \frac{-\partial_{22}u(t,x)x}{\partial_2 u(t,x)}$ is the relative risk aversion of the investor. Taking the limit as $s \uparrow t$ on both sides of this equation and using $\lim_{s\uparrow t} \mathcal{D}_s \hat{X}_t = \hat{\pi}_t' \sigma_t$, $\lim_{s\uparrow t} \mathcal{D}_s \xi_t = -\xi_t \theta_t'$ and the commutativity of the conditional expectation and Malliavin derivative operator then leads to

$$\hat{\pi}_t' = \hat{X}_t \left[\frac{1}{R(t, I(t, \hat{y}\xi_t))}\theta_t' + \frac{\mathbf{E}_t[\mathcal{D}_t J_{t,T}]}{\mathbf{E}_t[J_{t,T}]} \right] \sigma_t^{-1}.$$

But since

$$\mathcal{D}_t J_{t,T} = \frac{\partial_2 J(T, \hat{y}\xi_T)}{J(t, \hat{y}\xi_t)}\mathcal{D}_t \hat{y}\xi_T - \frac{J(T, \hat{y}\xi_T)}{J(t, \hat{y}\xi_t)}\frac{\partial_2 J(t, \hat{y}\xi_t)}{J(t, \hat{y}\xi_t)}\mathcal{D}_t \hat{y}\xi_t$$

where $\mathcal{D}_t \hat{y}\xi_T = -\hat{y}\xi_T(\theta_t' + H_{t,T}')$ with

$$H_{t,T} = \int_t^T \mathcal{D}_t r_s ds + \int_t^T dW_s' \mathcal{D}_t \theta_s + \int_t^T ds \theta_s' \mathcal{D}_t \theta_s = \int_t^T \mathcal{D}_t r_s ds + \int_t^T (dW_s^{\mathbf{Q}})' \mathcal{D}_t \theta_s$$

and since

$$\frac{\partial_2 J(t,y)}{J(t,y)}y = 1 + \frac{y\partial_2 I(t,y)}{I(t,y)} = 1 - \frac{1}{R(t,I(t,y))} := \alpha(t,y)$$

the second term in the expression for the optimal portfolio can be written

$$\begin{aligned}
\frac{\mathbf{E}_t[\mathcal{D}_t J_{t,T}]}{\mathbf{E}_t[J_{t,T}]} &= \frac{1}{\mathbf{E}_t[J_{t,T}]}\mathbf{E}_t\left[\frac{\partial_2 J(T,\hat{y}\xi_T)}{J(t,\hat{y}\xi_t)}\mathcal{D}_t\hat{y}\xi_T - \frac{J(T,\hat{y}\xi_T)}{J(t,\hat{y}\xi_t)}\frac{\partial_2 J(t,\hat{y}\xi_t)}{J(t,\hat{y}\xi_t)}\mathcal{D}_t\hat{y}\xi_t\right] \\
&= -\frac{1}{\mathbf{E}_t[J_{t,T}]}\mathbf{E}_t\left[\frac{\partial_2 J(T,\hat{y}\xi_T)}{J(t,\hat{y}\xi_t)}\hat{y}\xi_T\left(\theta_t' + H_{t,T}'\right)\right] + \frac{\partial_2 J(t,\hat{y}\xi_t)}{J(t,\hat{y}\xi_t)}\hat{y}\xi_t\theta_t' \\
&= -\frac{1}{\mathbf{E}_t[J_{t,T}]}\mathbf{E}_t\left[\frac{J(T,\hat{y}\xi_T)}{J(t,\hat{y}\xi_t)}\alpha(T,I(T,\hat{y}\xi_T))\left(\theta_t' + H_{t,T}'\right)\right] + \alpha(t,I(t,\hat{y}\xi_t))\theta_t' \\
&= \left(\alpha_t - \mathbf{E}_t\left[\frac{J_{t,T}}{\mathbf{E}_t[J_{t,T}]}\alpha_T\right]\right)\theta_t' - \mathbf{E}_t\left[\frac{J_{t,T}}{\mathbf{E}_t[J_{t,T}]}\alpha_T\int_t^T \mathcal{D}_t r_s ds\right] \\
&\quad -\mathbf{E}_t\left[\frac{J_{t,T}}{\mathbf{E}_t[J_{t,T}]}\alpha_T\int_t^T [dW_s + \theta_s ds]'\mathcal{D}_t\theta_s\right]
\end{aligned}$$

where $\alpha_t \equiv \alpha(t,I(t,\hat{y}\xi_t))$ (note that $\alpha_T = \alpha(T,I(T,\hat{y}\xi_T)) = \alpha(T,\widehat{X}_T)$)

Finally using $\frac{J_{t,T}}{\mathbf{E}_t[J_{t,T}]} = \frac{\xi_T}{\xi_t}\frac{\hat{X}_T}{\hat{X}_t}$ where $\frac{d\mathbf{Q}}{d\mathbf{P}}_{|\mathcal{F}_t} = B_t\xi_t$, we obtain

$$\begin{aligned}
\hat{\pi}_t' &= \frac{\hat{X}_t}{R(t,\hat{X}_t)}\mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\frac{R(t,\hat{X}_t)}{R(T,\hat{X}_T)}\right]\theta_t'\sigma_t^{-1} \\
&\quad +\hat{X}_t\frac{1-R(t,\hat{X}_t)}{R(t,\hat{X}_t)}\mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\frac{R(t,\hat{X}_t)}{R(T,\hat{X}_T)}\left(\frac{R(T,\hat{X}_T)-1}{R(t,\hat{X}_t)-1}\right)\int_t^T \mathcal{D}_t r_s ds\right]\sigma_t^{-1} \\
&\quad +\hat{X}_t\frac{1-R(t,\hat{X}_t)}{R(t,\hat{X}_t)}\mathbf{E}_t^{\mathbf{Q}}\left[\frac{\frac{\hat{X}_T}{B_T}}{\frac{\hat{X}_t}{B_t}}\frac{R(t,\hat{X}_t)}{R(T,\hat{X}_T)}\left(\frac{R(T,\hat{X}_T)-1}{R(t,\hat{X}_t)-1}\right)\int_t^T (dW_s^{\mathbf{Q}})'\mathcal{D}_t\theta_s\right]\sigma_t^{-1}.
\end{aligned}$$

Now note that the chain rule of Malliavin calculus gives

$$\begin{cases} \mathcal{D}_t\theta_s = \partial_2\theta(s,Y_s)\mathcal{D}_t Y_s \\ \mathcal{D}_t r_s = \partial_2 r(s,Y_s)\mathcal{D}_t Y_s \end{cases}.$$

Furthermore (1) and Nualart (1995), section 2.2, p. 99-108, imply that $\mathcal{D}_t Y_s = (\mathcal{D}_{1t}Y_s, ..., \mathcal{D}_{dt}Y_s)$ solves d systems (one for each of the d Malliavin derivatives) of d stochastic differential equation

$$\begin{aligned}
\mathcal{D}_{kt}Y_s &= \mathcal{D}_{kt}Y_t + \int_t^s \mathcal{D}_{kt}\mu^Y(v,Y_v)dv + \mathcal{D}_{kt}\int_t^s\left(\sum_{j=1}^d \sigma_{\cdot j}^Y(v,Y_v)dW_{jv}\right) \\
&= \sigma_{\cdot k}^Y(t,Y_t) + \int_t^s \partial_2\mu^Y(v,Y_v)\mathcal{D}_{kt}Y_v dv + \int_t^s \mathcal{D}_{kt}\left(\sum_{j=1}^d \sigma_{\cdot j}^Y(v,Y_v)dW_{jv}\right) \\
&= \sigma_{\cdot k}^Y(t,Y_t) + \int_t^s \partial_2\mu^Y(v,Y_v)\mathcal{D}_{kt}Y_v dv + \int_t^s\left(\sum_{j=1}^d \partial_2\sigma_{\cdot j}^Y(v,Y_v)\mathcal{D}_{kt}Y_v dW_{jv}\right)
\end{aligned}$$

$$= \sigma^Y_{\cdot k}(t, Y_t) + \int_t^s \partial_2 \mu^Y(v, Y_v) \mathcal{D}_{kt} Y_v dv + \int_t^s \left(\sum_{j=1}^d \partial_2 \sigma^Y_{\cdot j}(v, Y_v) dW_{jv} \right) \mathcal{D}_{kt} Y_v$$

for $k = 1, ..., d$. The solutions of these systems of linear equations are as stated in the theorem using the fact that the quadratic variation of the martingale part is $\sum_{j=1}^d \partial_2 \sigma^Y_{\cdot j}(v, Y_v)(\partial_2 \sigma^Y_{\cdot j}(v, Y_v))' dv$ where $\sigma^Y_{\cdot j}$ denotes the j^{th} column of the matrix σ^Y. ■

Proof of Proposition 4: Following the arguments of Doss (1977) we consider a function $F : [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\partial_2 F = \frac{1}{\sigma}$. Using $\partial_{22} F = (\partial_2 \frac{1}{\sigma}) = -\frac{\partial_2 \sigma}{\sigma^2}$ and Ito's lemma implies that

$$dF(t, Y_t) = \left[\frac{\mu}{\sigma} - \frac{1}{2} \partial_2 \sigma + \partial_1 F \right] (t, Y_t) dt + dW_t.$$

so that $F(t, Y_t)$ has the decomposition $F(t, Y_t) = N_t + W_t$ where

$$dN_t = \left[\frac{\mu}{\sigma} - \frac{1}{2} \partial_2 \sigma + \partial_1 F \right] (t, Y_t) dt.$$

Since F has an inverse G given by $G(t, F(t, y)) = y$ we can write $Y_t = G(t, N_t + W_t)$ and therefore

$$dN_t = \left[\frac{\mu}{\sigma} - \frac{1}{2} \partial_2 \sigma + \partial_1 F \right] (t, G(t, N_t + W_t)) dt.$$

with $N_0 = F(0, y)$. Then since from assumptions (i) and (ii) G is continuously differentiable and by theorem 2.2.1 of Nualart (1995) which needs assumption (iii) the process is in the domain of the Malliavin derivative operator $N \in \mathbb{D}^{1,2}$ we have for $t \leq s$ that

$$\mathcal{D}_t Y_s = \partial_2 G(s, N_s + W_s) Z_{t,s}$$

where

$$dZ_{t,s} = \partial_2 \left[\frac{\mu}{\sigma} - \frac{1}{2} \partial_2 \sigma + \partial_1 F \right] (s, G(s, N_s + W_s))(\partial_2 G(s, N_s + W_s)) Z_{t,s} ds$$

with $Z_{t,t} = 1$. Solving this linear SDE for $Z_{t,s}$ and using the relations for derivatives of F and its inverse G produces the result stated. ■.

Proof of Proposition 5: Since $dY_t = \mu(t, Y_t) dt + \sigma(t, Y_t) dW_t$ we have

$$\int_0^t \theta(s, Y_s) dW_s = -\int_0^t [\frac{\theta}{\sigma} \mu](s, Y_s) ds + \int_0^t [\frac{\theta}{\sigma}](s, Y_s) dY_s.$$

Then for ψ such that $\partial_2 \psi \sigma = \theta$ we have that

$$\psi(t, Y_t) - \psi(0, Y_0) = \int_0^t [\partial_1 \psi + \frac{1}{2} \partial_{22} \psi \sigma^2](s, Y_s) ds + \int_0^t [\frac{\theta}{\sigma}](s, Y_s) dY_s.$$

But $\partial_{22} \psi = \frac{\partial_2 \theta}{\sigma} - \frac{\theta}{\sigma} \frac{\partial_2 \sigma}{\sigma}$ and therefore

$$\int_0^t \theta(s, Y_s) dW_s = -\int_0^t [\frac{\theta}{\sigma} \mu](s, Y_s) ds + \psi(t, Y_t) - \psi(0, Y_0) - \int_0^t [\partial_1 \psi + \frac{1}{2} [\partial_2 \theta \sigma - \theta \partial_2 \sigma]](s, Y_s) ds$$

Using this expression for the stochastic integral in the expression of the SPD provides (18).

To establish (19) use $\int_t^T \mathcal{D}_t\theta_s[dW_s + \theta_s ds] = \mathcal{D}_t\{\int_0^t \theta_s[dW_s + \frac{1}{2}\theta_s ds]\}$, substitute the expression for $\int_0^t \theta(s, Y_s)dW_s$ above on the right hand side, and compute the Malliavin derivative of the expression in bracket. ■

Proof of Corollary 7: Substituting $\gamma_r, \gamma_\theta = 0$ in the expressions for the Malliavin derivatives in Proposition 4 gives $\mathbf{D}_t r_v = \sigma_r \exp[-\kappa_r(v-t)]$ and $\mathbf{D}_t\theta_v = \sigma_\theta \exp[-\kappa_\theta(v-t)]$. Since R is constant and $\mathbf{D}_t r_v$ is deterministic we can then write

$$a(t, r_t, S_t) = E_t\left[\frac{\xi_{t,T}^{1-1/R}}{E_t\left[\xi_{t,T}^{1-1/R}\right]}\left(\int_t^T \sigma_r \exp\left[-\kappa_r(v-t)\right]dv\right)\right] = \int_t^T \sigma_r \exp\left[-\kappa_r(v-t)\right]dv.$$

Substituting the expression for $\mathbf{D}_t\theta_v$ in $b(t, r_t, \theta_t)$ gives the formula in the lemma. ■

Proof of equations (42)-(44): We conjecture that the individual price of W_2-risk is null. The SPD is then given by the formula in theorem 3 where (r, θ) satisfy (39)-(40). Since (r, θ) is independent of W_2-risk, optimal wealth $\widehat{X}_T = I(T, \widehat{y}\xi_T)$ is independent of W_2. The Martingale representation theorem and the Clark-Ocone formula imply the existence of a unique financing portfolio which is given by (42)-(44). ■

Proof of equation (49): Theorem 3 implies that the optimal portfolio is given by

$$\hat{\pi}_t = \hat{X}_t(\sigma_t')^{-1}\left[\frac{1}{R}\theta_t + (\frac{1}{R}-1)a(t, r_t, \theta_t) + (\frac{1}{R}-1)b_1(t, r_t, \theta_t) + (\frac{1}{R}-1)b_2(t, r_t, \theta_t)\right]$$

where

$$a(t, r_t, \theta_t)' \equiv \mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_t r_s ds\right]$$

$$b_i(t, r_t, \theta_t)' \equiv \mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_t\theta_{is} dW_{is}^{\mathbf{Q}}\right]; i = 1, 2.$$

Straightforward computations give the Malliavin derivatives

$$(\mathcal{D}_t r_s)' = \begin{bmatrix} \mathcal{D}_{1t}r_s \\ \mathcal{D}_{2t}r_s \end{bmatrix} = \begin{bmatrix} -\sigma_r r_s^{\frac{1}{2}} \exp\left(-\frac{1}{2}\int_t^s(\kappa_r - \frac{\sigma_r^2}{4})\frac{1}{r_s}ds - \frac{1}{2}\kappa_r(s-t)\right) \\ 0 \end{bmatrix}$$

$$(\mathcal{D}_t\theta_{1s})' = \begin{bmatrix} \mathcal{D}_{1t}\theta_{1s} \\ \mathcal{D}_{2t}\theta_{1s} \end{bmatrix} = \begin{bmatrix} \sigma_1^\theta \frac{\alpha}{\sqrt{2\alpha}} e^{-\kappa_1(s-t)} + \int_t^s e^{-\kappa_1(s-v)}\delta_{1r}\mathcal{D}_{1t}r_v dv \\ \sigma_1^\theta \frac{\rho_\theta}{\sqrt{2\alpha}} e^{-\kappa_1(s-t)} \end{bmatrix}$$

$$(\mathcal{D}_t\theta_{2s})' = \begin{bmatrix} \mathcal{D}_{1t}\theta_{2s} \\ \mathcal{D}_{2t}\theta_{2s} \end{bmatrix} = \begin{bmatrix} \sigma_2^\theta \frac{\rho_\theta}{\sqrt{2\alpha}} e^{-\kappa_2(s-t)} + \int_t^s e^{-\kappa_2(s-v)}\delta_{2r}\mathcal{D}_{1t}r_v dv \\ \sigma_1^\theta \frac{\alpha}{\sqrt{2\alpha}} \end{bmatrix}$$

which leads to

$$a(t,r_t,\theta_t) = \begin{bmatrix} \mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_{1t}r_s ds\right] \\ \mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_{2t}r_s ds\right] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_{1t}r_s ds\right]$$

$$\begin{aligned} b_1(t,r_t,\theta_t) &= \begin{bmatrix} \alpha \\ \rho_\theta \end{bmatrix} \sigma_1^\theta \frac{1}{\sqrt{2\alpha}}\mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T e^{-\kappa_1(s-t)}dW_{1s}^{\mathbf{Q}}\right] \\ &+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta_{1r}\mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T\int_t^s e^{-\kappa_1(s-v)}\mathcal{D}_{1t}r_v dv dW_{1s}^{\mathbf{Q}}\right] \end{aligned}$$

and a symmetric expression for $b_2(t,r_t,\theta_t)$.

Defining the determinant of the volatility matrix $\Delta = \sigma_1\sigma_2\left(\rho_1\sqrt{1-\rho_2^2} - \rho_2\sqrt{1-\rho_1^2}\right)$ we can write the mean-variance term as

$$\frac{1}{R}(\sigma_t')^{-1}\theta_t = \frac{1}{R}\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} & -\sigma_2\rho_2 \\ -\sigma_1\sqrt{1-\rho_1^2} & \sigma_1\rho_1 \end{bmatrix}\begin{bmatrix} \theta_{1t} \\ \theta_{2t} \end{bmatrix}.$$

Substituting the expression for $a(t,r_t,\theta_t)$ gives the IR-hedge

$$(\frac{1}{R}-1)(\sigma_t')^{-1}a(t,r_t,\theta_t) = (\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} \\ -\sigma_1\sqrt{1-\rho_1^2} \end{bmatrix}\mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T \mathcal{D}_{1t}r_s ds\right].$$

Finally, substituting $b_1(t,r_t,\theta_t)$ provides the MPR(θ_1)-hedge

$$\begin{aligned} &(\frac{1}{R}-1)(\sigma_t')^{-1}b_1(t,r_t,\theta_t) \\ = \quad &(\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2}\alpha - \rho_\theta\sigma_2\rho_2 \\ -\sigma_1\sqrt{1-\rho_1^2}\alpha + \rho_\theta\sigma_1\rho_1 \end{bmatrix}\sigma_1^\theta\frac{1}{\sqrt{2\alpha}}\mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T e^{-\kappa_1(s-t)}dW_{1s}^{\mathbf{Q}}\right] \\ &+(\frac{1}{R}-1)\frac{1}{\Delta}\begin{bmatrix} \sigma_2\sqrt{1-\rho_2^2} \\ -\sigma_1\sqrt{1-\rho_1^2} \end{bmatrix}\delta_{1r}\mathbf{E}_t\left[\frac{\xi_{t,T}^{1-1/R}}{\mathbf{E}_t\left[\xi_{t,T}^{1-1/R}\right]}\int_t^T\int_t^s e^{-\kappa_1(s-v)}\mathcal{D}_{1t}r_v dv dW_{1s}^{\mathbf{Q}}\right]. \end{aligned}$$

A symmetric expression holds for the MPR(θ_2)-hedge. ■

12.2 Appendix B: A representation of Malliavin derivatives of multivariate diffusion processes.

Consider a d-dimensional process Y which satisfies the system of SDEs

$$dY_t = \mu(t,Y_t)dt + \sigma(t,Y_t)dW_t; Y_0 = y$$

where W is a d-dimensional Brownian motion process. For any $d \times 1$ vector of functions $f(t,Y)$ let $\partial_1 f$ represent the $d \times 1$ vector of first derivatives relative to time and $\partial_2 f$ the $d \times n$ matrix whose rows are composed of the gradients relative to Y of the elements of f. The Malliavin derivative of Y has the following alternative representation.

Proposition 8 *If the following conditions hold*

(i) differentiability of drift: $\mu \in \mathcal{C}^1([0,T] \times \mathbb{R}^d)$

(ii) differentiability of volatility: $\sigma \in \mathcal{C}^2([0,T] \times \mathbb{R}^d)$

(iii) growth condition: $\mu(t,0)$ *and* $\sigma(t,0)$ *are bounded for all* $t \in [0,T]$

(iv) invertibility condition: $det(\sigma(t,y)) \neq 0$ *for all* $t \in [0,T]$ *and* $y \in \mathbb{R}^d$

(v) volatility condition: the Lie algebra of the vector fields generated by the columns of σ, $\mathcal{L}\{\sigma_1, \ldots, \sigma_d\}$ *is Abelian, i.e.* $(\partial_2 \sigma_i)\sigma_j = (\partial_2 \sigma_j)\sigma_i$ *for all* $i,j = 1,\ldots,d$ *where* $\partial_2 \sigma_j$ *is the* $d \times d$ *Jacobian matrix with respect to* y *of the* $d \times 1$ *vector function* σ_j.

then we have for $t \leq s$ *that*

$$\mathcal{D}_t Y_s = \sigma(s, Y_s) Z_{t,s}$$

where the $d \times d$ *process* $Z_{t,s}$ *satisfies*

$$dZ_{t,s} = \left[\partial_2 \left[(\sigma)^{-1}\mu + \frac{1}{2}H\right] + \partial_1 (\sigma)^{-1}\right](s, Y_s)\sigma(s, Y_s) Z_{t,s} ds \tag{50}$$

subject to the boundary condition $Z_{t,t} = I_d$ *($d \times d$-identity matrix) where*

$$H = (I \otimes \mathbf{1}')(K \odot (\mathbf{1} \otimes \sigma'\sigma))\mathbf{1} \tag{51}$$

with K *for the Jacobian matrix of* σ^{-1} *given by*

$$K = -\frac{1}{2}[(\sigma \otimes \sigma')^{-1}\partial_2(\sigma') + [(\partial_2 \sigma')'(\sigma' \otimes \sigma)^{-1}]_\nu]. \tag{52}$$

The operators $\otimes$ *and* $\odot$ *represent, respectively, the Kronecker and Hadamard products,*[22] *whereas the stack operator* $[\cdot]_\nu$ *operates on a* $d \times d^2$ *matrix* $B = [B_1, \ldots, B_d]$ *where* B_i *are* d*-dimensional square matrices as follows:* $[B]_\nu = [(B_1)', \ldots, (B_d)']'$.

Assumption (v) in this proposition guarantees that there exists F such that $\partial_2 F = \sigma^{-1}$. Since by (iv) F has an inverse G, say, condition (v) could equivalently be written as $\partial_j G_i(t,z) = \sigma_{i,j}(t, G(t,z))$. The assumption is automatically satisfied if the state variables do not interact with each other, i.e. if $\sigma_j(t, Y_t) = \sigma_j(t, Y_t^j)$ for $j = 1, ..., d$. The one dimensional case treated earlier falls in this category.

[22] The Kronecker product of a vector Y and a matrix $A = [a_{ij}]$ is $X \otimes A = [Y a_{ij}]$. The Hadamard product of two matrices A and B is $A \odot B = [a_{i,j} b_{i,j}]$, i.e. the matrix composed of the direct products of the corresponding elements in the two matrices.

Proof of Proposition 8: The proof parallels the one dimensional case. Assumption (v) ensures the existence of a $d \times 1$ vector of functions $F : [0,T] \times \mathbb{R}_+^d \mapsto \mathbb{R}_+^d$ such that $\partial_2 F = \sigma^{-1}$. Using $\partial_{22}F = \partial_2 \sigma^{-1}$ we get by the identification theorem for Hessian matrices of vector functions (theorem 6.7. of Magnus and Neudecker (1988)) that

$$\partial_{22}F(t,y) = -\frac{1}{2}[(\sigma \otimes \sigma')^{-1}\partial_2(\sigma') + [(\partial_2\sigma')'(\sigma' \otimes \sigma)^{-1}]_\nu](t,y) \tag{53}$$

where the stack operator $[\cdot]_\nu$ acts in the following manner: for a $d \times d^2$ matrix $B = [B_1, \ldots, B_d]$ where B_i are d-dimensional square matrices we have $[B]_\nu = [(B_1)', \ldots, (B_d)']'$. The use of the stack operator is necessary to guarantee that the components $\partial_{22}F_i(t,y)$ which arise in blocks in $\partial_{22}F(t,y)$ remain symmetric.

Using Ito's lemma applied to each element of F we get

$$dF_i(t,Y_t) = [\partial_1 F_i + \partial_2 F_i \mu + \frac{1}{2} trace(\partial_{22}F_i \sigma'\sigma)](t,Y_t)dt + [\partial_2 F_i \sigma](t,Y_t)dW_t$$

for $i = 1, ..., d$. Stacking these SDEs for $i = 1, \ldots, d$ one below the other gives for $F(t,Y_t)$,

$$dF(t,Y_t) = \left[\sigma^{-1}\mu + \frac{1}{2}H + \partial_1 F\right](t,Y_t)dt + dW_t$$

where $H' = [trace(\partial_{22}F_1\sigma'\sigma), \ldots, trace(\partial_{22}F_d\sigma'\sigma)]$. To obtain the expression (51) for H note that $trace(AB') = \mathbf{1}'(A \odot B)\mathbf{1}$ where $\odot$ is the Hadamard product, i.e. $A \odot B = [[a_{i,j}b_{i,j}]]$. Now we can write the matrix H as follows

$$H = [\mathbf{1}'((\partial_{22}F_1)' \odot \sigma'\sigma)\mathbf{1}, \ldots, \mathbf{1}'((\partial_{22}F_d)' \odot \sigma'\sigma)\mathbf{1}]'$$

which is equivalent to

$$H = [(I \otimes \mathbf{1}')[((\partial_{22}F_1)' \odot \sigma'\sigma), \ldots, ((\partial_{22}F_d)' \odot \sigma'\sigma)]'\mathbf{1}.$$

But since $\partial_{22}F = [(\partial_{22}F_1)', \ldots, (\partial_{22}F_d)']'$ we get

$$[((\partial_{22}F_1)' \odot \sigma'\sigma), \ldots, ((\partial_{22}F_d)' \odot \sigma'\sigma)]' = (\partial_{22}F \odot (\mathbf{1} \otimes \sigma'\sigma))$$

and therefore $H = (I \otimes \mathbf{1}')(\partial_{22}F \odot (\mathbf{1} \otimes \sigma'\sigma))\mathbf{1}$ where $\partial_{22}F(t,y)$ is as given in (53). Thus, H is obtained by multiplying the Hessian of each element of F element by element with the matrix $\sigma'\sigma$ then summing over all elements and arranging the result in a column vector whose first element is obtained by performing the same operation for F_1, the second for F_2 and so on until F_d. This establishes (51).

Thus, using these expressions we see that $F(t,Y_t) = N_t + W_t$ where

$$dN_t = \left[\sigma^{-1}\mu + \frac{1}{2}H + \partial_1 F\right](t,Y_t)dt.$$

Since the determinant of the Jacobian $\partial_2 F$ differs from 0 (assumption (iv)) the vector $F(t,y)$ has a unique inverse G defined by $G(t,F(t,y)) = y$. We can then write $Y_t = G(t, N_t + W_t)$ and therefore

$$dN_t = \left[\sigma^{-1}\mu + \frac{1}{2}H + \partial_1 F\right](t, G(t, N_t + W_t))dt$$

with $N_0 = F(0, y)$. Then since from assumptions (i)-(ii) G is continuously differentiable and by theorem 2.2.1 of Nualart (1995), which requires assumption (iii), the process is in the domain of the Malliavin derivative operator $N \in \mathbb{D}^{1,2}$ we have for $t \leq s$ that

$$\mathcal{D}_t Y_s = \partial_2 G(s, N_s + W_s) Z_{t,s}$$

where

$$dZ_{t,s} = \partial_2 \left[\sigma^{-1}\mu + \frac{1}{2}H + \partial_1 F \right] (s, G(s, N_s + W_s))(\partial_2 G(s, N_s + W_s)) Z_{t,s} ds$$

with $Z_{t,t} = I_d$. Since $\partial_2 G(s, N_s + W_s)\partial_2 F(s, Y_s) = I$ we have that $\partial_2 G(s, N_s + W_s) = \sigma(s, Y_s)$. Substituting in the equation above leads to the result in the proposition. ■

12.3 Appendix C: the MRGID model.

We consider the following interest rate - market price of risk model with interaction in the drift of the MPR

$$dr_t = \kappa_r(\overline{r} - r_t)dt + \sigma_r\sqrt{r_t}dW_t, \qquad r_0 \text{ given} \tag{54}$$

$$d\theta_t = \left(\kappa_\theta(\overline{\theta} - \theta_t) + \delta_\theta r_t\right) dt + \sigma_\theta dW_t, \qquad \theta_0 \text{ given} \tag{55}$$

where $(\kappa_r, \overline{r}, \sigma_r, \kappa_\theta, \bar{\theta}, \delta_\theta, \sigma_\theta)$ are nonnegative constants. The transition from the general model with state variables Y to the model (54)-(55) with state variables (r, θ) is immediate since the Malliavin derivative $\mathbf{D}_t\theta_v$ can now be computed directly from the process (55). Taking account of the specific structure (54)-(55) then leads to

Proposition 9 *In the financial market (54)-(55) the optimal portfolio is given by (5) with*

$$\mathbf{D}_t r_v = \sqrt{r}_v \sigma_r \exp\left[-\frac{1}{2}\int_t^v \left(\kappa_r(1 + \overline{r}\frac{1}{r_u}) - \frac{1}{4}\sigma_r^2(\frac{1}{r_u}) \right) du \right]$$

$$\mathbf{D}_t\theta_v = \sigma_\theta e^{-\kappa_\theta(v-t)} + \delta_\theta \int_t^v e^{-\kappa_\theta(v-s)}\mathcal{D}_t r_s ds.$$

The SPD is then

$$\xi_t = \exp\left[-\int_0^t [r_s + (\frac{1}{2} - \frac{\kappa_\theta}{\sigma_\theta})\theta_s^2 - \frac{\kappa_\theta}{\sigma_\theta}\bar{\theta}\theta_s - \frac{\delta}{\sigma_\theta}\theta_s r_s]ds] - \frac{1}{2}(\theta_t^2 - \theta_0^2) + \frac{1}{2}\sigma_\theta t \right]$$

and the stochastic integral for the MPR-hedge becomes

$$\int_t^T \mathcal{D}_t\theta_s[dW_s + \theta_s ds] = \int_t^T \left[[(1 + 2\frac{\kappa_\theta}{\sigma_\theta}) - \frac{\kappa_\theta}{\sigma_\theta}\bar{\theta} + \frac{\delta}{\sigma_\theta}r_s]\mathcal{D}_t\theta_s + \frac{\delta}{\sigma_\theta}\theta_s\mathcal{D}_t r_s \right] ds + \frac{1}{\sigma_\theta}\theta_T\mathcal{D}_t\theta_T - \theta_t.$$

12.4 Appendix D: asymptotic laws of state variables estimators.

In this appendix we report theorems from Detemple, Garcia and Rindisbacher (2000) providing the asymptotic laws of estimators of functionals of Brownian motions. The proofs of these results are based on Kurtz and Protter (1991) and Jacod and Protter (1998). Consider the SDE of the vector of state variables Y_t after the Doss transformation

$$d\widehat{Y}_t = \widehat{m}(\widehat{Y}_t)dt + \sum_{j=1}^{d} dW_t^j \tag{56}$$

with

$$\hat{m}(Y_s) = [(\sigma^Y)^{-1}]_i' \mu^Y + \frac{1}{2} tr[\partial[(\sigma^Y)^{-1}]_i' \sigma^Y (\sigma^Y)'](Y_s), \tag{57}$$

and let $\widehat{Y}_T^N$ denote the estimator of $\widehat{Y}_T$ based on a Euler scheme. Our next theorem characterizes the estimation error.

Theorem 10 *The asymptotic law of the estimator of the state variables Y is given by*

$$U_{t,T}^{\widehat{Y}^N} \equiv N(\widehat{Y}_T^N - Y_T) \Rightarrow \hat{U}_{t,T}^Y$$

where

$$\hat{U}_{t,T}^Y = -\sigma^Y(Y_T)\hat{\Omega}_{t,T} \int_t^T \hat{\Omega}_{t,s}^{-1} \partial \hat{m}(Y_s) \left[\frac{1}{2}(\hat{m}(Y_s)ds + dW_s) + \frac{1}{\sqrt{12}} dB_s \right]. \tag{58}$$

with

$$\hat{\Omega}_{t,v} = \exp\left(\int_t^v [(\partial\hat{m})\sigma^Y](Y_s)ds \right). \tag{59}$$

In addition to providing an explicit expression for the asymptotic law of the estimator, theorem 10 also demonstrates a speed of convergence of order $1/N$. These results can be contrasted with those obtained when state variables are estimated before transformation. Applying a Euler scheme to estimate the solution of (1) leads to

$$U_t^{\widetilde{Y}^N} \equiv \sqrt{N}(\widetilde{Y}_t^N - Y_t) \Rightarrow \tilde{U}_t^Y$$

where

$$\tilde{U}_{t,v}^Y = -\frac{1}{\sqrt{2}} \tilde{\Omega}_{t,v} \int_t^v \tilde{\Omega}_{t,s}^{-1} \sum_{h,j=1}^{d} [(\partial\sigma_{\cdot j}^Y)\sigma_{\cdot h}^Y](Y_s) dB_s^{h,j}. \tag{60}$$

with

$$\tilde{\Omega}_{t,v} = \exp\left(\int_t^v [\partial\mu^Y(Y_s) - \frac{1}{2}\sum_{j=1}^{d}(\partial\sigma_{\cdot j}^Y(Y_s))^2]ds + \sum_{j=1}^{d}\int_t^v \partial\sigma_{\cdot j}^Y(Y_s)dW_s^j \right). \tag{61}$$

In this case the resulting speed of convergence is $1/\sqrt{N}$. These results illustrate the increase in the speed of convergence achieved by using the Doss transformation. They also highlights the fact that the limit law is different and involves an exponential of a bounded total variation process instead of a stochastic integral. DGR (2000) provides similar theorems for the Malliavin derivatives and the functionals that appear in the hedging terms $a(t, Y_t)$ and $b(t, Y_t)$. The increased rate of convergence is important when computing conditional estimators of the hedging terms based on an approximation of the dynamic evolution of the state variables.

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Table 1- Comparison of the speeds of convergence of the discretization schemes when the IR follows a MRSR process.

N		r		$\mathcal{D}r$
	Euler	Euler-Transform	Euler	Euler-Transform
2	0.000115598	5.49255e-06	5.81463e-07	3.47457e-07
4	0.000111128	3.37985e-06	3.58341e-07	2.13681e-07
8	8.74541e-05	1.82631e-06	2.33208e-07	1.15422e-07
16	6.50156e-05	9.41716e-07	1.6312e-07	5.9616e-08
32	4.66084e-05	4.7979e-07	1.16983e-07	3.03396e-08
64	3.336e-05	2.40698e-07	8.29213e-08	1.52396e-08
128	2.3761e-05	1.20386e-07	5.97503e-08	7.63041e-09
256	1.68824e-05	5.83759e-08	4.18739e-08	3.69586e-09
512	1.19618e-05	2.53747e-08	3.00371e-08	1.60477e-09

Table 2 - Unconstrained monthly estimates of the bivariate interest rate-MPR process with constant stock volatility

Parameters	ML estimates	Standard Errors
κ_{rM}	0.0265	0.0107
$\overline{r}$	0.0053	0.0007
κ_θ	0.6528	0.0482
$\overline{\theta}$	0.0846	0.0084
σ_r	0.0049	0.0002
σ_θ	0.1052	0.0039
$\rho_{r\theta}$	-0.1651	0.0539

Table 3 - Constrained (with $\rho_{r\theta}$ set at -0.9) monthly estimates of the bivariate interest rate-MPR process with constant stock volatility

Parameters	ML estimates	Standard Errors
κ_{rM}	0.0824	0.0116
$\overline{r}$	0.0050	0.0005
κ_θ	0.6950	0.0507
$\overline{\theta}$	0.0871	0.0161
σ_r	0.0105	0.0004
σ_θ	0.2125	0.0080

Table 4 - Constrained (with $\rho_{r\theta}$ set at -0.9) monthly estimates of the bivariate interest rate-MPR process with constant stock volatility with r_{t-1} in the drift of MPR

Parameters	ML estimates	Standard Errors
κ_{rM}	0.0005	0.0185
$\overline{r}$	0.0051	0.0010
κ_{θ}	0.7771	0.0484
$\overline{\theta}$	0.2675	0.0348
σ_r	0.0105	0.0004
σ_{θ}	0.2050	0.0073
δ	-26.2469	4.9686

Table 5 - Unconstrained monthly estimates of the bivariate interest rate-MPR process with a GARCH stock conditional variance

Parameters	ML estimates	Standard Errors
κ_{rM}	0.0290	0.0106
$\overline{r}$	0.0053	0.0006
κ_{θ}	0.5975	0.0464
$\overline{\theta}$	0.0882	0.0083
σ_r	0.0049	0.0002
σ_{θ}	0.0979	0.0035
$\rho_{r\theta}$	-0.1863	0.052

Table 6 - Constrained (with $\rho_{r\theta}$ set at -0.9) monthly estimates of the bivariate interest rate-MPR process with a GARCH stock conditional variance

Parameters	ML estimates	Standard Errors
κ_{rM}	0.0947	0.0128
$\overline{r}$	0.0050	0.0004
κ_{θ}	0.6826	0.0507
$\overline{\theta}$	0.0900	0.0147
σ_r	0.0104	0.0004
σ_{θ}	0.1928	0.0070

Table 7 - Shares of the portfolio in the stock and Hedging Components for Model 1.

$R = 2$	Investment horizon	1	2	3	4	5
	Stock demand	25.4	26.1	27.0	29.2	30.5
	MPR-hedge	-1.7	-3.0	-3.9	-3.5	-3.7
	Interest rate hedge	2.1	4.1	5.9	7.6	9.2

$T = 1$	Risk aversion	0.5	1	1.5	4	5
	Stock demand	113.0	50.0	33.2	14.4	12.3
	MPR-hedge	17.2	0.0	-1.6	-1.3	-1.24
	Interest rate hedge	-4.3	0.0	1.4	3.2	3.4

Table 8 - Dividend-Price Ratio Model - Shares of the portfolio in the stock and Hedging Components for Model 1 ($\sigma = 0.20$).

$R = 4$	Investment horizon	1	2	3	4	5
	Stock demand	30.18	39.92	46.88	52.45	57.27
	MPR-hedge	13.90	20.13	23.82	26.35	28.33
	Interest rate hedge	3.78	7.29	10.56	13.60	16.43

$T = 1$	Risk aversion	0.5	1	1.5	2	5
	Stock demand	81.42	50.0	40.86	36.49	28.94
	MPR-hedge	-13.56	0.0	5.85	8.98	14.91
	Interest rate hedge	-5.02	0.0	1.68	2.52	4.03

Table 9 - Multiasset model - Shares invested
in the two Funds and Hedging Components (case $\rho_2 < 0$).

Returns Correlation: $\rho = 0$							
Fund	ρ_θ	MV-Comp.	IR-H	MPR1-H	MPR2-H	H-Comp.	Holdings
1	-0.9	13.68594	1.62469	-0.46143	-2.86238	-1.69912	11.98682
2		-11.18734	-16.18569	-0.54965	-2.24462	-18.97997	-30.16731
1	-0.6	13.68594	1.61552	-1.85189	3.16808	2.93171	16.61764
2		-11.18734	-16.09436	-8.47951	1.43846	-23.13540	-34.32275
1	-0.3	13.68594	1.61200	-0.56593	6.75339	7.79946	21.48540
2		-11.18734	-16.05926	-12.29448	1.77988	-26.57386	-37.76121
1	0.0	13.68594	1.61197	1.28385	6.38567	9.28149	22.96742
2		-11.18734	-16.05898	-12.79016	0.67745	-28.17169	-39.35903
1	0.3	13.68594	1.61506	2.55858	4.96426	9.13790	22.82384
2		-11.18734	-16.08978	-10.21850	-0.23006	-26.53834	-37.72568
1	0.6	13.68594	1.61690	3.73219	2.63805	7.98715	21.67309
2		-11.18734	-16.10812	-8.57007	-0.57762	-25.25582	-36.44316
1	0.9	13.68594	1.62484	1.22769	-3.01071	-0.15819	13.52775
2		-11.18734	-16.18717	-1.46995	1.47530	-16.18182	-27.36916

Returns Correlation: $\rho = .5$							
Fund	ρ_θ	MV-Comp.	IR-H	MPR1-H	MPR2-H	H-Comp.	Holdings
1	-0.9	20.14629	10.97144	-0.14402	-1.56618	9.26124	29.40754
2		-12.91800	-18.68958	-0.63468	-2.59186	-21.91612	-34.83412
1	-0.6	20.14629	10.90953	3.04477	2.33741	16.29171	36.43800
2		-12.91800	-18.58412	-9.79127	1.66099	-26.71440	-39.63239
1	-0.3	20.14629	10.88574	6.53376	5.72556	23.14507	43.29136
2		-12.91800	-18.54359	-14.19641	2.05522	-30.68478	-43.60277
1	0.0	20.14629	10.88555	8.66978	5.99446	25.54979	45.69608
2		-12.91800	-18.54326	-14.76876	0.78225	-32.52978	-45.44777
1	0.3	20.14629	10.90643	8.45946	5.09711	24.46300	44.60929
2		-12.91800	-18.57883	-11.79928	-0.26565	-30.64375	-43.56175
1	0.6	20.14629	10.91886	8.68116	2.97161	22.57163	42.71792
2		-12.91800	-18.60001	-9.89584	-0.66698	-29.16283	-42.08083
1	0.9	20.14629	10.97244	2.07654	-3.86265	9.18633	29.33262
2		-12.91800	-18.69129	-1.69734	1.70352	-18.68511	-31.60311

Returns Correlation: $\rho = .9$							
Fund	ρ_θ	MV-Comp.	IR-H	MPR1-H	MPR2-H	H-Comp.	Holdings
1	-0.9	36.78663	35.04645	0.67354	1.77253	37.49252	74.27915
2		-25.66581	-37.13294	-1.26099	-5.14958	-43.54350	-69.20931
1	-0.6	36.78663	34.84868	15.65741	0.19780	50.70389	87.49052
2		-25.66581	-36.92339	-19.45354	3.30010	-53.07684	-78.74264
1	-0.3	36.78663	34.77270	24.82088	3.07813	62.67170	99.45833
2		-25.66581	-36.84288	-28.20578	4.08337	-60.96529	-86.6311
1	0.0	36.78663	34.77207	27.69419	4.98680	67.45306	104.23969
2		-25.66581	-36.84222	-29.34296	1.55420	-64.63098	-90.2967
1	0.3	36.78663	34.83877	23.65871	5.43931	63.93679	100.7234
2		-25.66581	-36.91289	-23.44310	-0.52779	-60.88378	-86.5495
1	0.6	36.78663	34.87849	21.42850	3.83078	60.13777	96.9244
2		-25.66581	-36.95497	-19.66131	-1.32517	-57.94146	-83.6072
1	0.9	36.78663	35.04965	4.26297	-6.05705	33.25558	70.0422
2		-25.66581	-37.13633	-3.37232	3.38461	-37.12405	-62.78985

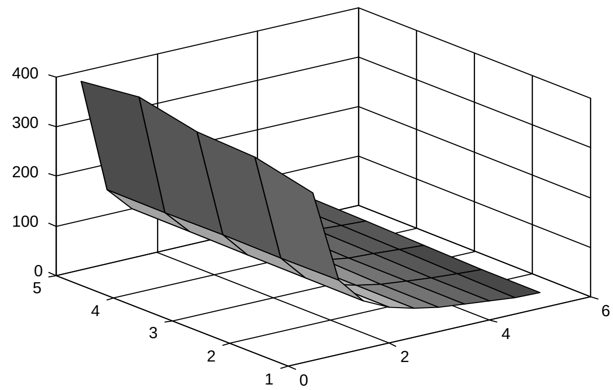

Figure 1: Share of portfolio invested in stock as a function of time and risk aversion.

R=2	Investment horizon	1	2	3	4	5
	Stock demand	72.7	73.2	74.4	76.7	78.3

T=1	Risk aversion	0.5	1	3	4	5
	Stock demand	339.1	150.0	48.9	37.3	30.5

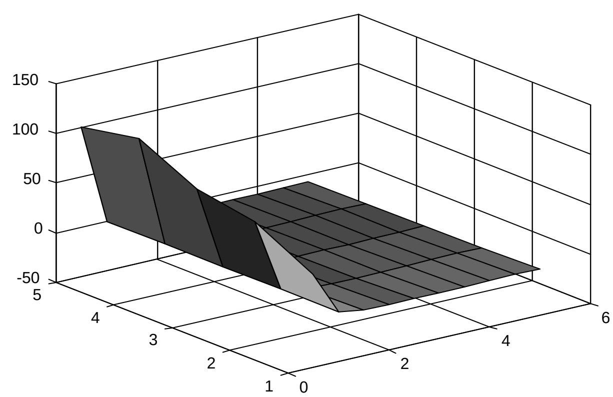

Figure 2: Share of the MPR hedge as a function of time and risk aversion.

R=2	Investment horizon	1	2	3	4	5
	MPR-hedge	-4.5	-6.3	-7.4	-7.3	-8

T=1	Risk aversion	.05	1	3	4	5
	MPR-hedge	43.5	0.0	–4.1	–3.5	-3.5

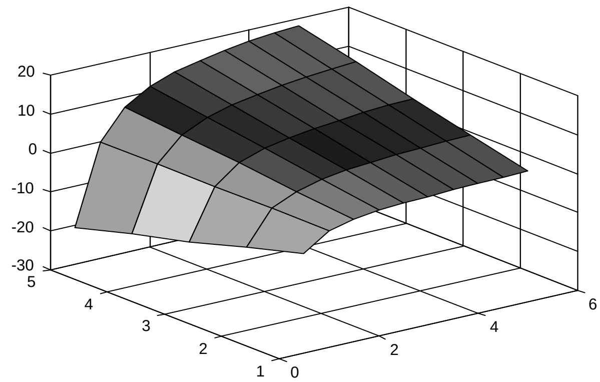

Figure 3: Share of the interest rate hedge as a function of time and risk aversion.

R=2	Investment horizon	1	2	3	4	5
	Interest rate hedge	2.2	4.5	6.7	9.0	11.3

T=1	Risk aversion	0.5	1	3	4	5
	Interest rate hedge	-4.4	0.0	3.0	3.4	3.6

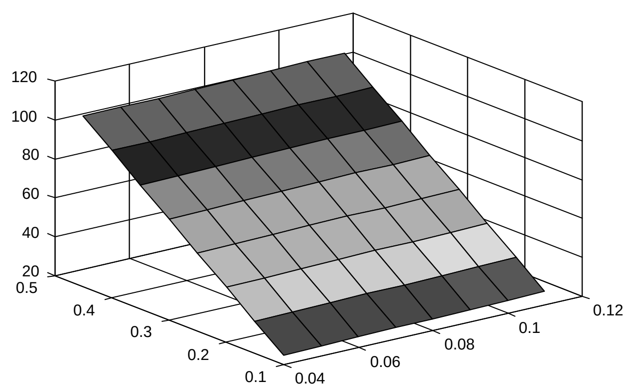

Figure 4: Stock demand behavior relative to r_0 and θ_0. Interest rate varies between 0.04 and 0.08; MPR between .05 and .40.

r=6%	MPR	0.10	0.20	0.40
	Stock demand	25.4	49.1	96.4

MPR=.25	Interest rate (%)	4	6	8
	Stock demand	60.3	60.9	61.5

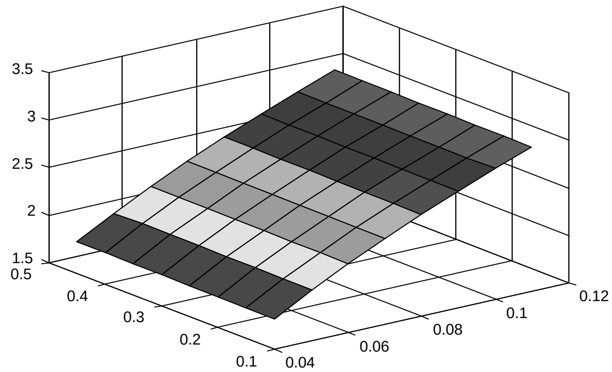

Figure 5: Interest rate hedge behavior relative to r_0 and θ_0. Interest rate varies between 0.04 and 0.08; MPR between .05 and .40.

r=6%	MPR	0.10	0.20	0.40
	Interest rate hedge	2.2	2.2	2.2

MPR=.25	Interest rate(%)	4	6	8
	Interest rate hedge	1.8	2.2	2.6

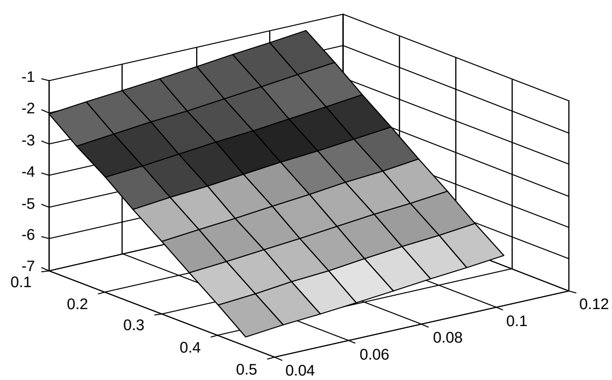

Figure 6: MPR -hedge behavior relative to r_0 and θ_0. Interest rate varies between 0.04 and 0.11; MPR between .10 and .45.

r=6%	MPR	0.10	0.20	0.40
	MPR-hedge	-1.8	-3.2	-5.8

MPR=.25	IR(%)	4	6	8
	MPR-hedge	-4.0	-3.8	-3.6

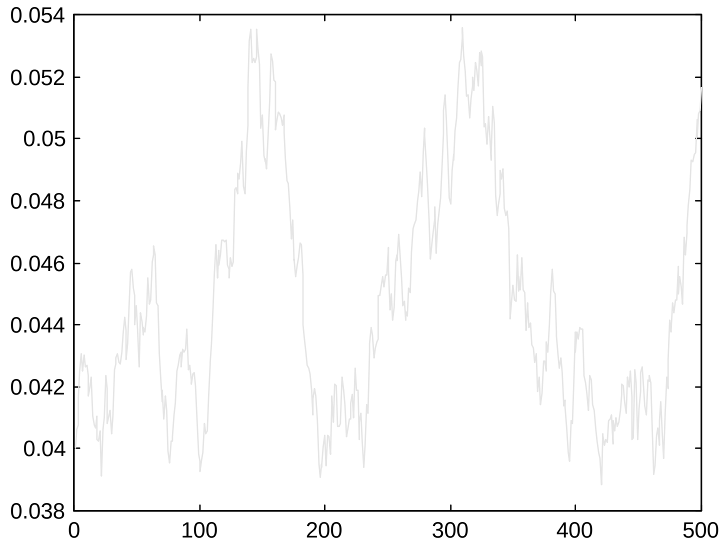

Figure 7: Simulated Path for Interest Rate.

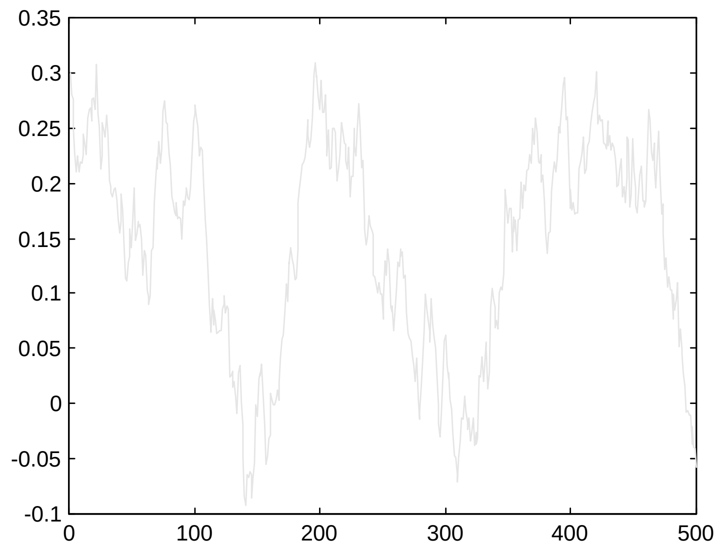

Figure 8: Simulated Path for Market Price of Risk.

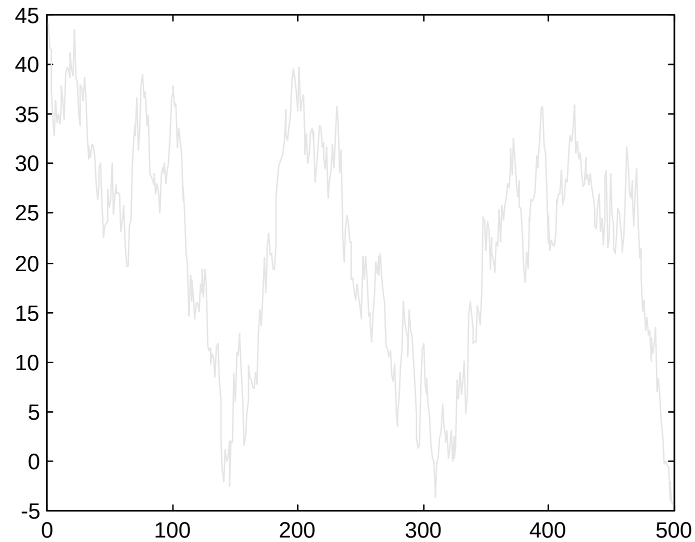

Figure 9: Fixed 5-year Horizon - Share in Stocks - R = 4.

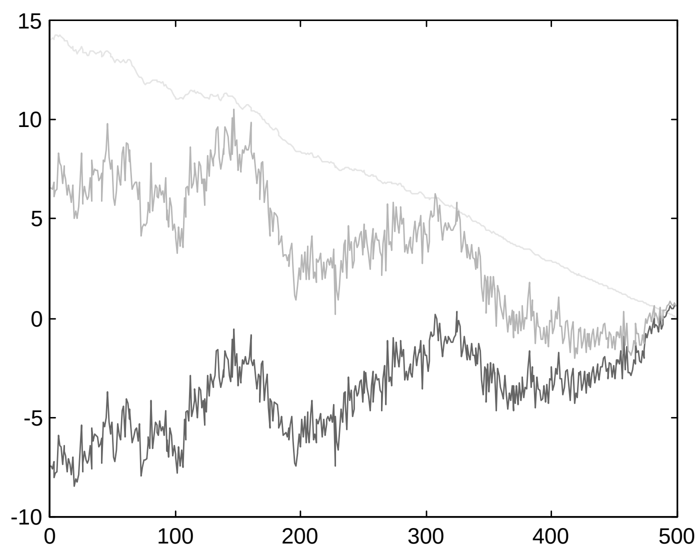

Figure 10: Fixed 5-year Horizon - Hedging Shares (top to bottom): Interest Rate, Total, MPR-R = 4.

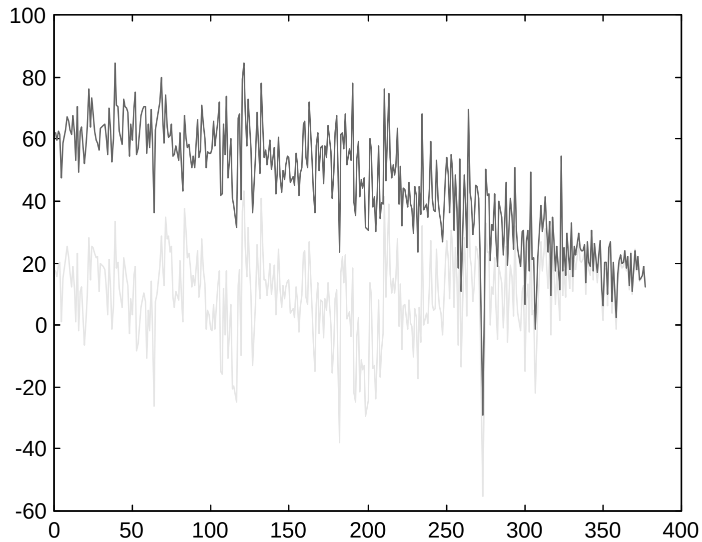

Figure 11: Share of Stock in Portfolio with (top) and without (bottom) hedging - Fixed Horizon of 31.5 years (our sample).

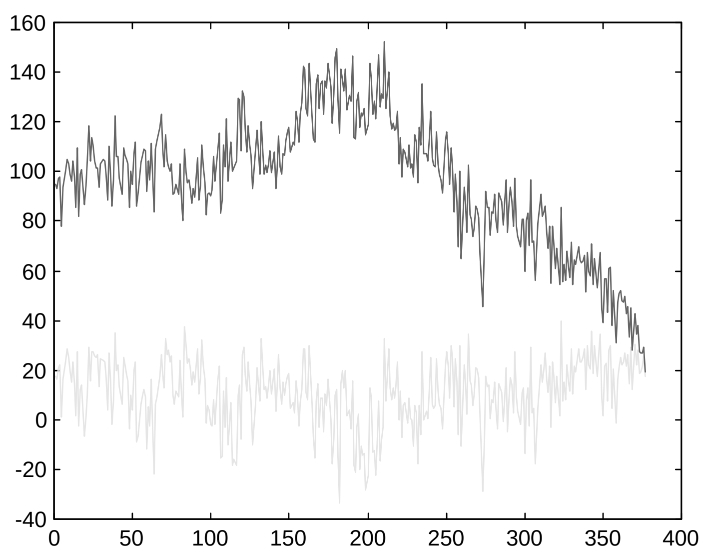

Figure 12: Model with Stochastic Dividends - Share of Stock in Portfolio with (top) and without (bottom) hedging - Fixed Horizon of 31.5 years (our sample).

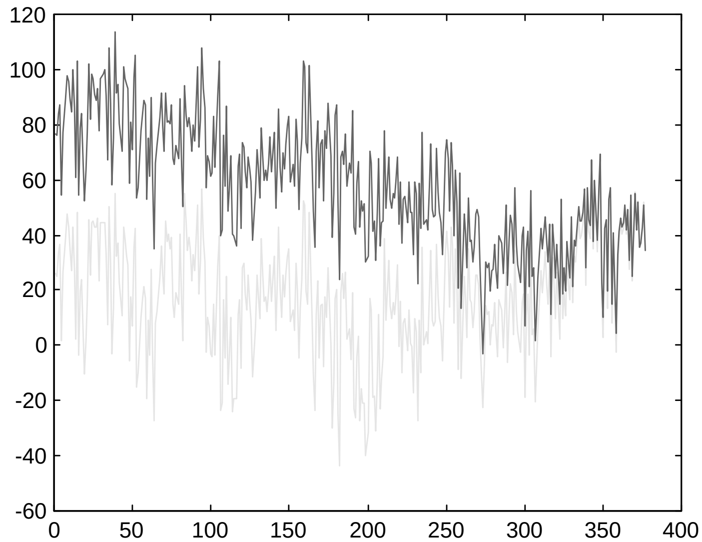

Figure 13: Model with Stochastic Volatility - Share of Stock in Portfolio with (top) and without (bottom) hedging - Fixed Horizon of 31.5 years (our sample).

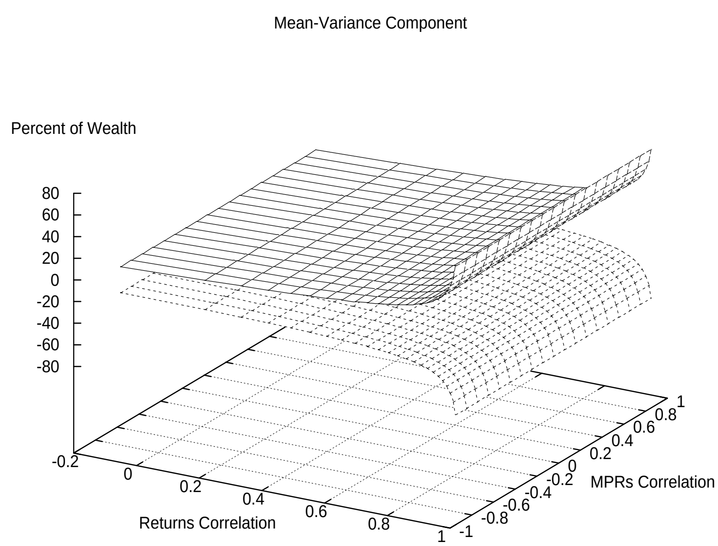

Figure 14: Mean-Variance Component: Fund 1 (plain) and Fund 2 (dotted line)

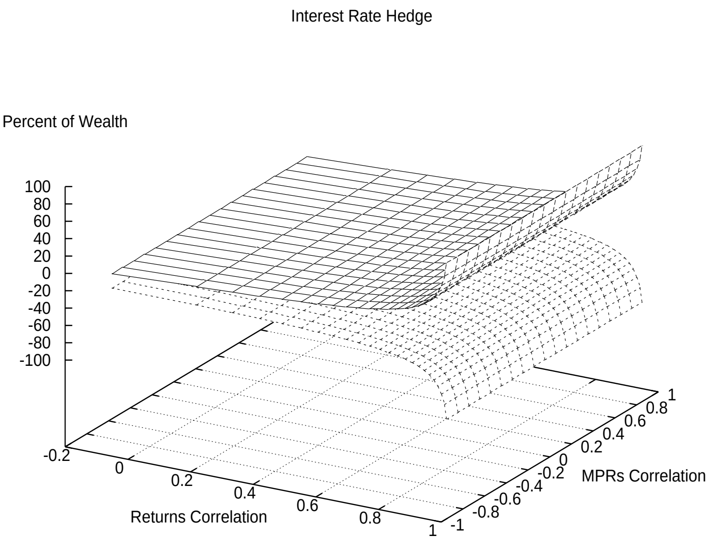

Figure 15: Interest Rate Hedge: Fund 1 (plain) and Fund 2 (dotted line)

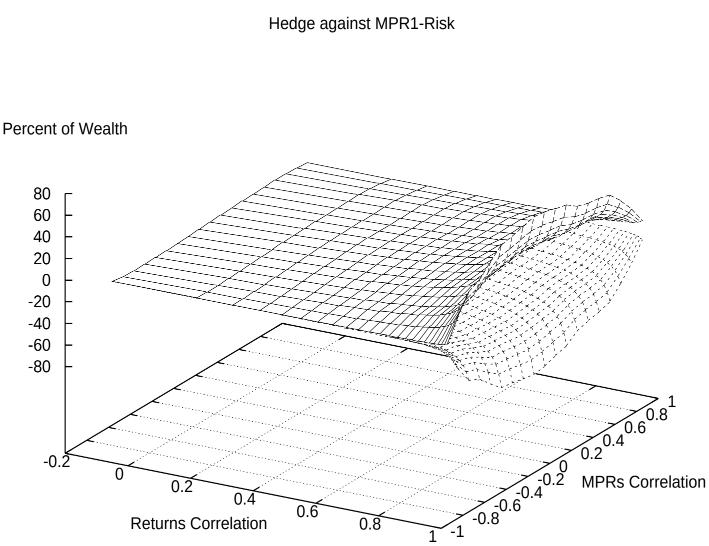

Figure 16: Hedge against MPR1-Risk: Fund 1 (plain) and Fund 2 (dotted line)

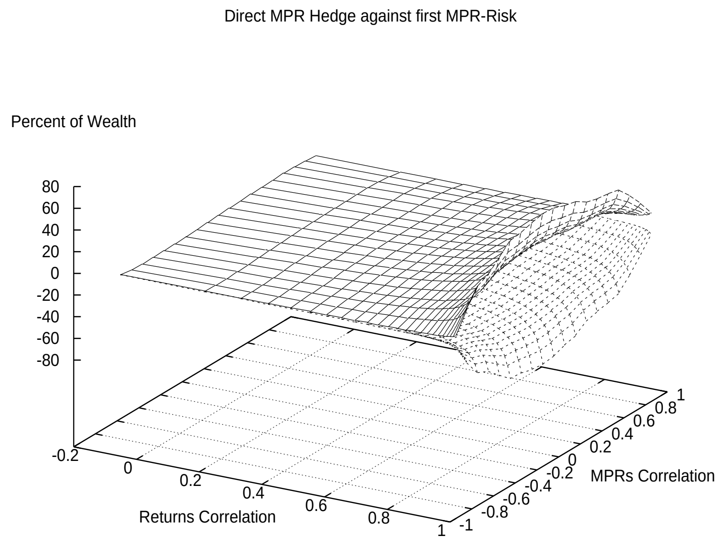

Figure 17: Direct MPR Hedge against first MPR-Risk: Fund 1 (plain) and Fund 2 (dotted line)

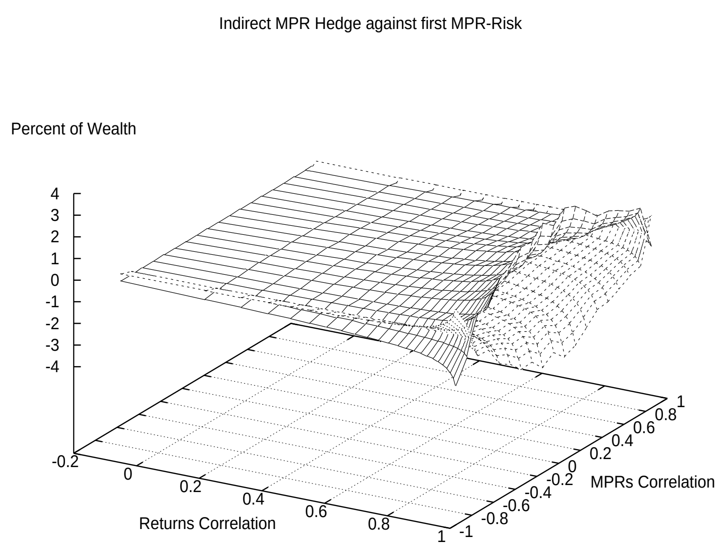

Figure 18: Indirect MPR Hedge against first MPR-Risk: Fund 1 (plain) and Fund 2 (dotted line)

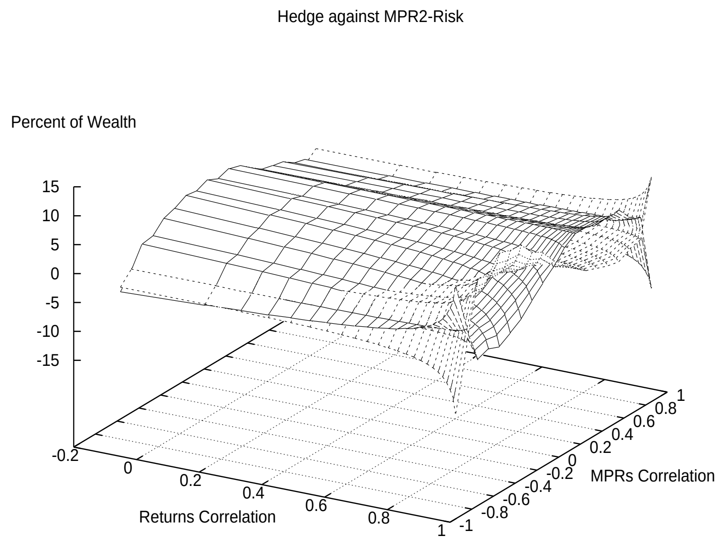

Figure 19: Hedge against MPR2-Risk: Fund 1 (plain) and Fund 2 (dotted line)

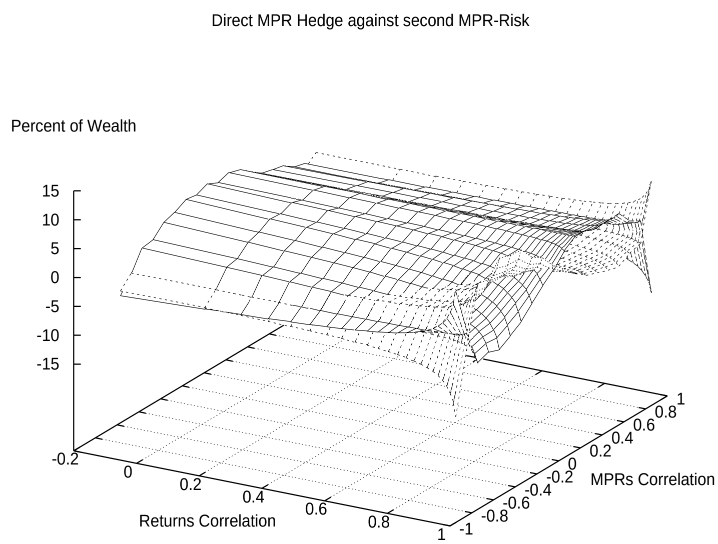

Figure 20: Direct MPR Hedge against second MPR-Risk: Fund 1 (plain) and Fund 2 (dotted line)

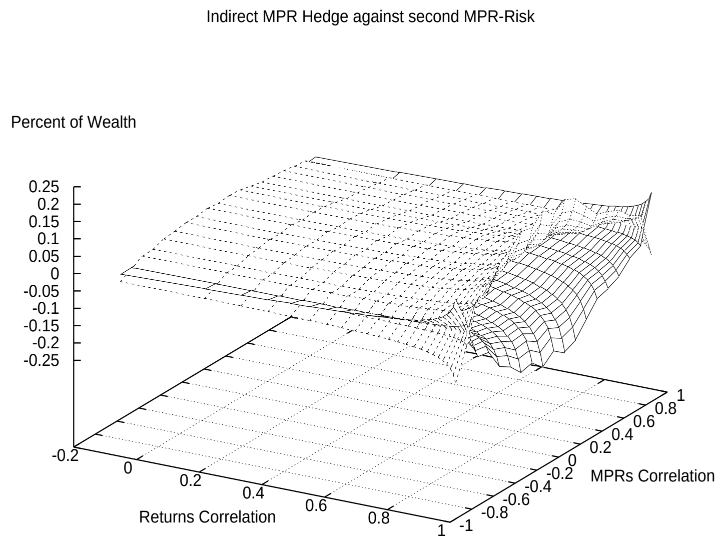

Figure 21: Indirect MPR Hedge against second MPR-Risk: Fund 1 (plain) and Fund 2 (dotted line)

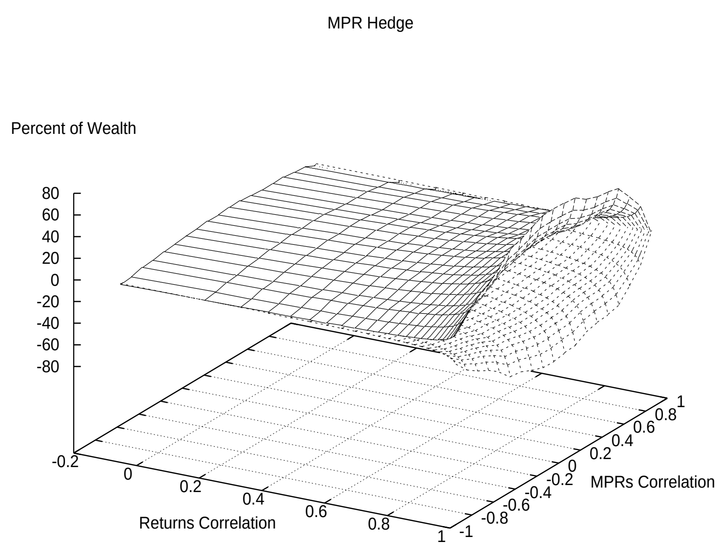

Figure 22: MPR Hedge: Fund 1 (plain) and Fund 2 (dotted line)

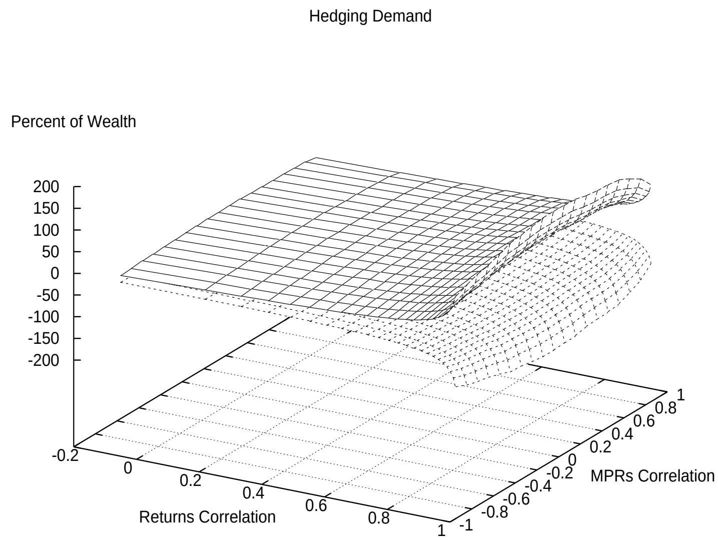

Figure 23: Hedge Demand: Fund 1 (plain) and Fund 2 (dotted line)

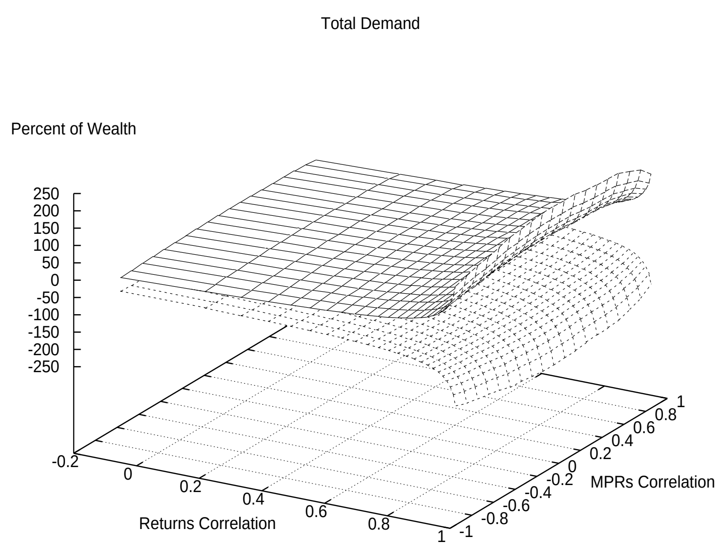

Figure 24: Total Demand: Fund 1 (plain) and Fund 2 (dotted line)