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ARBITRAGE-FREE INTERPOLATION OF THE SWAP CURVE

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We suggest an arbitrage free interpolation method for pricing zero-coupon bonds of arbitrary maturities from a model of the market data that typically underlies the swap curve; that is short term, future and swap rates. This is done first within the context of the Libor or the swap market model. We do so by introducing an independent stochastic process which plays the role of a short term yield, in which case we obtain an approximate closed-form solution to the term structure while preserving a stochastic implied short rate. This will be discontinuous but it can be turned into a continuous process (however at the expense of closed-form solutions to bond prices). We then relax the assumption of a complete set of initial swap rates and look at the more realistic case where the initial data consists of fewer swap rates than tenor dates and show that a particular interpolation of the missing swaps in the tenor structure will determine the volatility of the resulting interpolated swaps. We give conditions under which the problem can be solved in closed-form therefore providing a consistent arbitrage-free method for yield curve generation.

Keywords: Term structure modelling; Libor and swap market models; HJM.

1. Introduction

Yield curves are constructed in practice from market quoted rates of simple compounding with accrual periods of no less than a day. In particular, swap curves are constructed by combinations of bootstrap and interpolation methods from the following market data:

- Short-term interest rates,
- Interest rate futures,
- Swap rates.

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For example the GBP curve may stretch out to 52 years, and the interest rate futures are short sterling futures, but there are only a few data points beyond 10 years (for example we may have swap rates with six month payments for 12, 15, 20, 40 and 52 years) hence the need for bootstrapping methods. Constructing the yield curve however is a black art, covered briefly in Sec. 4.4 of Hull [11] but not generally described in detail in textbooks. Methods include linear interpolations and cubic splines; see for example the survey by Hagan and West [9].

On the other hand market practitioners interested in pricing interest rate derivatives will need to specify an arbitrage-free model for the evolution of the yield curve. So as Björk and Christenssen rightly point out in [3] a question immediately arises: if you choose to implement a particular yield curve generator, which is constantly being applied to newly arriving market data in order to recalibrate the parameters of the model, will the yield curve generator be consistent with the arbitrage-free model specified? That is, if the output of the yield curve generator is used as an input to the arbitrage-free model, will the model then produce yield curves matching the ones produced by the generator? We clarify this point and give an example in the next section.

A number of authors, starting from the work of Björk and Christenssen (see [1], Filipović [7] and Filipović and Teichmann [8]), have studied this problem within the HJM [10] framework in an infinite-dimensional space by looking at specific classes of functions and asking whether these functions are invariant under the HJM dynamics. They obtain some negative results but later extended the class of functions by using an infinite-dimensional version of the Frobenius theorem. They give conditions under which these can be reduced to a finite-dimensional state vector but don't relate this vector to market observables. Other authors (see [5] for the most general set up) have studied conditions on the volatility under which the HJM model admits a reduction to a finite-dimensional Markovian process. But again this Markov process is not identified to market observables.

In contrast to their work the approach of this paper is to model a finite number of market observables and then extend the model to a whole yield curve model in an arbitrage free way. That is we want to find the price at time $t < T$ of a zero-coupon bond $p(t, T)$ with arbitrary maturity T not equal to any of the tenor dates and starting from the dynamics of a Libor or a swap market model. To do that we need at least the continuous time dynamics of some numéraire asset N_t which would be a function of market data and define $p(t, T)$ by the following expectation formula under the $\mathbb{N}$-martingale measure

$$p(t, T) = N(t)\mathbb{E}^{\mathbb{N}}[1/N(T)|\mathcal{F}_t]. \tag{1.1}$$

The first attempts at modelling market observables directly was the work of Sandmann and Sondermann [16] and then extended by Miltersen *et al.* [14, 17] who focused their attention on nominal annual rates. Models of Libor rates were carried out by Brace *et al.* [4], Musiela and Rutkowski [15] and Jamshidian [12] who explicitly points out his desire to depart from the spot rate world. Some models were embedded in the HJM methodology as in [4, 14, 17] and others were simply modelling a finite set of Libor rates but then pricing products that were dependent on these given rates without any need for interpolation, e.g. [12, 15]. Only Schlögl in [18] looks at arbitrage-free interpolations for Libor market models as the underlying model for yield curve dynamics.

In this paper we extend his results to the case where the underlying market consists of some short term yields and a swap market model, that is a process for yields of bonds of short maturities, three or six months, and a collection of observed spot swap rates and their volatilities.

The contents of this paper are as follows. In Sec. 2 we give an example of how simple interpolation algorithms create arbitrage opportunities. We find an arbitrage free mapping from yields to the short rate and show how one could compute in theory the trading strategy that produces arbitrage. In Sec. 3 we outline the co-terminal and co-initial swap market models and introduce a novel interpolation of market rates that allows a simultaneous treatment of the Libor and swap market model while preserving the stochastic nature of the implied short rate and providing an approximate closed form solution for the term structure. Section 4 constructs the term structure from a more realistic market model where there are fewer swap rates than tenor dates. It introduces a consistent bootstrapping procedure that yields the implied volatilities of the "bootstrapped" rates in closed form. This method fits in with the interpolations carried out in Sec. 3 and so we can then obtain the HJM dynamics for the forward rates but driven purely by an SDE process on the market data.

2. Yield Curve Generators and Arbitrage Opportunities

We give an example to illustrate the consistency problem to show how a linear interpolation can introduce arbitrage opportunities. Consider the following bootstrapping procedure on an arbitrage-free model for two zero-coupon bonds maturing at times $T_1 < T_2$. That is let $y_1(t), y_2(t)$ be the yields, so that

$$p(t, T_1) = e^{-y_1(t)(T_1 - t)}, \quad p(t, T_2) = e^{-y_2(t)(T_2 - t)}.$$

Take $T \in (T_1, T_2)$ and define the interpolated price as

$$p(t, T) = e^{-y(t)(T-t)},$$

where

$$y(t) = \frac{T_2 - T}{T_2 - T_1} y_1(t) + \frac{T - T_1}{T_2 - T_1} y_2(t) = (1 - \alpha(T)) y_1(t) + \alpha(T) y_2(t).$$

Taking the T_2-bond as numéraire, absence of arbitrage demands that $p(t, T)/p(t, T_2)$ be a martingale in the T_2-forward measure. Write

$$\frac{p(t, T)}{p(t, T_2)} = \phi(t, y_1(t), y_2(t)) = \exp((\beta_1 + t) y_1(t) + (\beta_2 + \beta_3 t) y_2(t)),$$

where $\beta_1 = -(1 - \alpha)T, \beta_2 = (1 - \alpha), \beta_3 = T + T_2, \beta_4 = -(1 + \alpha)$.

If $y(t) = (y_1(t), y_2(t))$ is a continuous semimartingale with decomposition $y_i(t) = M_i(t) + A_i(t)$, then by the Itô formula

$$\begin{aligned} d\phi/\phi &= (\beta_2 y_1 + \beta_4 y_2)dt + (\beta_1 + \beta_2 t)dA_1 + (\beta_3 + \beta_4 t)dA_2 + (\beta_1 + \beta_2)^2 d\langle y_1\rangle \\ &\quad + (\beta_3 + \beta_4 t)^2 d\langle y_2\rangle + 2(\beta_1 + \beta_2 t)(\beta_3 + \beta_4 t)d\langle y_1, y_2\rangle + dM(t) \\ &\equiv dA(t) + dM(t), \end{aligned}$$

where $M(t)$ is a local martingale. For absence of arbitrage, $A(t)$ must vanish. However, the coefficients β_i depend on T, and it is not generically the case that $A(t) \equiv 0$ for *all* T, given a fixed model for $y(t)$. Thus there will be arbitrage opportunities in the model if we are prepared to trade zero-coupon bonds at interpolated prices. In other words, a linear interpolation method to construct a yield curve is not consistent with a model of the market yields. Presumably market friction in the form of bid-ask spreads is too great to allow these opportunities to be realized in practice.

We want to explore the above example a bit further. Since the short end of the yield curve is constructed from yields of zero-coupon bonds we model the yields directly. Assume the following model, the strong solution to an SDE under the $\mathbb{P}_1$-forward measure, for the yield $y(t)$ of a zero-coupon bond maturing at time $T_1 > 0$,

$$dy(t) = \mu(y(t))dt + \sigma(y(t))dw_1(t),$$

so that

$$p(t, T_1) = \exp(-y(t)(T_1 - t)).$$

We have the following arbitrage free mapping from the yield $y(t)$ to the short rate $r(t)$ defined as

$$r(t) = -\lim_{T\searrow t} \frac{\partial \ln p(t,T)}{\partial T}.$$

Proposition 2.1. *The implied short rate $r(t)$ for $t \in [0, T_1)$ is given by*

$$r(t) = y(t) - (T_1 - t)\mu - 1/2(T_1 - t)^2\sigma^2, \tag{2.1}$$

where μ is the drift of $y(t)$ under the $\mathbb{P}_1$-measure. For $0 < T < T_1$ the implied risk-neutral measure $\mathbb{P}^*$ is given by

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}_1}\right|_{\mathcal{F}_T} = \exp\left(\int_0^T \tau_1\sigma dw_s^1 - \int_0^T 1/2|\tau_1\sigma|^2 ds\right). \tag{2.2}$$

Proof. The short rate is a function of $y(t)$, $r(t) = r(t, y(t))$. By numéraire invariance we require

$$\begin{aligned} p(t,T) &= B_1(t)\mathbb{E}^1[1/B_1(T)|\mathcal{F}_t] \\ &= e^{-y(t)(T_1-t)}\mathbb{E}^1[e^{y(T)(T_1-T)}|\mathcal{F}_t] = \mathbb{E}^*[e^{-\int_t^T r(s)ds}|\mathcal{F}_t], \end{aligned} \tag{2.3}$$

where $\mathbb{P}^*$ denotes the risk-neutral measure.

Now let $\tau_1 = T_1 - t$ and denote $p(t,T) = F(t,y(t))$. By Feynman-Kac formula the right hand side of (2.3) is the probabilistic representation of the PDE determining $F(t,y(t))$ with terminal condition $F(T,y) = 1$. To obtain such PDE we apply the product rule to $e^{y(t)(T_1-t)}F(t,y(t))$ and we cancel the drift term

$$\begin{aligned} d(e^{y(t)(T_1-t)}F) &= e^{y(t)\tau_1}dF + Fd(e^{y(t)\tau_1}) + d\langle F, e^{y(t)\tau_1}\rangle \\ &= \frac{1}{B_1(t)}\left(\frac{\partial F}{\partial t} + (\mu + \tau_1\sigma^2)\frac{\partial F}{\partial y} + 1/2\frac{\partial^2 F}{\partial y^2}\sigma^2\right)dt \\ &\quad + \frac{1}{B_1(t)}\frac{\partial F}{\partial y}\sigma dw_1(t) \\ &\quad + F\frac{1}{B_1(t)}[-y(t) + \tau_1\mu + 1/2\tau_1^2\sigma^2]dt + F\frac{1}{B_1(t)}\tau_1\sigma dw_1(t). \end{aligned}$$

That is

$$d(e^{y(t)(T_1-t)}p(t,T)) = (\tilde{\mathcal{A}}_t F - (y(t) - \tau_1\mu - 1/2\tau_1^2\sigma^2)F)dt + (\cdots)dw_1, \qquad (2.4)$$

where $\tilde{\mathcal{A}}_t = \partial_t + (\mu + \tau_1\sigma^2)\partial_y F + 1/2\sigma^2\partial_{yy}^2 F$. This identifies simultaneously the short rate as $r(t) = y(t) - \tau_1\mu - 1/2\tau_1^2\sigma^2$ and the risk-neutral measure given by (2.2). The "market prices" of risk are $-\tau_1\sigma$, that is the volatility of the $B_1(t)$ over the period $[0,T_1)$. □

In fact Eq. (2.3) defines an arbitrage free value for $p(t,T)$ for $t \le T \le T_1$. For example assuming a Gaussian process for $y(t)$ we obtain the following

Proposition 2.2. *Assume we are given positive constants a, b, σ and a Brownian motion $w(t)$ under the $\mathbb{P}_1$ forward-measure. Define $y(t)$ as the strong solution to*

$$dy(t) = (a - by(t))dt + \sigma dw_t.$$

Then the arbitrage-free price of a zero-coupon bond maturing at $T \le T_1$ is given by

$$p(t,T) = \exp(n(t,T) - m(t,T)y(t)), \qquad (2.5)$$

with

$$\begin{aligned} m(t,T) &= (T_1 - t) - (T_1 - T)e^{-b(T-t)}, \\ n(t,T) &= (T_1 - T)\frac{a}{b}(1 - e^{-b(T-t)}) + \frac{(T_1 - T)^2\sigma^2}{4b}(1 - e^{-2b(T-t)}). \end{aligned}$$

Proof. By standard results the distribution of $y(T)(T_1 - T)$ given $y(t)$ with $t < T$ is $N((T_1 - T)k, (T_1 - T)^2 s^2)$ with

$$\begin{aligned} k &= y(t)\exp(-b(T-t)) + \frac{a}{b}(1 - \exp(-b(T-t))), \\ s^2 &= \frac{\sigma^2}{2b}(1 - \exp(-2b(T-t))). \end{aligned}$$

With these assumptions Eq. (2.3) gives us

$$p(t,T) = e^{-y(t)(T_1-t)}\mathbb{E}^1[e^{y(T)(T_1-T)}|\mathcal{F}_t] = e^{-y(t)(T_1-t)}e^{(T_1-T)k + (T_1-T)^2 s^2/2},$$

giving Eq. (2.5). □

In the next section we show that given a model specified under the forward measure there exists a unique short rate independent of maturity which corresponds to a finite variation process representing a savings account. We use results from Björk [2].

2.1. *Self-financing trading strategies and zero-coupon bond pricing*

Equation (2.3) defines an arbitrage free term structure for $T < T_1$ which depends on the drift and volatility of the yield $y(t)$. We don't need to assume any function for the short rate. In what follows we repeat the steps in Björk's description of short rate models (Sec. 21.2 in [2]) to find an alternative derivation of the term structure PDE in terms of self-financing trading strategies using the $p(t,T)$ and $B_1(t)$ as traded assets which will also allow us to obtain the hedging parameters. Letting $p(t,T) = F(t, y(t))$ and applying Itô to F and $B_1(t) = \exp(-y(t)(T_1 - t))$ we have

$$dp(t,T) = (F_t + \mu F_y + 1/2\sigma^2 F_{yy})dt + \sigma F_y dw_1, \tag{2.6}$$

$$dB_1(t)/B_1(t) = (y(t) - \tau_1\mu + 1/2\tau_1^2\sigma^2)dt - \sigma\tau_1 dw_1. \tag{2.7}$$

Denote the drift and diffusion of p by $m_T = F_t + \mu F_y + 1/2\sigma^2 F_{yy}$ and $b_T = \sigma F_y$, and similarly denote the drift and volatility of $B_1(t)$ by m_1 and b_1 respectively.

Assume we are interested in pricing a zero-coupon bond with maturity $T < T_1$. We form a portfolio based on T and T_1 bonds. Let $u_T(t)$ and $u_1(t)$ denote the proportions of total value held in bonds $p(t,T)$ and $B_1(t)$ respectively, held in a self-financing portfolio at time t. The dynamics of the portfolio are given by

$$dV = V\left(u_T(t)\frac{dp(t,T)}{p(t,T)} + u_1(t)\frac{dB_1(t)}{B_1(t)}\right). \tag{2.8}$$

Substitute (2.6) and (2.7) into (2.8) to obtain

$$dV = V(u_T m_T + u_1 m_1)dt + V(u_T b_T + u_1 b_1)dw_1. \tag{2.9}$$

Let the portfolio weights solve the system

$$\begin{aligned} u_T + u_1 &= 1, \\ u_T b_T + u_1 b_1 &= 0. \end{aligned} \tag{2.10}$$

The first equation is the self-financing property and the second makes the dw_1-term in (2.8) vanish. The value of the SFTS (u_T, u_1) is the solution to (2.10) and is given by

$$u_T = -\frac{b_1}{b_T - b_1}, \quad u_1 = \frac{b_T}{b_T - b_1}.$$

Substitute the solution into (2.9) to obtain

$$dV = V\left(\frac{m_1 b_T - m_T b_1}{b_T - b_1}\right)dt.$$

Absence of arbitrage requires to set the drift above equal to the spot rate r. That is

$$\frac{m_1 b_T - m_T b_1}{b_T - b_1} = r,$$

which can be rewritten as

$$\frac{m_T - r}{b_T} = \frac{m_1 - r}{b_1}. \tag{2.11}$$

The ratio above is independent of the bond and is usually known as the "market prices of risk". Since the risk comes from the randomness in $B_1(t)$ we set the above ratio equal to its volatility $b_1 = -\sigma\tau_1$. Substituting this on the right hand side of (2.11) we obtain

$$r(t) = y(t) - (T_{i+1} - t)\mu - 1/2(T_{i+1} - t)^2\sigma^2,$$

so (2.11) becomes

$$\frac{m_T - r}{b_T} = b_1. \tag{2.12}$$

Equation (2.12) is in fact another way of writing the PDE appearing in the proof of Proposition 2.1 which in the usual short rate models corresponds to the well known Vasicek PDE.

From this discussion we will show next that if $p(t, T)$ is given by linear interpolation of the yields, the implied short rate depends on the maturity date T and so we can create an arbitrage opportunity.

2.2. *Arbitrage opportunities in a linear interpolation*

We now illustrate how to compute an arbitrage opportunity in a bond market where bonds are obtained by log-linear interpolation from a set of benchmark rates. Consider again the small market introduced in the beginning of the section. Let $t < T_1$ and $T \in [T_1, T_2]$ where $T_1 < T_2$ and let $\tau = T - t$ and $\tau_2 = T_2 - t$. We are given a system under the T_2 bond forward measure $\mathbb{P}_2$

$$dy_1 = \mu_{1,2}dt + \sigma_1 dw_2,$$

$$dy_2 = \mu_2 dt + \sigma_2 dw_2,$$

where

$$\mu_{1,2} = (y_1 - y_2 + \tau_2\mu_2 + 1/2\sigma)/\tau_1,$$

$$\sigma = \tau_1\sigma_1 + \tau_2\sigma_2 - 2\tau_1\tau_2\sigma_1\sigma_2,$$

so that $p(t, T_1)/p(t, T_2)$ is a $\mathbb{P}_2$-martingale. Define the interpolated yield for a bond maturing at time T by

$$y(t, T) = \alpha(T)y_1(t) + (1 - \alpha(T))y_2(t),$$

where

$$\alpha(T) = \frac{T_2 - T}{T_2 - T_1}.$$

From Itô it follows that

$$dy(t,T) = \mu_y dt + \sigma_y dw_2(t),$$

where the drift μ_y and volatility σ_y are given by

$$\mu_y = \alpha(T)\mu_{1,2} + (1-\alpha(T))\mu_2, \quad \sigma_y = \alpha(T)\sigma_1 + (1-\alpha(T))\sigma_2.$$

The new bond price is

$$p(t,T) = \exp(-y(t,T)(T-t)).$$

Equations (2.6) and (2.7) now read as

$$dp(t,T) = (y(t,T) - \mu_y\tau + 1/2(\sigma_y\tau)^2)dt - \sigma_y\tau dw_2(t),$$

$$dp(t,T_2) = (y_2(t) - \mu_2\tau_2 + 1/2(\sigma_2\tau_2)^2)dt - \sigma_2\tau_2 dw_2(t),$$

and we can construct a finite variation process like (2.9). Choose proportions of wealth V_t in bonds $B_1(t)$ and $p(t,T)$ be

$$u_T = -\frac{\sigma_2\tau_2}{\sigma_y\tau - \sigma_2\tau_2}, \quad u_1 = \frac{-\sigma_y\tau}{\sigma_y\tau - \sigma_2\tau_2}. \tag{2.13}$$

Hence the system in (2.9) is

$$\frac{dV_t}{V_t} = \frac{-m_2\sigma_y\tau + m_T\sigma_2\tau_2}{\sigma_2\tau_2 - \sigma_y\tau}dt.$$

Letting the spot rate $r(t)$ be

$$\begin{aligned} r(t) &= \frac{-m_2\sigma_y\tau + m_T\sigma_2\tau_2}{\sigma_2\tau_2 - \sigma_y\tau} \\ &= y_2 + \frac{\alpha(T)\tau}{\sigma_2\tau_2 - \sigma_y\tau}\left(\tau_2(\sigma_1\mu_2 - \sigma_2\mu_{1,2}) + \sigma_2\tau_2(y_1 - y_2) + \frac{\sigma_2\tau_2(y_1-y_2)}{\tau}\right) \\ &\quad - \frac{\sigma_2\tau_2\sigma_y\tau}{\tau}, \end{aligned}$$

this depends on the maturity T. Denote by $\phi(t,T)$ and $\phi(t,T_1)$ the number of units of bonds with maturity T and T_1 respectively. They are related from the bond portfolios by $u(t,T) = \phi(t,T)V_T/P(t,T)$ and $u(t,T_1) = \phi(t,T_1)V_T/P(t,T_1)$. Hence from (2.13) these trading strategies are

$$\phi(t,T) = \frac{T-t}{T_1 - T}\frac{V(0)\exp(\int_0^t r(s)ds)}{p(t,T)},$$

$$\phi(t,T_1) = \frac{t-T}{T-T_1}\frac{V(0)\exp(\int_0^t r(s)ds)}{p(t,T_1)}.$$

Since the implied spot rate is a smooth function of T we can find maturities T' and T'' such that $r(t,T') < r(t,T'')$ and so we can borrow money at the cheaper rate and invest in the higher rate "savings account" for as long as $t < \min(T', T'')$ to obtain a profit.

3. Interpolation of Swap Market Models

The data used to construct the zero curve usually from two years onwards consists of swap rates. We now give a brief outline of the swap market model and introduce some notation following Jamshidian [12].

3.1. *Co-terminal swap market model*

We are given a tenor structure $0 < T_1 < \cdots < T_n$ with accrual factors $\theta \in \mathbb{R}^n_+$, $\theta_i = T_{i+1} - T_i$, so that θ_i is the time interval $T_{i+1} - T_i$ expressed according to some day-count convention. For $t \leq T_i$ denote by $B_i(t)$ the price of a zero coupon bond at time t with maturity date T_i and $S_i(t)$ the forward swap rate starting at date T_i and with reset dates T_j for $j = i, \ldots, n-1$. Forward swap rates are related to zero coupon bonds by

$$S_i(t) = \frac{B_i(t) - B_n(t)}{A_i(t)}, \quad 0 \leq t \leq T_i, \tag{3.1}$$

where $A_i(t) \equiv \sum_{j=i+1}^{n} \theta_{j-1} B_j(t)$, the "annuity". We want to obtain prices of zero-coupon bonds from these rates. Towards that aim one can show algebraically from (3.1) and using induction that if we let

$$v_{ij} \equiv v_{ij,n} \equiv \sum_{k=j}^{n-1} \theta_k \prod_{l=i+1}^{k} (1 + \theta_{l-1} S_l), \tag{3.2}$$

$$v_i \equiv v_{ii}, \quad 1 \leq i \leq j \leq n-1, \tag{3.3}$$

we can then express the ratios B_i/B_n (see [12]) for $0 \leq t \leq T_i$, $i = 1, \ldots, n-1$ as

$$\frac{A_i(t)}{B_n(t)} = v_i(t), \tag{3.4}$$

$$\frac{B_i(t)}{B_n(t)} = (1 + v_i(t) S_i(t)). \tag{3.5}$$

We want to be able to interpret $B_i(t)$ as the price of a zero-coupon bond maturing at time T_i, so by setting $B_i(T_i) = 1$ we can deduce from (3.5), for $i = 1, \ldots, n-1$

$$B_n(T_i) = 1/(1 + v_i(T_i) S_i(T_i)). \tag{3.6}$$

We define the auxiliary process $Y_i(t)$

$$Y_i(t) := \frac{B_i(t)}{B_n(t)} = (1 + v_i(t) S_i(t)). \tag{3.7}$$

These Y_i process will be martingales under the $\mathbb{P}_n$ forward measure. From (3.5) we have that $Y_i/Y_{i+1} = 1 + \theta_i L_i$ where $L_i(t)$ denotes the forward Libor rate set at T_i and paying at T_{i+1} and we obtain the relation between Libor and swap rates

$$1 + \theta_i L_i(t) = \frac{1 + v_i(t)S_i(t)}{1 + v_{i+1}(t)S_{i+1}(t)}. \tag{3.8}$$

Notice that a model on the forward swap rates can generate a negative Libor rate $L_k(t)$ for some $k = 1, \dots, n-1$ if it happens that

$$S_k(t) < \frac{v_{k+1}(t)S_{k+1}(t)}{v_k(t)}. \tag{3.9}$$

See for example [6] and [13] for a more detailed discussion.

The swap market model is described by the following: a set of forward swap rates $S_i(0)$ for $i = 1, \dots, n-1$, an $n-1$ dimensional vector of bounded, measurable, locally Lipschitz functions $\psi_i(t, S(t)) \in \mathbb{R}^d$. The swap rate follows a positive martingale under the corresponding "annuity" measure, that is the measure corresponding to using the "annuity" $A_i(t)$ as the numéraire. More precisely, given a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_{in})$ supporting a d-dimensional Brownian motion w_{in}, the swap rate $S_i(t)$ is given by the strong solution to

$$dS_i(t) = S_i(t)\psi_i(t, S(t))dw_{in}(t).$$

In particular $\mathbb{P}_{n-1,n} = \mathbb{P}_n$, the terminal measure. The Radon-Nikodym derivative for the change of measure to the $\mathbb{P}_n$-forward measure is given by

$$\frac{d\mathbb{P}_n}{d\mathbb{P}_{in}} = \frac{A_i(0)}{B_n(0)}\frac{B_n(T_i)}{A_i(T_i)} = \frac{v_i(0)}{v_i(T_i)}. \tag{3.10}$$

One can use backward induction to deduce the form of the drift term for all swap rates under the $\mathbb{P}_n$ measure. We recall from Jamshidian [12].

Proposition 3.1. *(Jamshidian) If we are given a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_n)$, supporting a Brownian motion $w_n \in \mathbb{R}^d$, and data consisting of an $n-1$ dimensional vector of bounded, measurable, locally Lipschitz functions $\psi_i(t, S) \in \mathbb{R}^d$ and initial forward swap rates $S_i(0)$ for $i = 0, \dots, n-1$, then the SDE*

$$dS_i = -S_i\psi_i \sum_{j=i+1}^{n-1} \frac{\theta_{j-1}S_j\psi_j^t}{(1+\theta_{j-1}S_j)}\frac{v_{ij}}{v_i}dt + S_i\psi_i dw_n, \tag{3.11}$$

has a strong solution and v_i and $S_i v_i$ are $\mathbb{P}_n$-martingales for all i where v_i is defined by (4.6).

Similarly we can derive the SDE for $S_i(t)$ taking any other annuity A_k as the numéraire for any $k = 1, \dots, i-1, i+1, \dots, n-1$. Similarly to (3.10) we can define the forward measure $\mathbb{P}_{kn}$ by $d\mathbb{P}_{kn}/d\mathbb{P}_{in} = cv_k/v_i$ where $c = v_{in}(0)/v_{kn}(0)$. The process for $t \leq T_i$, $S_i(t)$ is then given under the $\mathbb{P}_{kn}$ measure by

$$\frac{dS_i(t)}{S_i(t)} = \psi_i(t)\left(\sum_{l=k+1}^{n-1} \frac{\theta_{l-1}S_l(t)\psi_l(t)}{1+\theta_{l-1}S_l(t)}\frac{v_{kl}(t)}{v_k(t)} - \sum_{j=i+1}^{n-1} \frac{\theta_{j-1}S_j(t)\psi_j(t)}{1+\theta_{j-1}S_j(t)}\frac{v_{ij}(t)}{v_i(t)}\right)dt + \psi_i(t)dw_{kn}. \quad (3.12)$$

Notice that by setting $l = n-1$ we recover (3.11). We will make use of these dynamics in Sec. 4.

It seems natural to find an arbitrage-free interpolation of the yield curve by extending the swap market model.

3.2. *Interpolation*

We want to price zero-coupon bonds given a numéraire asset with price process $N(t)$ by applying (1.1) which we rewrite below,

$$p(t,T) = N(t)\mathbb{E}^i[1/N(T)|\mathcal{F}_t].$$

The problem with market models is that it doesn't give us continuous time dynamics for any bond in the tenor structure. Recall that given a Libor or a swap market model we can obtain the process for any ratio of zero coupon bonds by defining the auxiliary process $Y_i^j(t)$ with $i < j$, by

$$Y_i^j(t) \equiv \prod_{m=i}^{j-1}(1+\theta_m L_m(t)) = \frac{B_i(t)}{B_j(t)}, \quad (3.13)$$

in the LMM and

$$Y_i^j(t) \equiv \frac{1+v_i(t)S_i(t)}{1+v_j(t)S_j(t)} = \frac{B_i(t)}{B_j(t)}, \quad (3.14)$$

in the SMM for $j < n$ and $Y_i^n(t) = 1 + v_i(t)S_i(t)$. Furthermore from Proposition 3.1 the process $Y_i^j(t)$ follows a $\mathbb{P}_j$-martingale. We want to interpret B_i as the zero-coupon bond maturing at time T_i, that is we need $B_i(T_i) = 1$. We thus require $B_i(T_i)/B_j(T_i) = 1/B_j(T_i)$ for $i < j$ and, since $B_i(T_i)/B_j(T_i) = Y_i^j(T_i)$ we can define the random variables $B_j(T_i)$ uniquely by

$$B_j(T_i) = \frac{1}{Y_i^j(T_i)}. \quad (3.15)$$

Summarizing, a market model provides us with the continuous time process of any ratio of zero coupon bonds as in (3.13) and (3.14) but we only have the prices of each individual bond at a finite number of dates as Eq. (3.15) shows.

To apply (1.1) for pricing zero coupon bonds we need the *continuous* time dynamics of a numéraire, say $B_n(t)$, so we need to extend the definition of $B_n(.)$

given in (3.15) with $j = n$ to times $t \in (T_i, T_{i+1})$ for $i = 1, \ldots, n-1$ by interpolating the random variables in (3.15) and setting $B_n(T_n) = 1$. Note that in this context we mean interpolation with respect to present date, that is in the t parameter, and not with respect to maturity, that is in the T parameter.

This is equivalent to defining "short bonds" as Schlögl [18] suggests, that is arbitrarily defining $B_i(t)$ for *all* $i = 1, \ldots, n$ where $t \in (T_{i-1}, T_i)$ but letting $B_i(T_{i-1}) = (1 + \theta_{i-1} L_{i-1}(T_{i-1}))^{-1}$ and $B_i(T_i) = 1$.

These approaches are equivalent because having defined $B_n(t)$ for $t \in [0, T_n]$, it automatically defines processes for bonds $B_i(t)$ for $0 \leq t \leq T_i$ and $T_i \in \mathbb{T}$ since from, say (3.14), we can set

$$B_i(t) = Y_i(t) B_n(t). \tag{3.16}$$

Alternatively having all "short bonds" we can obtain $B_n(t)$ for all t. Say for example $t \in [T_i, T_{i+1}]$, then we take the previously defined bond $B_{i+1}(t)$ and let

$$B_n(t) = B_{i+1}(t)/Y_{i+1}(t).$$

Schlögl proposes two interpolations for Libor market models. The first one is a linear interpolation which produces a piecewise deterministic short rate for $t \in [T_i, T_{i+1})$

$$B_{i+1}(t) = \frac{1}{1 + \alpha(t) L_i(T_i)},$$

where

$$\alpha(t) = \frac{T_{i+1} - t}{T_{i+1} - T_i},$$

and a second interpolation which introduces volatility in the short rate. It exploits the fact that any Libor rate, say L_i, has a known transition probability distribution under two forward measures $\mathbb{P}_i$ and $\mathbb{P}_{i+1}$ corresponding to using B_i and B_{i+1} as numéraires respectively. The linear interpolation will be extended to the swap market model. However the second interpolation proposed in [18] used to introduce volatility in the short rate can not be extended to the swap market model. Therefore we propose an additional method which works well for the swap market model and also fits in with the Libor market model. In addition since only martingale properties will be used the method also works for Lévy market models.

We use the fact that the data for the short end of the curve consists of yields to suggest the following interpolation process for $B_n(t)$

$$B_n(t) = \frac{1}{e^{y(t)(T_{k+1}-t)} Y_{k+1}(t)}, \quad t \in [T_k, T_{k+1}], \tag{3.17}$$

where $y(t)$ is the strong solution of some SDE driven by a Brownian motion under the $\mathbb{P}_{k+1}$-forward measure

$$dy(t) = \mu_{k+1}(y(t))dt + \sigma(y(t))dw_{k+1},$$

with initial condition

$$y(T_k) = \frac{\ln(1 + \theta_k L_k(T_k))}{T_{k+1} - T_k}, \quad \text{for } k = 0, \dots, n-1.$$

The boundary conditions $B_n(T_k) = 1/Y_k(T_k)$ and $B_n(T_n) = 1$ are indeed satisfied since we have

$$e^{y(T_k)(T_{k+1}-T_k)} = 1 + \theta_k L_k(T_k) = \frac{1 + v_k(t)S_k(t)}{1 + v_{k+1}(t)S_{k+1}(t)}. \tag{3.18}$$

The interpolation in (3.17) is equivalent to interpolating the "short bonds" since applying (3.16) with $i = k+1$

$$B_{k+1}(t) = Y_{k+1}(t)B_n(t) = e^{-y(t)(T_{k+1}-t)} \quad \text{for } t \in [T_k, T_{k+1}].$$

The process $y(t)$ is defined for $t \in [0, T_n]$ but the idea is to interpret it as a short term yield, hence we can use the volatility of the yields at the short end of the curve to specify the volatility σ which remains unchanged for all t. Unfortunately the drift depends on the section of the tenor structure where the process lies, so for $t \in [T_k, T_{k+1})$ the process is evolved under the $\mathbb{P}_{k+1}$ measure and is not specified at this point.

We can use the results from the second section to define a short rate for $t \in [T_k, T_{k+1})$ by

$$r(t) = y(t) - (T_{k+1} - t)\mu_{k+1} - 1/2(T_{k+1} - t)^2\sigma^2.$$

This implied short rate however is discontinuous. At $t = T_k-$

$$r(T_k-) = y(T_k-),$$

and at $t = T_k$

$$r(T_k) = y(T_k) - (T_{k+1} - T_k)\mu_{k+1} - 1/2(T_{k+1} - T_k)^2\sigma^2,$$

where $y(T_k) = \frac{1}{T_{k+1}-T_k}\ln(1 + \theta_k L_k(T_k))$. Hence

$$\Delta r(T_k) = (y_k - y_{k-}) - \tau_{k+1}\mu_{k+1} - 1/2\tau_{k+1}^2\sigma^2.$$

We have the ability to freely choose the drift for the $y(t)$ process which allows us to rule out discontinuities in the short rate. For example we could choose a process for $y(t)$ with $t \in [T_k, T_{k+1})$ such as

$$dy(t) = (a_k + \mu(y(t)))dt + \sigma(y(t))dw_{k+1}(t),$$

where a_k is a $\mathcal{F}_{T_k}$ measurable random variable that satisfies

$$a_k = \frac{(y_k - y_{k-}) - \Delta_k \frac{1}{2}\sigma^2(y_k)}{\Delta_k} - \mu(y_k),$$

and SDE becomes

$$dy(t) = \left(\frac{1}{\Delta_k}(y_k - y_{k-}) - \frac{1}{2}\sigma^2(y_k)\right)dt + \sigma(y(t))dw_{k+1}(t).$$

However from looking at the expectation definition above and the reset condition at each T_k this assumption would complicate the calculations such as the ones in Sec. 2.1.

We extend the self-financing strategies found in Sec. 2.1 to bonds with maturity $T \in [T_i, T_{i+1})$. Denote by Δ_j the number of bonds B_j held in the portfolio and let it be

$$\Delta_j(t) = \mathbb{I}_{[T_j,T_{j+1})}(t), \quad j = 1, \ldots, i, \tag{3.19}$$

$$\Delta_T(t) = \frac{u_T V(t)}{p(t,T)}, \quad \text{for } t \in [T_i, T], \tag{3.20}$$

$$\Delta_{i+1} = \frac{u_{i+1} V(t)}{B_{i+1}(t)}, \tag{3.21}$$

with (u_T, u_{i+1}) as in (2.10) with $i+1$ instead of 1. We call the implied numéraire corresponding to this SFTS the rolling horizon numéraireand its martingale measure is denoted by $\mathbb{P}^{SL}$ since this corresponds to the Spot Libor measure in Jamshidian [12]. Note that it will be used for swap market models as well.

For $T \in [T_i, T_{i+1})$ the price of a zero-coupon bond $p(t,T)$ using this rolling horizon numéraire is given in the following proposition

Proposition 3.2. *Define* $q_{j-1} = e^{-y_{T_{j-1}}(T_j - T_{j-1})} = B_j(T_{j-1})$ *for* $j = 1, \ldots, n$. *Then for* $k < i$, $t \in [T_k, T_{k+1}]$ *and* $T \in [T_i, T_{i+1})$, *then price of a zero-coupon bond* $p(t,T)$ *under* $\mathbb{P}^{SL}$ *is given by*

$$\begin{aligned} p(t,T) &= B_{i+1}(t)\mathbb{E}^{SL}[1/B_{i+1}(T)] \\ &= B_k(t)\mathbb{E}^{k+1}(q_{k+1}\mathbb{E}^{k+2}(q_{k+2}\cdots\mathbb{E}^i(q_i\mathbb{E}^{i+1} \\ &\quad \times (1/B_{i+1}(T)|\mathcal{F}_{T_i})|\mathcal{F}_{T_{i-1}})\cdots|\mathcal{F}_{T_{k+1}})|\mathcal{F}_t), \end{aligned} \tag{3.22}$$

and for $k = i$

$$p(t,T) = B_{i+1}(t)\mathbb{E}^{i+1}[1/B_{i+1}(T)|\mathcal{F}_t].$$

Proof. The second equality in (3.22) follows from the constructions of the rolling horizon numéraireand $\mathbb{P}^{SL}$ given in Eqs. (3.19) to (3.21). Now $p(s,T)/B_{i+1}(s)$ is a P_{i+1}-martingale for $s \in [T_i, T_{i+1}]$, so by definition

$$p(T_i, T) = B_{i+1}(T_i)\mathbb{E}^{i+1}\left[\frac{1}{B_{i+1}(T)}\middle|\mathcal{F}_{T_i}\right].$$

Now $p(s,T)/B_i(s)$ is a P_i-martingale for $s \in [T_{i-1}, T_i]$, hence substituting the above expression in the expectation formula

$$\begin{aligned} p(T_{i-1}, T) &= B_i(T_{i-1})\mathbb{E}^i[p(T_i,T)|\mathcal{F}_{T_{i-1}}] \\ &= B_i(T_{i-1})\mathbb{E}^i\left[B_{i+1}(T_i)\mathbb{E}^{i+1}\left[\frac{1}{B_{i+1}(T)}\middle|\mathcal{F}_{T_i}\right]\middle|\mathcal{F}_{T_{i-1}}\right] \end{aligned}$$

$$[3pt] = B_i(T_{i-1})\mathbb{E}^i[B_{i+1}(T_i)\mathbb{E}^{i+1}[e^{y(t)(T_{i+1}-T)}|\mathcal{F}_{T_i}]|\mathcal{F}_{T_{i-1}}]$$
$$= q_i\mathbb{E}^i[q_{i+1}\mathbb{E}^{i+1}[e^{y(t)(T_{i+1}-T)}|\mathcal{F}_{T_i}]|\mathcal{F}_{T_{i-1}}].$$

The result follows by backward iteration. □

3.3. *Examples*

In this section we give three explicit formulas for zero-coupon bonds of arbitrary maturity. The first one rewrites the linear interpolation as in Schlögl [18] in the context of the SMM and shows that the implied spot rate has zero volatility between tenor dates. The next two uses a stochastic process for $y(t)$.

3.3.1. *Linear interpolation*

We apply the linear interpolation carried out by Schlögl in [18] to the swap market model to obtain closed form solutions to zero-coupon bond prices. Define $B_n(t)$ as follows: for $t \in [T_i, T_{i+1})$,

$$B_n(t) = \frac{1}{(1+\alpha(t)(y(T_i)-1))(1+v_{i+1}(t)S_{i+1}(t))},$$

where

$$\alpha(t) = \frac{T_{i+1}-t}{T_{i+1}-T_i}, \quad \text{and} \quad y(T_i) = \frac{1+v_i(T_i)S_i(T_i)}{1+v_{i+1}(T_i)S_{i+1}(T_i)}.$$

With this definition one can check that indeed we meet the boundary conditions $B_n(T_j) = (1+v_j(T_j)S_j(T_j))^{-1}$ for $j = i, i+1$ as required by relation (3.15). We have the following result.

Proposition 3.3. *With the above definition of $B_n(t)$ we have, for any $T_k \le t < T_{k+1} \le T_i \le T \le T_{i+1}$,*

$$p(t,T) = \frac{1+v_{i+1}(t)S_{i+1}(t)+\alpha(T)(v_i(t)S_i(t)-v_{i+1}(t)S_{i+1}(t))}{(1+\alpha(t)(y(T_k)-1))(1+v_{k+1}(t)S_{k+1}(t))}. \tag{3.23}$$

Proof.

$$\begin{aligned}\frac{p(t,T)}{B_n(t)} &= \mathbb{E}^n_t[B_n(T)^{-1}]\\ &= \mathbb{E}^n_t[(1+\alpha(T)(y(T_i)-1))(1+v_{i+1}(T)S_{i+1}(T))]\\ &= \mathbb{E}^n_t[1+\alpha(T)\mathbb{E}^n_{T_i}[(y(T_i)-1)(1+v_{i+1}(T)S_{i+1}(T))]\\ &= \mathbb{E}^n_t[1+v_{i+1}(T_i)S_{i+1}(T_i)+\alpha(T)(v_i(T_i)S_i(T_i)-v_{i+1}(T_i)S_{i+1}(T_i))]\\ &= 1+v_{i+1}(t)S_{i+1}(t)+\alpha(T)(v_i(t)S_i(t)-v_{i+1}(t)S_{i+1}(t)).\end{aligned}$$

The result now follows. □

The implied spot rate is piecewise deterministic.

Proposition 3.4. *We have for $t \in [T_k, T_{k+1})$ and $T \in [T_i, T_{i+1})$ the forward rate $f(t,T)$ and the spot rate $r(t)$ given by*

$$f(t,T) = \frac{(v_i(t)S_i(t) - v_{i+1}(t)S_{i+1}(t))}{(T_{i+1} - T_i)[1 + v_{i+1}(t)S_{i+1}(t) + \alpha(T)(v_i(t)S_i(t) - v_{i+1}(t)S_{i+1}(t))]},$$

and

$$r(t) = \frac{(v_k(T_k)S_k(T_k) - v_{k+1}(T_k)S_{k+1}(T_k))}{(T_{k+1} - T_k)[1 + v_{k+1}(T_k)S_{k+1}(T_k) + \alpha(t)(v_k(T_k)S_k(T_k) - v_{k+1}(T_k)S_{k+1}(T_k))]}.$$

Proof. Forward rates are related to bond prices by $f(t,T) = -\partial \ln p(t,T)/\partial T$ and since $\partial\alpha(T)/\partial T = -1/(T_{i+1} - T_i)$ the first result follows. We then obtain the short rate by setting $t = T \in (T_k, T_{k+1})$. □

In the next two examples we add some additional volatility and compute approximate solutions to (3.22) by assuming the familiar Ornstein-Ulhenbeck and CIR processes for $y(t)$.

3.3.2. *Example: Ornstein-Uhlenbeck*

Note that for ease of notation in the next two examples the pricing formula for zero-coupon bonds will be derived using the discount factor $(1 + \theta_i L_i(T_i))$ and its martingale properties but this really should be seen in terms of the auxiliary processes $Y_i(T_i)/Y_{i+1}(T_{i+1})$ so that the derivation works equally well for the SMM.

Now we proceed with the example. Assume an Ornstein-Uhlenbeck process for $y(t)$ under the $\mathbb{P}_{i+1}$-measure

$$dy(t) = (a - by(t))dt + \sigma dw_{i+1}(t),$$

where a, b, σ are constants and

$$y(T_i) = \frac{\ln(1 + \theta_i L_i(T_i))}{T_{i+1} - T_i}.$$

We can compute an approximate solution to the term structure in (3.22) for $t \in [T_k, T_{k+1})$ and $T \in [T_i, T_{i+1})$,

$$p(t,T) = \frac{\exp(A_i(T_i, T, T_{i+1}) - y(t)(T_{k+1} - t))}{\tilde{Y}^i_{k+1}(t)},$$

where

$$A_i(T) = (T_{i+1} - T)\frac{a}{b}(1 - e^{-b(T-T_i)}) + (T_{i+1} - T)^2\frac{\sigma^2}{4b}(1 - e^{-2b(T-T_i)}) \tag{3.24}$$

$$C_i(T) = (T_{i+1} - T_i) - (T_{i+1} - T)e^{-b(T-T_i)}, \tag{3.25}$$

$$\tilde{Y}^i_{k+1}(t) = \prod_{j=k+1}^{i-1} (1 + \theta_j L_j(t))(1 + \theta_i C_i(T) L_i(t)/(T_{i+1} - T_i)).$$

In particular we have at time zero and $T \in [T_i, T_{i+1})$ we have

$$p(0,T) = \frac{\exp\left((T_{i+1} - T)\frac{a}{b}(1 - e^{-b(T-T_i)}) + (T_{i+1} - T)^2 \frac{\sigma^2}{4b}(1 - e^{-2b(T-T_i)})\right)}{(1 + \theta_i(1 - \alpha(T)e^{-b(T-T_i)})L_i(0))Y^i_0(0)}. \tag{3.26}$$

Indeed the first expectation in the chain is

$$p(T_i, T) = e^{-y(T_i)(T_{i+1}-T_i)}\mathbb{E}^{i+1}(e^{y(t)(T_{i+1}-T)}|\mathcal{F}_{T_i}).$$

We know from standard theory that for $t > T_i$ $y(t)$ is normally distributed with

$$y(t) \sim N\left(e^{-b(t-T_i)}y(T_i) + \frac{a}{b}(1 - e^{-b(t-T_i)}), \frac{\sigma^2}{2b}(1 - e^{-2b(t-T_i)})\right),$$

and that if a variable is normally distributed with $Y \sim N(m, s^2)$ we have that $E[e^Y] = \exp(m + s^2/2)$.

Combining these results we have

$$p(T_i, T) = e^{-y(T_i)(T_{i+1}-T_i)}\mathbb{E}^{i+1}(e^{y(t)(T_{i+1}-T)}|\mathcal{F}_{T_i}) = \exp(A_i - C_i y(T_i)),$$

with A_i and C_i as in (3.24) and (3.25). According to the boundary condition this is equal to

$$e^{A_i - C_i y(T_i)} = e^{A_i(T)}(1 + \theta_i L_i(T_i))^{-\frac{C_i}{\Delta_i}}.$$

We make the following approximation

$$(1 + \theta_i L_i(T_i))^{\frac{C_i(T)}{\Delta_i}} \approx (1 + \tilde{\theta}_i(T) L_i(T_i)),$$

where $\tilde{\theta}_i(T) = \frac{\theta_i C_i(T)}{\Delta_i}$. Note that C will be a positive function as long as $b > \ln \alpha(T)/(T - T_i)$. Note $\tilde{\theta}_i(T_i) = 0$ and $\tilde{\theta}_i(T_{i+1}) = \theta_i$.

The next problem becomes that of computing

$$\mathbb{E}^i\left(\frac{1}{1 + \tilde{\theta}_i(T)L_i(T_i)}\middle| \mathcal{F}_{T_{i-1}}\right).$$

We make a second approximation $(1 + \tilde{\theta}_i(t)L_i(t))^{-1} \approx B_{i+1}(t)/B_i(t)$ so that

$$\mathbb{E}^i\left(\frac{1}{1 + \tilde{\theta}_i(T)L_i(T_i)}\middle| \mathcal{F}_{T_{i-1}}\right) \backsimeq \frac{1}{1 + \tilde{\theta}_i(T)L_i(T_{i-1})},$$

since B_{i+1}/B_i is a $\mathbb{P}_i$-martingale. The next expectation becomes

$$\mathbb{E}^{i-1}\left(\frac{1}{(1+\theta_{i-1}L_{i-1}(T_{i-1}))(1+\tilde{\theta}_i(T)L_i(T_{i-1}))}\middle|\mathcal{F}_{i-2}\right)$$
$$=\frac{1}{(1+\theta_{i-1}L_{i-1}(T_{i-2}))(1+\tilde{\theta}_i(T)L_i(T_{i-2}))},$$

and so on. Hence our bond pricing formula simply becomes

$$p(t,T)=\frac{e^{A_i}}{e^{y(t)(T_{k+1}-t)}\prod_{j=k+1}^{i-1}(1+\theta_j L_j(t))(1+\theta C_i L_i(t)/\Delta_i)}.$$

In particular we obtain (3.26) at time zero.

The formula is an interpolation of today's Libor rates and function of the yield and volatility of the "short rate" $y(t)$.

Remark. The above formula works just as well if our initial data consists of swap rates by using

$$1+\theta_k L_k(0)=\frac{1+v_k(0)S_k(0)}{1+v_{k+1}(0)S_{k+1}(0)},$$

where v_k is defined in (3.3). We then have (3.26) expressed as

$$p(0,T)=\frac{\exp\left((T_{i+1}-T)\frac{a}{b}(1-e^{-b(T-T_i)})+(T_{i+1}-T)^2\frac{\sigma^2}{4b}(1-e^{-2b(T-T_i)})\right)}{\left(1+(1-\alpha(T)e^{-b(T-T_i)})\left[\frac{1+v_i(0)S_i(0)}{1+v_{i+1}(0)S_{i+1}(0)}-1\right]\right)\frac{1+v_0(0)S_0(0)}{1+v_i(0)S_i(0)}}. \tag{3.27}$$

Figures 1 and 2 show examples of all three interpolations of Libor market models studied so far; the two by Schlögl [18] and the above interpolation under the Orstein-Uhlenbeck assumption [formula (3.26)]. Figure 1 displays the instantaneous forwards $f(t)$, where

$$f(t)=-\frac{\partial}{\partial T}\ln P(0,T),$$

and Fig. 2 the discount factors implied by all three interpolations. The line "First interpolation" corresponds to the linear interpolation, the line "Second interpolation" is Schlögl's second interpolation with a flat term structure of Libor volatilities at 20%. The line "Third interpolation" corresponds to the Ornstein-Uhlenbeck interpolation with coefficients $a=b=0.1$ and $\sigma=0.2$.

The three interpolations produce continuous discount factors with respect to their maturity. Note that there is the implicit assumption that the parameters a,b for the Ornstein-Uhlenbeck process are the same under all forward measures so the humped shape is the same between tenors but this need not be true.

It can be seen from the plots that Schlögl's interpolations produce better behaved forward curves than the Ornstein-Uhlenbeck interpolation. It is also important to note that in Schlögl's interpolations a Libor Market Model alone is all that is needed to compute discount factors for *any* maturity. On the other hand, in the formula presented above (3.26) an additional stochastic process $x(t)$ is required to compute

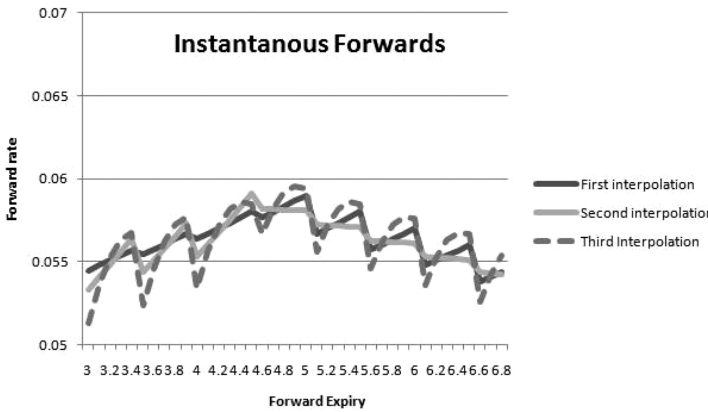

Fig. 1. Comparison of the instantaneous forwards given by the three interpolations with quarterly accrual factor. The horizontal axis represents forward's expiry T and the vertical axis the instantaneous forward rate $-\frac{\partial}{\partial T}\ln P(0,T)$. The "First interpolation" line are forwards from the linear interpolation, the line "Second interpolation" are from Schlögl's second interpolation with Libor volatilities $\lambda = 20\%$ for all rates. The "Third interpolation" line are forwards from the Ornstein-Uhlenbeck interpolation with parameters $a = b = 0.1$ and $\sigma = 0.2$.

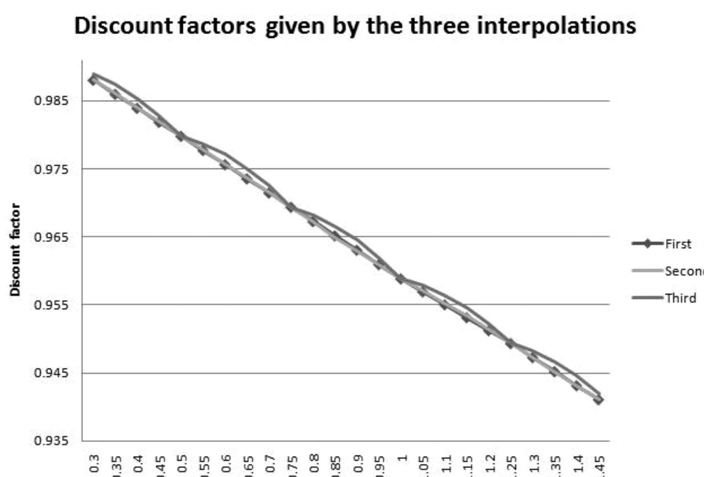

Fig. 2. Comparison of the three interpolations with $L_i(0) = 4\%$ and $\theta_i = 0.25$ for $i = 1, \ldots, 6$. The horizontal axis represents maturity T and the vertical axis the discount factor $P(0,T)$. The "First" line represents the linear interpolation. The "Second" line represents Schlögl's second interpolation with Libor volatilities $\lambda = 20\%$ for all Libor rates. The "Third" line represents the Ornstein-Uhlenbeck interpolation with parameters $a = b = 0.1$ and $\sigma = 0.2$.

discount factors. However the main advantage of the method presented in this section is that it is applicable to a swap market model while also producing a stochastic short rate. Perhaps a refinement of any of the approximations made to arrive at formula (3.26) will produce better behaved forward curves.

3.3.3. *Example: CIR model*

This example computes bond prices with a CIR model for $y(t)$ to rule out the possibility of negative short term rates.

Assume (a, b, σ) are given constants and define $y(t)$ as the strong solution to

$$dy(t) = (a - by(t))dt + \sigma\sqrt{y(t)}dw_t.$$

We can compute an approximate solution to the term structure in (3.22) for $t \in [T_k, T_{k+1})$ and $T \in [T_i, T_{i+1})$ given by

$$p(t, T) = \frac{\exp(m_i(T) - y(t)(T_{k+1} - t))}{\tilde{Y}^i_{k+1}(t)},$$

where, denoting $\Delta_i \equiv T_{i+1} - T_i$,

$$m_i(T) = \frac{a}{2}(T^2 - T_i^2 - 2T_{i+1}(T - T_i) + \int_{T_i}^{T} n_i(s)ds, \tag{3.28}$$

$$n_i(T) = \frac{\Delta_i(T_{i+1} - T)\sigma^2(e^{bT_i} - e^{bT}) - 2b((T_{i+1} - T)e^{bT_i} - \Delta_i e^{bT})}{\sigma^2(T_{i+1} - T)(e^{bT_i} - e^{bT}) + 2be^{bT}}, \tag{3.29}$$

$$\tilde{Y}^i_{k+1}(t) = \prod_{j=k+1}^{i-1}(1 + \theta_j L_j(t))(1 + \theta n_i(T)L_i(t)/\Delta_i).$$

As before we need to compute

$$p(T_i, T) = e^{-y(T_i)(T_{i+1}-T_i)}\mathbb{E}^{i+1}(e^{y(t)(T_{i+1}-T)}|\mathcal{F}_{T_i}).$$

This time we use Riccati equations. Assume the solution F is of the form $F(t, y(t)) = \exp(m_i(t) - n_i(t)y(t))$, then by Feynman-Kac the above is the probabilistic solution to the following PDE

$$F_t + F_y(a - by + \tau_{i+1}\sigma^2 y) + 1/2\sigma^2 y F_{yy} - rF = 0,$$
$$F(T, y) = 1,$$

where $r \equiv y - \tau(a - by) - 1/2\tau^2\sigma^2$ and $\tau = T - t$. By assumption on F it becomes

$$\dot{m} - \dot{n}y - n(a - by + \tau_{i+1}\sigma^2 y) + 1/2\sigma^2 y n^2 - ((1 + b\tau - 1/2\sigma^2\tau^2)y - \tau a) = 0.$$

We can separate into a system of ODE's with boundary conditions $n(T, T) = m(T, T) = 0$

$$\dot{n} = (b - \tau_{i+1}\sigma^2)n + 1/2\sigma^2 n^2 - (1 + b\tau_{i+1} - 1/2\tau_{i+1}^2\sigma^2),$$
$$\dot{m} = a(n - \tau_{i+1}).$$

We invoke Maple to give us an explicit solution which is given by (3.28) and (3.29). The rest of the proof follows along the same lines as the example in Sec. 3.3.2.

4. Missing Swap Rates

In reality market data provides us with fewer swap rates than underlying discount factors. Therefore in this section we study how to bootstrap the missing data in an arbitrage-free way so that the induced volatility process is consistent with the interpolation method. To be precise, we consider the problem where given a SMM, that is a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ supporting a d-dimensional Brownian motion w, but this time $\mathbb{P}$ being the physical measure, we have today's rates $\{S_i(0)\}_{i=1}^{n-1}$ and bounded measurable volatility functions $\{\psi_i(t)\}_{i=1}^{n-1}$, *except* the value $S_k(0)$ and the volatility function ψ_k for some $k \in \{1, \ldots, n-1\}$. We want to deduce a consistent method for obtaining these objects under arbitrage-free restrictions while preserving the positivity of the implied Libor rate $L_k(0)$. We first deal with the case where we are missing one swap rate and then move on to see how we can extend the method to allow for more missing swap rates. We give an example where we compute the value for the missing rates and their volatilities explicitly. Finally we put together these results to obtain an HJM model from market data.

4.1. *One missing swap*

Before we go on to the main result of this section it is worth discussing a motivating example. Let's see what happens if we try a naive interpolation method to find a missing swap rate. Say we have $(S_{n-1}(0), \psi_{n-1})$ and $(S_{n-3}(0), \psi_{n-3})$ with constant volatilities $\psi_1, \psi_3 > 0$ available and we define $S_{n-2}(t)$ by

$$S_{n-2}(t) = \frac{1}{2}(S_{n-3}(t) + S_{n-1}(t)).$$

To simplify notation let $n-1 \equiv 1, n-2 \equiv 2, n-3 \equiv 3$. If we apply Itô's rule to the above under the $\mathbb{P}_n$-forward measure corresponding to the bond B_n as the numéraire, we have

$$dS_2(t) = \frac{1}{2}S_3(t)\psi_3\mu_{3,n}dt + \frac{1}{2}(\psi_3 S_3 + \psi_1 S_1)dw_n, \tag{4.1}$$

so that $\psi_2 := \frac{1}{2S_2}(\psi_3 S_3 + \psi_1 S_1)$. The process for $S_2(t)$ under $\mathbb{P}_n$ ought to be

$$dS_2 = -\frac{\theta_1\theta_2\psi_1\psi_2 S_1 S_2}{\theta_2 + \theta_1(1 + \theta_2 S_1)}dt + \psi_2 dw_n. \tag{4.2}$$

But from (4.1) it's

$$dS_2 = -\frac{S_3\psi_3}{2}\left(\frac{\theta_2 S_2\psi_2}{1+\theta_2 S_2}\frac{v_{3,2}}{v_3} + \frac{\theta_1 S_1\psi_1}{1+\theta_1 S_1}\frac{v_{3,1}}{v_3}\right)dt + \psi_k dw_n. \tag{4.3}$$

We need to compare the drift of (4.2) with that of (4.3) after substituting in our interpolation for S_2 in (4.3). Simple numerical experiments show that these are not equal. For example taking $\psi_1 = \psi_3 = 1/2$ and $S_1 = S_2 = 0.03$ gives a drift of $-3.7e-5$ when it should be $1.1e-4$. These are very small values but they raise the question, at least at the theoretical level, of how can swap rates be bootstrapped without creating arbitrage opportunities. The reason for this inconsistency is that if we start from an arbitrary interpolation function for the missing swap as a function of the available data we are simultaneously specifying a volatility and a drift term for the missing swap rate S_k and this drift may not coincide with the drift condition derived by Jamshidian [12] required for an arbitrage-free term structure.

In order to find our missing swap we first need the dynamics of $S_i(t)$ for $t \leq T_i < T_k$ under the forward swap measure $\mathbb{P}_{k,n}$ given by (3.12) in Sec. 3.1 and which we rewrite below

$$dS_i(t) = S_i(t)\psi_i\mu_{i,k}dt + S_i(t)\psi_i dw_{k,n}, \tag{4.4}$$

where

$$\mu_{i,k} \equiv \sum_{l=k+1}^{n-1} \frac{\theta_{l-1}S_l(t)\psi_l(t)}{1+\theta_{l-1}S_l(t)}\frac{v_{kl}(t)}{v_k(t)} - \sum_{j=i+1}^{n-1} \frac{\theta_{j-1}S_j(t)\psi_j(t)}{1+\theta_{j-1}S_j(t)}\frac{v_{ij}(t)}{v_i(t)}, \tag{4.5}$$

$w_{k,n}$ is a $\mathbb{P}_{k,n}$-Brownian motion, and for $1 \leq i \leq j \leq n-1$

$$v_{ij}(t) \equiv \sum_{k=j}^{n-1} \theta_k \prod_{l=i+1}^{k} (1+\theta_{l-1}S_l(t)), \quad v_i(t) \equiv v_{ii}(t). \tag{4.6}$$

It is important to note that $v_{ii}(t)$ at time $t \leq T_{i+1}$ is a deterministic function of $S_{i+1}(t), \ldots, S_{n-1}(t)$ only, so that $S_i(t)$ doesn't enter the expression.

We define the missing swap $S_k = f(t, S_{k+1}, \ldots, S_{n-1})$ as a function f to be determined of the remaining swap rates. This function represents an interpolating function and it implies a particular volatility ψ_k for the swap rate S_k. The next proposition gives the consistency conditions for the pair (f, ψ) so that the underlying economy of zero-coupon bonds B_i for all $i = 1, \ldots, n-1$ is arbitrage-free.

Proposition 4.1. *Let us assume be given bounded continuous $\mathbb{R}_+$-valued functions $\psi_1(t), \ldots, \psi_{k-1}(t), \psi_{k+1}(t), \ldots, \psi_{n-1}(t)$, a vector in $\mathbb{R}^{n-k-2}$ of initial values*

$$S_1(0), \ldots, S_{k-1}(0), S_{k+1}(0), \ldots, S_{n-1}(0),$$

and a bounded, continuous, $\mathbb{R}_+$-valued function $g_k : \mathbb{R}^{n-k-2} \to \mathbb{R}^+$ (the subscript k is there just to note that g_k is a particular boundary condition defined for time T_k). Define

$$\psi_k := \sum_{j=k+1}^{n-1} S_j\psi_j \frac{1}{f}\frac{\partial f}{\partial S_j}, \tag{4.7}$$

where the function $f \in C^{1,2}([0, T_k]; \mathbb{R}^{n-k-2})$ satisfies the following PDE (assuming a solution exists)

$$\frac{\partial f}{\partial t} + \mathcal{D}_{k+1}f = 0, \tag{4.8}$$
$$f(T_k, S_{k+1}, \ldots, S_{n-1}) = g_k(S_{k+1}, \ldots, S_{n-1}),$$

with

$$\mathcal{D}_{k+1} = \sum_{j=k+1}^{n-1} \psi_j S_j \mu_{jk} \partial_j + 1/2 \sum_{ij=k+1}^{n-1} \psi_i \psi_j \partial_{ij}^2.$$

Then, with B_n as the numéraire, that is w_n a $\mathbb{P}_n$-Brownian motion, $S_k(t)$ is given by the solution to

$$dS_k = -S_k \psi_k \sum_{j=k+1}^{n-1} \frac{\theta_{j-1} S_j \psi_j^t v_{kj}}{(1+\theta_{j-1} S_j) v_k} dt + S_k \psi_k dw_n. \tag{4.9}$$

Moreover we can represent the solution to (5.9) *by*

$$S_k(t) = \mathbb{E}^{k,n}[g_k(S_{k+1}(T_k), \ldots, S_{n-1}(T_k))|\mathcal{F}_t]. \tag{4.10}$$

We thus have a complete market model for all swaps S_i for $i = 1, \ldots, n-1$.

Proof. Evolve the system for $i = k+1, \ldots, n-1$ under the $\mathbb{P}_{k+1,n}$-forward measure as in (4.4) with $k+1$ instead of k

$$dS_i = S_i \psi_i \mu_{i,k+1} dt + S_i \psi_i dw_{k+1,n},$$

and define $S_k(t) = f(t, S_{k+1}(t), \ldots, S_{n-1}(t))$. Itô f and compare the drift and volatilities with what they should look like under the $\mathbb{P}_{k+1,n}$ measure below in (4.11)

$$\begin{aligned} df &= \left(\partial_t f + \sum_{j=k+1}^{n-1} \psi_j S_j \mu_{j,k+1} \partial_j f + 1/2 \sum_{ij=k+1}^{n-1} \psi_i \psi_j \partial_{ij}^2 f \right) dt \\ &\quad + \sum_{j=k+1}^{n-1} \psi_j S_j \partial_j f dw_{k+1,n}, \\ dS_k &= \psi_k S_k \mu_{k,k+1} dt + \psi_k S_k dw_{k+1,n}. \end{aligned} \tag{4.11}$$

The volatilities must coincide, hence set $\psi_k S_k = \sum_{j=k+1}^{n-1} \psi_j S_j \partial_j f$ so that

$$\psi_k = \sum_{j=k+1}^{n-1} \psi_j S_j \partial_j (\ln f),$$

and substitute it in the drift in (4.11) to obtain

$$\partial_t f + \sum_{j=k+1}^{n-1} \psi_j S_j (\mu_{j,k+1} - \mu_{k,k+1}) \partial_j f + 1/2 \sum_{ij=k+1}^{n-1} \psi_i \psi_j \partial_{ij}^2 f = 0.$$

The result (4.9) follows since

$$\mu_{j,k+1} - \mu_{k,k+1} = \left(\sum_{i=k+1}^{n-1} \frac{\theta_{i-1} S_i \psi_i}{1+\theta_{i-1}S_i} \frac{v_{ki}}{v_k} - \sum_{i=k+2}^{n-1} \frac{\theta_{i-1} S_i \psi_i}{1+\theta_{i-1}S_i} \frac{v_{k+1i}}{v_{k+1}} \right.$$
$$\left. + \sum_{i=k+1}^{n-1} \frac{\theta_{i-1} S_i \psi_i}{1+\theta_{i-1}S_i} \frac{v_{k+1i}}{v_{k+1}} - \sum_{i=j+1}^{n-1} \frac{\theta_{i-1} S_i \psi_i}{1+\theta_{i-1}S_i} \frac{v_{ji}}{v_j} \right)$$
$$= \mu_{jk}.$$

Notice that the differential generator is the one for S under the $\mathbb{P}_{k,n}$ measure hence the probabilistic representation follows by Feynman-Kac.

We then define the remaining swap rates as the strong solution to (4.9) where ψ_k is given by (4.7). In this way the corresponding SMM is consistent with an arbitrage free term structure, that is

$$S_i(t) = \frac{B_i(t) - B_n(t)}{\sum_{j=i+1}^{n} \delta_{j-1} B_j(t)}.$$

□

4.2. *More missing rates*

In reality we have more than one missing swap, say apart from missing $(S_k(0), \psi_k(t))$ we are also missing $(S_{k-1}(0), \psi_{k-1}(t))$, since the SMM is a triangular system, in the sense that the evolution of a particular swap rate depends only on the swap rates with later maturities, we can repeat the procedure carried out in Proposition 4.1 to define S_{k-1} as

$$S_{k-1}(t) = \mathbb{E}^{k-1,n}[g_{k-1}(S_k(T_{k-1}), \ldots, S_{n-1}(T_{k-1}))|\mathcal{F}_t], \tag{4.12}$$

where $S_k(t)$ is defined by the SDE in Proposition 4.1. One of the key results in Proposition 4.1 is that the boundary condition g is completely arbitrary for each missing swap rate. The next proposition introduces a function g_{k-j} for a generic missing swap rate S_{k-j} once all S_i, $i = k-j+1, \ldots, n-1$ have been defined that allows for an explicit computation of both the function f and the volatility ψ_{k-j}. Again note that the functions v_{k-j} depend only on $S_{k-j+1}, \ldots, S_{n-1}$.

Proposition 4.2. *Let $a_k, \ldots, a_{k-j}$ be constants such that*

$$a_{k-j} \geq a_{k-j+1} \geq \cdots \geq a_k \geq 1 \quad \text{and} \quad a_{k-j} < \frac{v_{k-j}(0)}{v_{k+1}S_{k+1}(0)},$$

and define

$$S_{k-j}(t) := f^{k-j}(t, S_{k+1}(t), \ldots, S_{n-1}(t)) = \frac{a_{k-j} v_{k+1}(t) S_{k+1}(t)}{v_{k-j}(t)}. \tag{4.13}$$

Then the volatilities ψ_{k-j} *are given by*

$$\begin{aligned}\psi_{k-j} &= -\sum_{i=k-j+1}^{k} \frac{\psi_i S_i}{v_{k-j}} \theta_{i-1}(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{i-2}S_{i-1})v_i \\ &\quad + \frac{\psi_{k+1}}{v_{k-j}}(v_{k-j} - S_{k+1}\theta_k(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{k-2}S_{k-1})v_{k+1}) \\ &\quad + \sum_{i=k+2}^{n-1} \frac{\psi_i S_i}{v_k^2}\left(v_{k-j}\frac{\partial_i v_{k+1}}{v_{k+1}} - \partial_i v_{k-j}\right). \end{aligned} \tag{4.14}$$

Proof. We change measure in (4.12) to the $\mathbb{P}_{k+1,n}$-forward measure

$$\frac{d\mathbb{P}_{k+1,n}}{d\mathbb{P}_{k-1,n}} = \frac{\Lambda(T_{k-1})}{\Lambda(t)} = \frac{v_{k+1}}{v_{k-1}}(T_{k-1})\frac{v_{k-1}}{v_{k+1}}(t).$$

Notice that from the definition of v_{k-1}, see (4.6), the change of measure depends only on $S_k, \ldots, S_{n-1}$ which have already been defined. Applying Bayes' rule we have

$$S_{k-1}(t) = \Lambda(t)\mathbb{E}^{k+1,n}\left[\left.\frac{g_{k-1}(S_k(T_{k-1}), \ldots, S_{n-1}(T_{k-1}))}{\Lambda(T_{k-1})}\right| \mathcal{F}_t\right].$$

Defining

$$g_{k-1}(S_{k+1}(T_{k-1}), \ldots, S_{n-1}(T_{k-1})) := a_{k-1}S_{k+1}(T_{k-1})\Lambda(T_{k-1}), \tag{4.15}$$

(4.12) simplifies to

$$S_{k-1}(t) = \frac{a_{k+1}v_{k+1}(t)}{v_{k-1}(t)}\mathbb{E}^{k+1,n}[S_{k+1}(T_{k-1})|S_{k+1}(t)] = a_{k-1}\frac{v_{k+1}}{v_{k-1}}S_{k+1}(t).$$

Notice that if we set $a_{k-1} = 1$ we have that the corresponding Libor rate L_{k-1} is zero. Similarly for the S_{k-j} missing swap we have

$$S_{k-j}(t) = a_{k-j}\frac{v_{k+1}(t)}{v_{k-j}(t)}S_{k+1}(t).$$

The constants a_i are set to satisfy the positive Libor constraint $v_{i-1}S_{i-1} \geq v_i S_i$ and $S_{i-1} \leq 1$, that is

$$a_{k-j} \geq a_{k-j+1} \geq \cdots \geq a_k \geq 1 \quad \text{and} \quad a_{k-j} < \frac{v_{k-j}(0)}{v_{k+1}S_{k+1}(0)}.$$

We can compute the volatilities in closed form. We have

$$f^{k-j} = \frac{a_{k-j}v_{k+1}S_{k+1}}{v_{k-j}},$$

$$f^{k-j}\psi_{k-j} = \sum_{i=k-j+1}^{n-1} \psi_i S_i \partial_i f^{k-j}.$$

Hence

$$f^{k-j}\psi_{k-j} = \sum_{i=k-j+1}^{k} \psi_i S_i a_{k-j} v_{k+1} S_{k+1} \partial_i \left(\frac{1}{v_{k-j}}\right) + \psi_{k+1} S_{k+1} a_{k-j} v_{k+1} \partial_i \left(\frac{S_{k+1}}{v_{k-j}}\right)$$
$$+ \sum_{i=k+2}^{n-1} \psi_i S_i a_{k-j} S_{k+1} \partial_i \left(\frac{v_{k+1}}{v_{k-j}}\right). \tag{4.16}$$

For $i > k-j$ we have

$$\partial_i v_{k-j} = \theta_{i-1}(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{i-2}S_{i-1})v_i.$$

Substitute this into Eq. (4.16) to obtain

$$f^{k-j}\psi_{k-j}$$
$$= - \sum_{i=k-j+1}^{k} \frac{a_{k-j}\psi_i S_i S_{k+1} v_{k+1}}{v_{k-j}^2} \theta_{i-1}(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{i-2}S_{i-1})v_i$$
$$+ \frac{a_{k-j}\psi_{k+1}S_{k+1}v_{k+1}}{v_{k-j}^2}(v_{k-j} - S_{k+1}\theta_k(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{k-2}S_{k-1})v_{k+1})$$
$$+ \sum_{i=k+2}^{n-1} \frac{a_{k-j}\psi_i S_i S_{k+1}}{v_{k-j}^2}(v_{k-j}\partial_i v_{k+1} - v_{k+1}\partial_i v_{k-j}).$$

Hence

$$\psi_{k-j} = - \sum_{i=k-j+1}^{k} \frac{\psi_i S_i}{v_{k-j}} \theta_{i-1}(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{i-2}S_{i-1})v_i$$
$$+ \frac{\psi_{k+1}}{v_{k-j}}(v_{k-j} - S_{k+1}\theta_k(1+\theta_{k-j}S_{k-j+1})\cdots(1+\theta_{k-2}S_{k-1})v_{k+1})$$
$$+ \sum_{i=k+2}^{n-1} \frac{\psi_i S_i}{v_k^2}\left(v_{k-j}\frac{\partial_i v_{k+1}}{v_{k+1}} - \partial_i v_{k-j}\right). \qquad \square$$

Remark. Equation (4.10) shows that we can define the missing swap rate S_k as a function à la arrears of the available swap rates which we rewrite here

$$S_k(t) = \mathbb{E}^{k,n}[g(S_{k+1}(T_k), \dots, S_{n-1}(T_k)|\mathcal{F}_t].$$

The resulting volatility was obtained using the PDE approach. We can use Clark-Ocone's formula to see the probabilistic relation between the interpolation (that is the choice of function g and the obtained volatility ψ_k. First we have that (4.10) can be expressed as

$$S_k(t) = \mathbb{E}^{kn}[S_{k+1}(T_k)|S_{k+1}(0)] + \int_0^t \mathbb{E}_s^{kn}[D_s S_k(t)]dw_{k,n}(s)$$
$$= \mathbb{E}^{kn}[S_{k+1}(T_k)|S_{k+1}(0)] + \int_0^t \mathbb{E}_s^{kn}[D_s S_{k+1}(T_k)]dw_{k,n}(s),$$

where the last equality follows by the tower property and interchange of derivative and conditional expectation, hence we can deduce that

$$\psi_k(t) = \frac{\mathbb{E}^{kn}[D_t g(S(T_k))|\mathcal{F}_t]}{\mathbb{E}^{kn}[g(S(T_k))|\mathcal{F}_t]},$$

where D_t denotes the Malliavin derivative. In particular it follows that ψ_k is not log-normal since it depends on all swap rates $S_{k+1}, \ldots, S_{n-1}$. Previously we obtained an explicit solution for both (S_k, ψ_k) because we can choose the function g to change to a more convenient martingale measure. But for more general functions g we can use Eq. (4.10) and Monte Carlo simulation to find $S_k(0)$ and obtain the today's yield curve. However Monte Carlo methods will have to be used to obtain also the missing volatilities.

5. Yield Curve Dynamics and Relation to HJM

In their paper [4], Brace *et al.* introduce the Libor Market Model by starting from an HJM model and showing how the HJM drift condition translates to a condition on the drift of the implied model for Libor rates. The drift in the LMM, as in the HJM, is a function of the volatilities specified for the Libor rates. In this section we take the opposite direction and use the previous results of this paper to obtain an HJM model from a model of Libor or swap rates and short-term rates. We use the bootstrapping method from Sec. 4 to fill in the missing swaps and then use Sec. 3 to interpolate between these values. At any one time, including the present date, the interpolation is a function of the present rates and the drift and volatility of the auxiliary process $y(t)$ (which is calibrated to one of the short-term rates); but as we will see below the whole curve will depend on the evolution of the market and bootstrapped rates. Hence we can generate rich movements in the yield curve.

The ingredients of the model are the following: a tenor structure $0 < T_1 < \cdots < T_n$ and accrual factors $\theta_i \approx T_{i+1} - T_i$ according to some day-count convention. We then have a co-terminal model of swap rates, that is today's swap rates, $S_i(0)$ for i in some subset of $\{1, \ldots, n-1\}$ that is fewer than dates in the tenor structure, and their volatilities ψ_i. The short end of the curve is made up of short-term interest rates and we assume for simplicity that we have two yields y_1, y_2, their volatilities (σ_1, σ_2) and their drift coefficients (a_1, b_1, a_2, b_2), (see the appendix for details).

That is $B_i(t) = e^{-y_i(t)(T_i - t)}$ for $i = 1, 2$ and $T_1 < T_2$. We use the process for the yield $y_1(t)$ as the auxiliary process used in Sec. 3 to construct a continuous time numéraire from a swap market model.

The randomness is modelled by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^{SL})$ supporting a $\mathbb{P}^{SL}$-Brownian motion w where $\mathbb{P}^{SL}$ is the measure that corresponds to using the rolling horizon numéraire introduced in Sec. 3. In the Proposition below we show that the bond price dynamics is a function of the short-term yield and the volatilities of the swap rates with reset date at each tenor date. At the tenor dates where we are missing a rate we plug in Eq. (4.14). This will be fine as long as we use the swap rates from (4.13) for today's interpolation of the yield curve. We need the following lemma which summarizes the properties of the Libor and swap market models.

Lemma 5.1. *For $k < i$ and $t \in [0, T_k]$ the process $Z_k^i(t) \equiv 1/Y_k^i(t)$ defined by*

$$Z_k^i(t) \equiv \frac{1}{\prod_{j=k}^{i-1}(1 + \theta_j L_j(t))} = \frac{B_i(t)}{B_k(t)},$$

follows a stochastic process under the $\mathbb{P}_k$-forward measure given by

$$dZ_k^i(t) = -Z_k^i(t)\gamma_k^i(t)dw_k(t), \tag{5.1}$$

where

$$\gamma_k^i(t) = \sum_{j=k}^{i} \frac{\theta_j L_j(t)\lambda_j(t)}{1 + \theta_j(t)L_j(t)}. \tag{5.2}$$

Under the assumptions of a swap market model, the process

$$Z_k^i(t) \equiv \frac{Y_i(t)}{Y_k(t)} \equiv \frac{1 + v_i(t)S_i(t)}{1 + v_k(t)S_k(t)} = \frac{B_i(t)}{B_k(t)},$$

follows under $\mathbb{P}_k$

$$dZ_k^i(t) = \frac{Z_k^i(t)}{Y_k(t)Y_i(t)}(Y_k(t)\gamma_i(t) - Y_i(t)\gamma_k(t))dw_k(t), \tag{5.3}$$

where

$$\gamma_i(t) = \sum_{j=i+1}^{n-1} \frac{\theta_{j-1}S_j(t)\psi_j(t)}{1 + \theta_{j-1}S_j(t)} \frac{v_{ij}(t)}{v_i(t)} + \psi_i(t). \tag{5.4}$$

Proof. The proof of the statement for the LMM goes as follows. From Jamshidian [12], $Y_k^i(t)$ under $\mathbb{P}_i$ is given by

$$dY_k^i(t) = Y_k^i(t)\gamma_k^i(t)dw_i(t),$$

we apply Itô's chain rule to $Z_k^i(t) \equiv 1/Y_k^i(t)$ to obtain

$$dZ_k^i(t) = -Z_k^i(t)\gamma_k^i(t)dw_i(t) + \frac{1}{2}Z_k^i(t)\gamma_k^i(t)^2 dt,$$

and changing the measure to $\mathbb{P}_k$ gives the result since Z_k^i is a martingale under that measure.

For the swap market model we apply Itô's product rule under the $\mathbb{P}_n$-measure

$$\begin{aligned} dZ_k^i(t) &= 1/Y_k(t)dY_i(t) + Y_i(t)d(1/Y_k(t)) + d\langle Y_i, (1/Y_k)\rangle_t \\ &= (1/Y_k^2(t))(Y_k(t)\gamma_i(t) - Y_i(t)\gamma_k(t))dw_n(t) \\ &\quad + \frac{1}{Y_k^2(t)}\left(\frac{Y_i(t)\gamma_k^2(t)}{2Y_k(t)}dt + \gamma_i(t)\gamma_k(t)\right)dt. \end{aligned}$$

We now change to the $\mathbb{P}_k$ measure under which Z_k^i is a martingale to obtain

$$dZ_k^i(t) = (1/Y_k^2(t))(Y_k(t)\gamma_i(t) - Y_i(t)\gamma_k(t))dw_k(t),$$

and the result follows after multiplying the right hand side above by $Z_k^i Y_k/Y_i$. □

We can now prove the following Proposition which gives the bond price dynamics using in addition to the Libor or swap market models the auxiliary process $y(t)$ defined in Sec. 3.2.

Proposition 5.1. *Within the context of the Libor market model the bond price dynamics for* $t \in [T_k, T_{k+1})$, $T \in [T_i, T_{i+1})$ *and* $\tau_k \equiv T_k - t$, *are given under the* $\mathbb{P}_{k+1}$*-forward measure by*

$$\begin{aligned} \frac{dp(t,T)}{p(t,T)} &= (y(t) - \mu_{k+1}\tau_{k+1} + (\sigma_1\tau_{k+1})^2/2 + \gamma_{k+1}^{i+1}(t,T)\sigma_1\tau_{k+1})dt \\ &\quad - (\gamma_{k+1}^{i+1}(t,T) + \sigma_1\tau_{k+1})dw_{k+1}, \end{aligned} \tag{5.5}$$

where $\gamma_{k+1}^{i+1}(t,T)$ *is defined as in* (5.2) *but with*

$$\tilde{\theta}_i(T) = \frac{\theta_i}{T_{i+1} - T_i}(T_{i+1} - T_i - (T_{i+1} - T)\exp(-a(T - T_i)). \tag{5.6}$$

From Sec. 3.2 *we have an implied spot rate given by*

$$r(t) = y(t) - \mu_{k+1}\tau_{k+1} - 1/2(\sigma_1\tau_{k+1})^2.$$

Let $\mathbb{P}^*$ *denote the associated risk-neutral measure, then with* $t \in [T_k, T_{k+1})$ *and* $T \in [T_i, T_{i+1})$ *the bond dynamics under this measure are given by*

$$\frac{dp(t,T)}{p(t,T)} = r(t)dt - (\gamma_{k+1}^{i+1}(t,T) + \sigma_1\tau_{k+1})dw_t^*. \tag{5.7}$$

For the swap market model the equations above read under the $\mathbb{P}_{k+1}$ *measure*

$$\begin{aligned} \frac{dp(t,T)}{p(t,T)} &= \left(r(t) + \frac{\sigma_1\tau_{k+1}}{Y_{i+1}Y_{k+1}}(Y_{k+1}\gamma_{k+1} - Y_{i+1}\gamma_{i+1})\right)dt \\ &\quad + \left(\frac{1}{Y_{i+1}Y_{k+1}}(Y_{k+1}\gamma_{k+1} - Y_{i+1}\gamma_{i+1}) + \sigma_1\tau_{k+1}\right)dw_{k+1}(t), \end{aligned}$$

and under the $\mathbb{P}^*$ *measure as*

$$\frac{dp(t,T)}{p(t,T)} = r(t)dt + \left(\frac{1}{Y_{i+1}Y_{k+1}}(Y_{k+1}\gamma_{k+1} - Y_{i+1}\gamma_{i+1}) + \sigma_1\tau_{k+1}\right)dw^*(t),$$

where

$$Y_j \equiv Y_j(t,T) = 1 + v_j(t,T)S_j(t),$$

$$\gamma_j \equiv \gamma_j(t,T) = \sum_{m=j+1}^{n-1} \frac{\theta_{m-1}S_m(t)\psi_m(t)}{1+\theta_{m-1}S_m(t)} \frac{v_{jm}(t,T)}{v_j(t,T)} + \psi_j(t), \quad j = i+1, k+1.$$

The dependence of the random variables v_j *above on* (t,T) *means that they are still defined by* (3.2) *but with* θ_i *replaced by* (5.6) *whenever it appears.*

Proof. We apply Itô's product rule to equation $p(t,T) = e^{-y(t)\tau_{k+1}}/\tilde{Y}_{k+1}^{i+1}(t)$. For ease of notation and as in the previous lemma let $Z(t) \equiv 1/\tilde{Y}_{k+1}^{i+1}(t)$. The proof of this lemma is not affected by changing θ_i to $\tilde{\theta}_i(T)$. We have

$$\begin{aligned} dp(t,T) &= Z_t de^{-y(t)\tau_{k+1}} + e^{-y(t)\tau_{k+1}} dZ_t + d\langle e^{-y(t)\tau_{k+1}}, Z\rangle_t \\ &= e^{-y(t)\tau_{k+1}} Z_t (d(-y\tau_{k+1})_t + \frac{1}{2} d\langle y\tau_{k+1}\rangle_t) + e^{-y(t)\tau_{k+1}} dZ_t + d\langle e^{-y\tau_{k+1}}, Z\rangle_t. \end{aligned} \tag{5.8}$$

To deal with the Libor market model we substitute Eq. (5.1) in for $Z(t)$ to obtain

$$\begin{aligned} \frac{dp(t,T)}{p(t,T)} &= (y(t) - \mu_{k+1}\tau_{k+1} + (\sigma_1\tau_{k+1})^2/2)dt \\ &\quad + \sigma_1\tau_{k+1}dw_{k+1} - \gamma_{k+1}^i dw_{k+1} + \gamma_{k+1}^i \sigma_1\tau_{k+1}dt. \end{aligned}$$

The result follows after rearranging. For the swap market model we let

$$Z(t) = \frac{1 + v_{i+1}(t)S_{i+1}(t)}{1 + v_{k+1}(t,T)S_{k+1}(t)},$$

and we substitute (5.3) in (5.8) to obtain the result.

The dynamics under the risk-neutral measure for both the Libor and swap market models follow from Sec. 5.3, where we show that the "market prices of risk" $\lambda(t)$ are given by

$$\lambda(t) = \frac{r(t) - m(t,T)}{b(t,T)},$$

where $m(t,T)$ and $b(t,T)$ are the drift and volatility of the bond $p(t,T)$. In this case this is

$$\lambda(t) = \frac{-(\sigma_1\tau_{k+1}^2 + \gamma_{k+1}^{i+1}\sigma_1\tau_{k+1})}{-(\sigma_1\tau_{k+1} + \gamma_{k+1}^{i+1})} = \sigma_1\tau_{k+1}.$$

By Girsanov theorem the risk-neutral Brownian motion $w^*(t)$ is then given by

$$dw^*(t) = dw_{k+1}(t) - \lambda(t)dt.$$

We now substitute the above for $w_{k+1}(t)$ in (5.5) to obtain the result in equation (5.7). A similar calculation works for the swap market model. □

Notice that from Eq. (5.6), for a fixed time t, the curve $P(t, T)$ is continuous as a function of maturity T.

Proposition 5.2. *(HJM) For $t \in [T_k, T_k + 1)$ and $T \in [T_i, T_{i+1})$ the dynamics of the forward rate under the risk-neutral measure $\mathbb{P}^*$ are given by*

$$df_t(t, T) = \alpha(t, T)dt + \beta(t, T)dw_t^*,$$

where

$$\alpha(t, T) = -(\gamma_{k+1}^i(t) + \sigma_1 \tau_{k+1}) \frac{\partial}{\partial T} \left[\frac{\tilde{\theta}_i(T) L_i(t)}{1 + \tilde{\theta}_i(T) L_i(t)} \right],$$

$$\beta(t, T) = \frac{\partial}{\partial T} \frac{\tilde{\theta}_i(T) L_i(t)}{1 + \tilde{\theta}_i(T) L_i(t)},$$

$$\tilde{\theta}_i = \frac{\theta_i}{T_{i+1} - T_i} (T_{i+1} - T_i - (T_{i+1} - T) \exp(-a(T - T_i)),$$

$$L_i(t) = \frac{1}{\theta_i} \left(\frac{1 + v_i(t) S_i(t)}{1 + v_{i+1}(t) S_{i+1}(t)} - 1 \right),$$

and for $t \in [0, T_1)$ and $T \in [T_1, T_2)$ by

$$df(t, T) = \left(-\frac{\partial^2 M(t, T)}{\partial t \partial T} + y_1(t) \frac{\partial^2 N_1(t, T)}{\partial t \partial T} + y_2(t) \frac{\partial^2 N_2(t, T)}{\partial t \partial T} \right) dt + \left(\frac{\partial N_1(t, T)}{\partial T} \sigma_1 + \frac{\partial N_2(t, T)}{\partial T} \sigma_2 \right) dw_t^*,$$

with M, N_1, N_2 given by (A.12), (A.13) and (A.14), see the Appendix for the derivations.

Proof. $f(t, T) = -\partial \ln p(t, T)/\partial T$.

$$d \ln p(t, T) = r dt - 1/2(\sigma\tau + \gamma)^2 dt - (\gamma + \tau\sigma) dw^*.$$

Differentiating the drift and volatility above with respect to T gives us the result. For the short end of the curve they are given where bond prices are given by

$$p(t, T) = \exp(M(t, T) - N_1(t, T) y_1(t) - N_2(t, T) y_2(t)).$$

Hence

$$f(t, T) = -\partial_T M + y_1(t) \partial_T N_1 + y_2(t) \partial_T N_2.$$

Differentiate this with respect to t. □

5.1. *Conclusions*

We now give a brief summary of the results obtained, some of its short comings and ideas for possible future research.

We have given an example of an interpolation of the yield curve which generates arbitrage opportunities and motivated by this we have started exploring a bootstrapping method, consistent with absence of arbitrage, for obtaining discount factors from market data consisting of short-term rates and swap rates. That is, we have found a deterministic function F and specified a numéraire asset $B^*(t)$, namely the "savings account" of Sec. 3 such that for an arbitrary maturity T,

$$p(t,T) = F(t,T,\mu,\sigma,S(0),\psi(t,S(0)),a), \quad \mu,\sigma \in \mathbb{R}^n, \quad S(0),\psi,a \in \mathbb{R}^k,$$

and $p(t,T)/B^*(t)$ is a $\mathbb{P}^*$-martingale, where $\mathbb{P}^*$ is the corresponding martingale measure. The parameters (μ_i,σ_i) are the drifts and volatilities of the short term yields, $S_i(0),\psi_i$ are the initial swap rates and their volatilities respectively, and a_i are the coefficients that are used in Proposition 4.2.

To calibrate this model we could obtain the volatilities for the underlying swap, and short term rates from liquid option prices, for example swaptions and bond options. However the drift for the auxiliary process $y(t)$ remains unspecified, in fact there are at least as many parameters as tenor dates. Also the parameters a_i are not specified either. At this point one could choose these parameters to match any shape of the yield curve. To reduce the number of parameters one could for example specify the short-term drifts so that the implied spot rate in Secs. 2 and 3 doesn't jump between tenor dates as remarked in Sec. 3.2. But in that case the examples in Secs. 3.3.2 and 3.3.3 would no longer hold so we leave the question unanswered.

Another important unanswered question, in our view, is concerned with Proposition 4.2, in the missing swaps section. We defined the missing swap S_k as a function $f(t,S_{k+1}(t),\ldots,S_{n-1}(t))$ of the swaps that would still be alive after T_k. This eases considerably the computations because the co-terminal swap market model is a "triangular" system in that under the $\mathbb{P}_{k,n}$ the swap rate S_k only depends on the swap rates that are still alive, that is $S_{k+1},\ldots,S_{n-1}$. However it would be more satisfactory to have the function f depend also on the swap rate S_{k-1}. In that case the model of yield curve obtained at present time would not need recalibration, at least until the maturity of the first instrument used, that is until T_1. However in order to find this function f the problem that we need to solve, in analogy with Proposition 4.2, is, for a function $g_k(S_{k-1},S_{k+1},\ldots,S_{n-1})$ free to be specified, solve

$$\frac{\partial f}{\partial t} + \mathcal{D}_{k+1} f = 0, \tag{5.9}$$

$$f(T_k,S_{k-1},S_{k+1},\ldots,S_{n-1}) = g_k(S_{k-1},S_{k+1},\ldots,S_{n-1}),$$

with

$$\mathcal{D}_{k+1} = \psi_{k-1}S_{k-1}\mu_{k-1,k}(f)\partial_{k-1} + \sum_{j=k+1}^{n-1}\psi_j S_j \mu_{jk}\partial_j + 1/2\sum_{ij=k-1}^{n-1}\psi_i\psi_j\partial_{ij}^2.$$

Now the operator $\mathcal{D}_{k+1}$ is quasi-linear through the dependence of the drift $\mu_{k-1,k}$ [given by (4.5)] on the function f. We are adding the rate S_{k-1} to the interpolation but this depends, under the $\mathbb{P}_{k-1,n}$-measure on the rate that we are trying to find, S_k. It's not obvious to us how to solve this problem as the trick that we used in the proof of Proposition 4.2 no longer applies.

Appendix. Bond Prices at the Short End of the Curve

In this appendix we derive a closed-form solution for the term structure at the short end of the curve. The rates available at the short-end of the curve consist of short-term yields, therefore we need a different derivation for this section of the curve. For this particular case we assume that our data consists of the yields $y_1(t)$ and $y_2(t)$ of two zero-coupon bond prices $B_1(t)$, $B_2(t)$ where $T_1 < T_2$. We are also given a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ supporting a two dimensional Brownian motion $w_1(t), w_2(t)$. At any $t \in (T_1, T_2)$ only the yield $y_2(t)$ remains alive. We assume the following process for it under the $\mathbb{P}_2$ measure

$$dy_2(t) = (a_2 - b_2 y_2(t))dt + \sigma_2 dw_2(t), \tag{A.1}$$

where a_2, b_2, σ_2 are constants with $\sigma_2^2 > 0$. For $t \in [0, T_1]$ we assume a similar process for y_1 under the $\mathbb{P}_1$ forward measure with (a_1, b_1, σ_1) constants and $\sigma_1^2 > 0$,

$$dy_1(t) = (a_1 - b_1 y_1(t))dt + \sigma_1 dw_1(t). \tag{A.2}$$

The process for y_2 for $t \in [0, T_1]$ under $\mathbb{P}_1$ is given however by

$$dy_2(t) = \mu_{2,1}(t)dt + \sigma_2 dw_{2,1}(t), \tag{A.3}$$

where $\mu_{2,1}$ is the drift under this forward measure, making sure that B_2/B_1 follows a $\mathbb{P}_1$-martingale. Denote by $w_{2,1}$ the Brownian motion driving the yield y_2 under the $\mathbb{P}_1$ measure with $dw_1 dw_{1,2} = \rho dt$.

Lemma A.1. *Assume $t \in [0, T_1]$ and $y_1(t)$ satisfies under the $\mathbb{P}_1$-forward measure*

$$dy_1(t) = (a_1 - b_1 y_1(t))dt + \sigma_1 dw_1(t). \tag{A.4}$$

Then under $\mathbb{P}_1$ $y_2(t)$ is given by the strong solution to

$$dy_2(t) = \frac{1}{\tau_2}(y_2(t) - y_1(t)(1 + b_1\tau_1) + a_1\tau_1 + \sigma)dt + \sigma_2 dw_{2,1}(t), \tag{A.5}$$

where $\tau_i = T_i - t$ and $\sigma := \frac{1}{2}(\tau_1^2\sigma_1^2 + \tau_2^2\sigma_2^2) - \tau_1\tau_2\rho\sigma_1\sigma_2$. Furthermore the solution to these are given by

$$\begin{aligned}
y_1(t) &= e^{-b_1 t} y_1(0) + \frac{a_1}{b_1}(1 - e^{-b_1 t}) + \sigma_1 \int_0^t e^{-b_1(t-u)} dw_1(u),\\
\tau_2 y_2(t) &= T_2 y_2(0) - y_1(0)h_1(t) + h_2(t) + \tilde{\sigma}(t),\\
&\quad + \sigma_1 \int_0^t u - T_1 + e^{-b_1 u}(T_1 - t)dw_1 + \sigma_2 \int_0^t T_2 - u dw_{2,1}(u),
\end{aligned}$$

where

$$h_1(t) := \frac{1}{b_1}(1 - e^{-b_1 t})(b_1 T_1 + t e^{-b_1 t}),$$

$$h_2(t) := \frac{a_1}{b_1}((T_1 - t)(1 - e^{-b_! t}) - b_1 t T_1) + a t^2/2,$$

$$\tilde{\sigma}(t) := a_1 t \left(T_1 - \frac{t}{2}\right) + \frac{t}{2}\sum_{i=1}^{2} \sigma_i^2 \left(T_i^2 + \frac{t^2}{3} - 2T_i\right)$$
$$+ \rho\sigma_1\sigma_2 t \left(T_1 T_2 + \frac{t^2}{3} - (T_1 + T_2)\right).$$

Hence $y_1(t) \sim N(m_1, s_1^2)$ *and* $y_2(t) \sim N(m_2, s_2^2)$ *with*

$$m_1(t) = y_1(0)e^{-b_1 t} + \frac{a_1}{b_1}(1 - e^{-b_1(t - T_i)}),$$

$$s_1^2(t) = \frac{\sigma^2}{2b_1}(1 - e^{-2b_1(t - T_i)}), \tag{A.6}$$

$$m_2(t) = \frac{1}{\tau_2}(T_2 y_2(0) - h_1(t) y_1(0) + h_2(t) + \tilde{\sigma}(t)), \tag{A.7}$$

$$s_2^2(t) = \frac{2\sigma_1(T_1 - t)}{b_1^2}(1 - e^{-b_1 t})((1 - b_1 T_1) - \rho\sigma_2(1 - b_1 T_2))$$
$$+ \frac{\sigma_1(T_1 - t)^2}{2b_1}(1 - e^{-2b_1 t}) - \frac{2\sigma_1(T_1 - t)t}{b_1}(1 + \rho\sigma_2)e^{-b_1 t}$$
$$+ t\left(\sigma_2 T_2^2 - \sigma_1 T_1^2 - 2\rho\sigma_1\sigma_2 T_1 T_2 + t(\sigma_1 T_1 - \sigma_2 T_2\right.$$
$$\left. + (T_1 + T_2)\rho\sigma_1\sigma_2) + \frac{t^2}{3}(\sigma_2 - \sigma_1 - 2\rho\sigma_1\sigma_2)\right). \tag{A.8}$$

Proof. We require the following ratios to follow martingales:

$$\frac{B_2(t)}{B_1(t)} = \exp(y_1(t)(T_1 - t) - y_2(t)(T_2 - t)). \tag{A.9}$$

Let

$$f(t, y_2, y_1) := \exp(y_1(t)(T_1 - t) - y_2(t)(T_2 - t)).$$

Applying Itô's Lemma we obtain

$$\frac{df(t, y(t))}{f(t, y(t))} = ((y_2 - y_1) - \mu_{2,1}(T_2 - t) + \mu_1(T_1 - t)$$
$$+ 1/2(T_2 - t)^2\sigma_2^2 + 1/2(T_1 - t)^2\sigma_1^1$$
$$- (T_2 - t)(T_1 - t)\rho\sigma_2\sigma_1)dt - \tau_2\sigma_2 dw_{2,1} + \tau_1\sigma_1 dw_1. \tag{A.10}$$

Setting $\mu_{2,1}$ as in (A.5) the drift term cancels and hence (A.9) will follow a martingale. The solution to (A.4) follows by standard theory. The solution to (A.5) is a bit more involved. Let $I(t) = \int_0^t 1/(T_2 - s)ds$ be an integrating factor. Then

$$d(e^{-I(t)}y_2(t)) = e^{-I(t)}dy_2(t) - \frac{y_2(t)}{T_2 - t}e^{-I(t)}dt$$
$$= \frac{e^{-I(t)}}{T_2 - t}(-y_1(t)(1 + b_1\tau_1) + a_1\tau_1 + \sigma)dt + e^{-I(t)}\sigma_2 dw_{2,1}.$$

A simple calculation gives $e^{I(t)} = T_2/(T_2 - t)$, hence

$$y_2(t) = e^{I(t)}y_2(0) + e^{I(t)}\int_0^t \frac{e^{-I(t)}}{T_2 - s}(-y_1(t)(1 + b_1\tau_1)$$
$$+ a_1\tau_1 + \sigma)dt + e^{I(t)}\int_0^t e^{-I(s)}\sigma_2 dw_{2,1}$$
$$= \frac{T_2}{T_2 - t}y_2(0) + \frac{1}{T_2 - t}\int_0^t -y_1(s)(1 + b_1\tau_1)$$
$$+ a_1\tau_1 + \sigma(s)ds + \frac{\sigma_2}{T_2 - t}\int_0^t (T_2 - s)dw_{2,1}.$$

We analyse the integrals in detail. First

$$\int_0^t y_1(s)ds = \frac{y_1(0)}{b_1}(1 - e^{-bt}) + \frac{a_1}{b_1}\left(t - \frac{1}{b_1}(1 - e^{-b_1 t})\right)$$
$$+ \frac{\sigma_1}{b_1}\int_0^t 1 - e^{-b_1(t-u)}dw_1(u)$$

$$\int_0^t y_1(s)sds = y_1(0)\int_0^t se^{-b_1 s}ds + \frac{a_1}{b_1}\int_0^t s(1 - e^{-b_1 s})ds$$
$$+ \sigma_1\int_0^t s\int_0^s e^{-b_1(s-u)}dw_1(u)ds$$
$$= \frac{1}{b_1}\left(y_1(0) - \frac{a_1}{b_1}\right)\left(\frac{1}{b_1}\left(1 - e^{-b_1 t}\right) - te^{-b_1 t}\right) + \frac{a_1 t^2}{2b_1}$$
$$+ \frac{\sigma_1}{b_1}\int_0^t u + \frac{1}{b_1} - \left(t + \frac{1}{b_1}\right)e^{-b_1(t-u)}dw_1(u),$$

$$\int_0^t a(T_1 - s) + \sigma ds = a_1\left(T_1 t - \frac{t^2}{2}\right) + \frac{1}{2}\sum_{i=1}^{2}\sigma_i^2\left(T_i^2 t + \frac{t^3}{3} - 2T_i t\right)$$
$$+ \rho\sigma_1\sigma_2(T_1 T_2 t + \frac{t^3}{3} - t(T_1 + T_2)).$$

Rearranging gives us the result. □

We then have the main result of this subsection.

Proposition A.1. *Assume a process for* $(y_1(t), y_2(t))$ *given by* (A.1), (A.4) *and* (A.5). *Then for* $t \in [0, T_1]$ *and* $T \in [T_1, T_2]$ *the term structure is given by*

$$p(t,T) = \exp(M(t,T) - N_1(t,T)y_1(t) - N_2(t,T)y_2(t)), \tag{A.11}$$

where

$$M(t,T) = A - T_1\frac{a_1}{b_1}(1 - e^{b_1 t} - \frac{C}{\tau_2}h_2(T_1) + 1/2(T_1^2 s_1^2 + C^2 s_2^2), \tag{A.12}$$

$$N_1(t,T) = e^{-b_1 T_1}T_1 - \frac{C}{\tau_2}h_2(T_1), \tag{A.13}$$

$$N_2(t,T) = CT_2/\tau_2, \tag{A.14}$$

$$C = (T_2 - T)\exp(-b_2(T - T_1)), \tag{A.15}$$

$$A = (T_2 - T)\frac{a_2}{b_2}(1 - e^{-b_2(T-T_1)}) + (T_2 - T)^2\frac{\sigma^2}{4b_2}(1 - e^{-2b_2(T-T_1)}). \tag{A.16}$$

Proof. We want to evaluate $p(0,T)$ where $T_1 < T < T_2$,

$$\begin{aligned} p(t,T) &= \mathbb{E}^1[e^{-y_1(T_1)T_1}\mathbb{E}^2[e^{y_2(T)(T_2-T)}|\mathcal{F}_1]|\mathcal{F}_t] \\ &= \mathbb{E}^1[e^{-y_1(T_1)T_1}e^{A-Cy_2(T_1)}|\mathcal{F}_t], \end{aligned} \tag{A.17}$$

with C and A given by (A.15) and (A.16) respectively. The equality in (A.17) is obtained by applying the result in Sec. 3.3.2 to the expectation in the $\mathbb{P}_2$-forward measure with the process for y_2 given in (A.1). We have the processes for (y_1, y_2) under the $\mathbb{P}_1$-forward measure given by (A.6) and (A.6). We can rewrite $(A.17)$ as

$$p(0,T) = e^{A(T)}\mathbb{E}^1[e^{y(T_1)}|\mathcal{F}_0],$$

where $y = -T_1 y_1(T_1) - Cy_2(T_1)$ implying that $y \sim N(m, s^2)$ with (using (A.6) to (A.8)

$$\begin{aligned} m &= -T_1\mathbb{E}^1 y_1 - C\mathbb{E}^1 y_2 \\ &= -T_1 m_1(T_1) - Cm_2(T_1), \\ s^2 &= T_1^2 var(y_1) + C^2 var(y_2) \\ &= T_1^2 s_1^2 + C^2 s_2^2. \end{aligned}$$

Hence

$$p(t,T) = \exp(A + m + s^2/2).$$

Rearranging we obtain the result in (A.11)

$$p(t,T) = \exp(M(t,T) - N_1(t,T)y_1(t) - N_2(t,T)y_2(t)).$$ □

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