Fast Numerical Valuation of American, Exotic and Complex Options

M.A.H. Dempster & J.P. Hutton
Department of Mathematics
University of Essex
Wivenhoe Park, Colchester, England CO4 3SQ
mahd@essex.ac.uk & hutto@essex.ac.uk

13 July 1995

Abstract

The purpose of this paper is to present evidence in support of the hypothesis that fast, accurate and parametrically robust numerical valuation of a wide range of derivative securities can be achieved by use of direct numerical methods in the solution of the associated PDE problems. Specifically, linear programming methods for American vanilla and exotic options, and explicit methods for a three stochastic state variable problem (a multi-period terminable diff swap) are explored and promising numerical results are discussed. The resulting value surface gives, simultaneously, valuation for many maturities and underlying prices, and the parameters required for risk analysis.

1 Introduction

This paper briefly presents evidence accumulated to date in support of the use of *direct* numerical methods for the solution of *partial differential equation* (PDE) type problems associated with valuation of derivative securities based on one or more underlying securities. Vanilla and exotic American options on a single underlying and a multi-period terminable differential swap involving domestic and foreign interest rates and the cross-currency rate are considered in detail. The numerical methods employed are comparable in accuracy and speed to alternatives, but can enjoy the added advantage of robustness of these properties to variation in contract parameters. This is an important property for methods employed in real-time trading information systems; one which is not possessed by most alternative methods based on tree structures, closed form multiple integral formulae or series formulae, Monte Carlo techniques or iterative numerical methods. Another important property possessed by all PDE solution methods for the derivative security value surface is the immediate recovery from the calculations of estimates of the partial derivatives of value with respect to the contract parameters [Carr 1993] needed for risk management, via simple difference approximations.

In the next section of the paper, the fundamental relationships amongst the abstract variational inequality, complementarity [Jaillet *et al* 1990] and *linear programming* (LP) [Dempster & Hutton 1995] formulations of the American put valuation problem are presented in terms of the Black-Scholes partial differential operator. Finite difference approximation is applied to this operator in §3 to yield an *ordinary* LP which is solved by time-stepping decomposition. Results are presented — using IBM's Optimization Systems Library (OSL) [IBM 1992] on an IBM RS6000/590/AIX3.2.5 workstation — which support the hypothesis of the abstract. This approach is extended in §4 to valuation of lookback and Asian options, with both continuous and discrete sampling [Wilmott *et al* 1993], and some computational results are presented. Section 5 outlines a PDE-based valuation technique for a multi-period terminable diff swap under a cross-currency extended Vasicek model [Babbs 1990,1993,1994], while §6 presents numerical results on a 10 year quarterly terminable contract. To our knowledge this represents the first numerical valuation of a cross-currency derivative based on a full term structure-consistent model. In §7, conclusions are drawn and directions for further work indicated.

This research was supported in part by the University of Essex, the EPSRC (UK) and HSBC Markets. The reader should consult [Hutton 1995 and Dempster and Hutton 1995] for more details. It is a pleasure to acknowledge both the general advice of M.J.P Selby and the extensive involvement of S.H Babbs in the research presented in §6. We are grateful to J.N.Dewynne who kindly made his PSOR C codes available to us to enable the comparative numerical results of §3.

2 Valuation of American options by LP

We consider the interesting case of an *American* put option with strike price K on an underlying security with geometric Brownian motion price process $\mathbf{S}$ with constant *volatility* σ and riskless rate r over the life of the option, under the Black-Scholes assumptions. Then on $[0, T]$ the arbitrage-free price process $\mathbf{X}$ is given by

$$\mathbf{X}(t) = \operatorname*{ess\,sup}_{\boldsymbol{\tau} \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}} \left[e^{-r(\boldsymbol{\tau} - t)} (K - \mathbf{S}_{\boldsymbol{\tau}})^{+} \mid \mathcal{F}_t \right], \tag{1}$$

where $\mathcal{T}_{t,T}$ denotes the set of *stopping* (exercise) times $\boldsymbol{\tau}$ with respect to the current information field $\mathcal{F}_t$ of the price process, $\widetilde{\mathbb{E}}[. \mid \mathcal{F}_t]$ denotes conditional expectation with respect to the risk-adjusted probability (equivalent martingale) measure and $(K - \mathbf{S}_{\boldsymbol{\tau}})^{+} := (K - \mathbf{S}_{\boldsymbol{\tau}}) \wedge 0$, the pointwise minimum of $(K - \mathbf{S}_{\boldsymbol{\tau}})$ and 0. Moreover, for hedging purposes, this price process possesses on $[0, T]$ a perfectly replicating (continuously rebalanced) portfolio of the form

$$\mathbf{X}(t) = \boldsymbol{\phi}_{\mathbf{1}}(t)\beta(t) + \boldsymbol{\phi}_{\mathbf{2}}(t)\mathbf{S}(t), \tag{2}$$

where $\boldsymbol{\phi}_{\mathbf{1}}(t)$ and $\boldsymbol{\phi}_{\mathbf{2}}(t)$ denote the positions at time t in a pure discount bond with maturity T (of value $\beta(t) \leq 1$) and the underlying security respectively.

If we define the *value* P of the option as its arbitrage-free price at time $t \in [0,T]$ when the price of the underlying is $x \geq 0$, then

$$P(x,t) := \sup_{\boldsymbol{\tau} \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}} \left[e^{-r(\boldsymbol{\tau}-t)} (K - \mathbf{S}_{\boldsymbol{\tau}})^{+} \,|\, \mathbf{S}(t) = x \right] \tag{3}$$

The optimal exercise time ρ is given by

$$\boldsymbol{\rho}(t) := \inf\{s \in [t,T] : \mathbf{X}(s) = (K - S(s))^{+}\}, \tag{4}$$

i.e. the first time the underlying price process $\mathbf{S}$ reaches the *optimal exercise boundary* $S^{*}(t)$ given by

$$S^{*}(t) := \sup\{x : P(x,t) = (K - x)^{+}\}. \tag{5}$$

The determination of this *free boundary* in $[0,\infty) \times [0,T]$ along with the option value P is equivalent to the solution of an abstract *variational inequality* (VI) [Wilmott *et al* 1993] involving the Black-Scholes parabolic partial differential operator, as was first observed in this context by [Jaillet *et al* 1990]. The problem (VI) has a unique solution by the Lions-Stampacchia theorem and is easily seen [see e.g. Hutton 1995] to have an equivalent formulation as an abstract (linear) order complementarity problem [Borwein and Dempster 1989].

Indeed, making a logarithmic change of the underlying price variable, $\xi = \log x$, the Black-Scholes operator becomes the constant coefficient parabolic operator $\mathcal{L} + \frac{\partial}{\partial t}$, with elliptic part

$$\mathcal{L} := \frac{\sigma^2}{2} \frac{\partial^2}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial}{\partial \xi} - r. \tag{6}$$

Denoting the option value in terms of log price by u, and considering u and $\frac{\partial u}{\partial t}$ as elements of appropriate (dual Sobolev) Hilbert spaces of functions L_1^2 and L_0^2 respectively, yields the abstract *order complementarity problem*

$$\text{(OCP)} \qquad \begin{aligned} & u \in L_1^2, \ \tfrac{\partial u}{\partial t} \in L_0^2 \\ & u - \psi \geq 0, \ \mathcal{L} + \tfrac{\partial u}{\partial t} \leq 0 \\ & \left(-\mathcal{L} - \tfrac{\partial u}{\partial t} \right) \wedge (u - \psi) = 0. \end{aligned} \tag{7}$$

(OCP) neatly expresses the main features of the option value, namely: u is always at least equal to the payoff ψ ($:= (K - e^{(\cdot)}$ here); before optimal exercise, when it exceeds the payoff, u satisfies the Black-Scholes PDE; at and after exercise u equals the payoff ψ.

It may be shown [Jaillet *et al* 1990] that the linear operator $\mathcal{L}$ is both *coercive*, i.e.

$$\langle v, \mathcal{L}v \rangle_0 \geq \alpha \|v\|_0^2 \quad \forall v \in L_0^2 \tag{8}$$

for some $\alpha > 0$, where $\langle .,. \rangle_0$ and $\|.\|_0$ denote the inner product and corresponding norm on L_0^2 respectively, and of type Z, i.e.

$$u \wedge v = 0 \Rightarrow u \wedge \mathcal{L}v \leq 0 \quad \forall u, v \in L_0^2. \tag{9}$$

Extending earlier results of [Cryer & Dempster 1980] from elliptic to parabolic operators, we have the following theorem [Dempster & Hutton 1995].

Theorem 1 *If $\mathcal{L}$ is an elliptic coercive type Z operator, then there exists a unique solution to the equivalent problems (OCP) and the abstract linear programme*

$$\text{(LP)} \qquad \inf_v \langle c, v\rangle_0 \quad \text{s.t. } v \in F \subset L_1^2, \tag{10}$$

where $c > 0$ in L_0^2 is arbitrary and F denotes the constraint set

$$F := \{v \in L_1^2 : v \geq \psi,\ \mathcal{L}v + \frac{\partial v}{\partial t} \leq 0\}. \tag{11}$$

■

The proof employs Laplace transforms to show that under the stated conditions on $\mathcal{L}$, the unique solution u of (OCP) is the coordinatewise *least element* of the constraint set F of (LP) given by (11). Hence minimizing any positive functional $c \in L_0^2$ on F yields $v = u$.

Upon discretizing the abstract problem (LP) by finite differences — equivalently, finite elements [Wilmott *et al* 1993] — over $[0, \infty) \times [0, T]$, an ordinary LP is obtained which may be solved by state-of-the-art linear programming techniques, to which we now turn.

3 Numerical Methods and Results for Options

By employing standard finite difference approximations on a uniform grid — *implicit*, *explicit* and *Crank-Nicolson* — to $\mathcal{L} + \frac{\partial}{\partial t}$ given by (6) in terms of time $T - t$ to maturity, a (finite dimensional) matrix operator M is obtained of the form

$$C = \begin{pmatrix} A & & & & \\ B & A & & & \\ & \ddots & \ddots & & \\ & & B & A & \\ & & & B & A \end{pmatrix}, \tag{12}$$

where A and B are at most tridiagonal matrices of order $I - 1$ whose entries are simple functions of the deal and market parameters, and hence C is an order $M(I-1)$ square matrix, where I and M are the number of space and time grid points in the corresponding localised domains $[L, U]$ and $[0, T]$ respectively. In terms of C, (OCP) and (LP) are approximated in terms of the vectors of discretised values $u, v \in \mathbb{R}^{M(I-1)}$, with discretised spatial boundary conditions $u(L, .) := \psi(e^L)$ and $u(U, .) := 0$, as

$$\text{(OCP}'\text{)} \qquad u \geq \psi, \quad Cu \geq \theta, \ (Cu - \theta) \wedge (u - \psi) = 0 \tag{13}$$

and

$$\text{(LP}'\text{)} \qquad \min c'v \quad \text{s.t. } Cv \geq \theta \quad v \geq \psi, \tag{14}$$

where ψ is the vector of discretised payoff values and θ is a vector determined by the terminal boundary condition (i.e. the payoff) and spatial boundary conditions. For the full formulation, see [Hutton 1995].

In practice, (LP′) decomposes into the time-stepping sequence of tridiagonal LPs,

$$\min 1'u^m \quad \text{s.t. } Au^m \geq \phi^m - Bu^{m-1} \quad m = 1, \ldots, M \tag{15}$$

where we have arbitrarily chosen $c = 1$, the $(I-1)$-vector of ones. (OCP′) may be decomposed in a similar manner.

The best iterative algorithm for solving (OCP′) is the projected *successive overrelaxation* (PSOR) algorithm of Cryer [see Wilmott *et al* 1993], while the *dual simplex* algorithm for (LP′) is best for the type of linear programme in question.

All computation was performed in double precision on an IBM RS6000/590 computer with 128 Mb of RAM, running under AIX 3.2.5. The LP algorithms used were from IBM's Optimisation Systems Library [IBM 1992], namely the simplex routine `EKKSSLV`. The basis for the first time step was generated by an initial call to the basis crash routine `EKKCRSH` at level 4 and successive time step LP problems were 'hot started' from the previous time step's optimal basis. Dewynne's PSOR algorithm C code was used with a relaxation parameter $\omega = 1.5$ and initial value equal to the previous time step's solution. For both algorithms convergence tolerance was set to 10^{-8}.

Figures 1 and 2 show the value surface and optimal exercise boundary respectively, computed by solving (LP′) for an American put stock option of maturity $T = 1$ year, strike $K = \$1$, riskless rate $r = .1$ and underlying volatility $\sigma = .4$, with a discretization of $M = 50$, $I = 50$, $L = -1.5$ and $U = 1.5$. Table 1 shows the accuracies, at current stock price \$1 (i.e. at the money), of Crank-Nicolson and implicit discretization schemes, relative to the first three terms of the analytic series expansion developed by [Geske & Johnson 1984], against whose computations we compare our solution. Table 2 displays comparative solution times (shown in Figure 3) for this option with underlying volatilities $\sigma = .2$ and .4 for the PSOR, simplex and explicit methods with time discretization $M = 1000$ and varying spatial discretization I. Times quoted there for PSOR and dual simplex algorithms are for the Crank-Nicolson scheme, while the *explicit scheme* is a straightforward recursive matrix-vector multiplication with a comparison of each value to the payoff function, corresponding to a choice of discretization such that the matrix A in (15) is diagonal, and equivalent to running the Cox-Ross-Rubinstein binomial tree algorithm from each spatial grid point *without* redundant calculations. With this method, however, in general we must choose the number of time steps M proportional to the square of the number of space steps I^2, a fact which eventually makes it uncompetitive compared to implicit methods in one dimension. For the standard Black-Scholes operator, the exact stability condition is that $M \geq \sigma^2 T I^2/(U-L)$. Note that, while PSOR times increase with volatility, the simplex solution time remains relatively stable. This robustness to parameter variation in the simplex method is amply demonstrated for the parameters r and σ in Table 3, and the corresponding Figure 4, which compare PSOR and simplex solution times.

Although the space and time discretizations used here are perhaps higher than those typically required in practice, the standardization and robustness of the LP method, together with the even faster computing times provided by either a purpose-written LP code for tridiagonal problems or the latest commercial simplex and interior point LP algorithms such as CPLEX, point towards the eventual employment of these methods in trading information systems. In this regard, it should be borne in mind that accurate estimates of the standard parameters for hedging and the simultaneous valuation of options of many different maturities and underlying prices are automatically available from any value surface (i.e. PDE-based) approach. An extra advantage of the LP method is the availability of standard parametric techniques for fast reevaluation of derivative value surfaces when strike and volatility parameters change and its direct applicability to time and underlying price dependent volatilities, however estimated.

4 Valuation of American exotics

In this section we demonstrate the generality of the LP approach to American option pricing by briefly outlining its straightforward extension to continuous and discretely sampled American *lookback* and (arithmetic average) *Asian* put options. These have *path dependent* strike prices given at exercise time $\boldsymbol{\tau}$ by

$$\mathbf{S}_{\boldsymbol{\tau}}^{\max} := \sup_{t\in[0,\boldsymbol{\tau}]} \mathbf{S}_t \quad \text{or} \quad \max_{t_i\in[0,\boldsymbol{\tau}]} \mathbf{S}_{t_i} \tag{16}$$

and

$$\bar{\mathbf{S}}_{\boldsymbol{\tau}} := \int_0^{\boldsymbol{\tau}} \mathbf{S}_s\, ds/\boldsymbol{\tau} \quad \text{or} \quad \sum_{t_i\in[0,\boldsymbol{\tau}]} \mathbf{S}_{t_i}/\#\{t_i \in [0,\boldsymbol{\tau}]\}. \tag{17}$$

respectively. Hence they make the corresponding arbitrage-free option value depend on a second state variable y representing the current value of (16) or (17). For notational simplicity, we let y denote the value of the running sum in (17), rather than the average itself.

Consider the first case of the (somewhat artificial) continuously sampled American lookback put. Making the *similarity transformation* $\xi = \log(y/x) \geq 0$, originally introduced by Babbs in the context of binomial tree valuation [Babbs 1992], this 2-state variable problem (where the value function solves (LP) with the usual non-transformed Black-Scholes operator and y entering as a parameter of the payoff) can be reduced to (LP) in ξ and t, with the slightly modified elliptic part of the partial differential operator given by $\mathcal{L} := \frac{\sigma^2}{2}\frac{\partial^2}{\partial\xi^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial\xi}$, and the *Neumann* spatial boundary condition $\frac{\partial u}{\partial\xi}(0,t) = 0$ (i.e. a discrete condition $u_0^m = u_1^m$ for $m = 1,\ldots,M$). This boundary condition arises because if $\xi = 0$, i.e. $y = x$, the probability that y is the final maximum is zero, and hence the option value is insensitive to small changes in y/x.

Figure 5 shows the value surface for such a deal, computed by the LP method for a 6 month stock option with riskless rate $r = .05$ and volatility $\sigma = .5$, with Crank-Nicolson discretization $M = I = 100$ and ξ localized to $[0,1]$. Table 4 shows this scheme's accuracy and solution time for varying space steps I, and the limiting modified binomial tree value computed in [Babbs 1992] is given there for comparison. Note that the accuracy is slightly degraded by the crude first order approximation to the Neumann boundary condition, and would be improved by a second order approximation, but this is an artifact of the discretization, not the solution algorithm.

The discretely sampled American lookback put option may be solved over the triangular domain $y \leq x$ in the original variables. As for the continuous case, between sampling dates the value function $P(x, y, t)$ solves (LP) with the standard non-transformed Black-Scholes operator, and y as a parameter of the payoff. Thus we may solve (LP$'$) in each intra-sampling interval $[t_i, t_{i+1})$, using parametric simplex method to rapidly recompute the solution at each time step for varying y, and then pass the initial value back to the preceeding intra-sampling interval $[t_{i-1}, t_i)$ as a terminal condition via the jump condition [Wilmott *et al* 1993] at sampling date t_i given by

$$P(x, y, t_i-) = P(x, \max\{x, y\}, t_i). \tag{18}$$

It has been shown by [Wilmott *et al* 1993] that the continuously sampled arithmetic average Asian option has a value of the form

$$v(x, y, t) = xu(y, t), \tag{19}$$

where u satisfies the parabolic partial differential inequality

$$\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial y} - ru + \frac{\partial u}{\partial t} \leq 0, \tag{20}$$

together with $u \geq \psi$, where

$$\psi(y, t) := \left(1 - \frac{y}{t}\right)^+ \tag{21}$$

on $[0, \infty) \times [0, T]$, which upon suitable localization and finite difference discretization once again gives an instance of (LP$'$). For the discretely sampled case, we may solve the same recursive sequence of intra-sampling date problems, with a similar jump condition

$$P(x, y, t_i-) = P(x, x + y, t_i) \tag{22}$$

at sampling dates t_i, where the current running sum value y varies over the same localisation interval $[L, U]$ as x, to give terminal conditions in each period.

The four path-dependent exotic American options considered have all been reduced from a two state variable problem to at worst a dynamic programming type backwards sequence of parametric LP problems in one state variable, and further numerical investigation of these techniques is in progress. We turn now to a complex European (i.e. Bermudan) option in 3 state variables for which no such reduction is possible.

5 Valuation of complex differential swaps

In this section, following [Babbs 1990,1994a,1994b], we consider the numerical valuation of a cross-currency interest rate-sensitive 10 year differential (diff) swap deal with 3-monthly deferred payments in terms of current rate differentials at successive intervals, together with the option to terminate the deal at payment dates at the cost of a penalty payment in foreign currency. The valuation of this multistage swap deal stretches current PDE and computer technology, involving, as it does, three underlying correlated stochastic state variables, namely 'domestic' and 'foreign' rates and the exchange rate, plus the time variable, and with multiple decision points.

Specifically, then, in a typical 3-month period $[t_{j-1}, t_j)$, $j = 1, \dots N$, of the swap deal the market maker receives (per unit of notional principal) the foreign (3 month LIBOR) rate $L_f(t_{j-1}, t_j)$, the counterparty receives the domestic rate $L_d(t_{j-1}, t_j)$, unless the counterparty chooses to terminate for cost X in foreign currency. So, at each 3-monthly decision point t_{j-1}, the counterparty *either*:

a) pays the market maker $XS(t_{j-1})$ units of domestic currency to terminate, *or*

b) agrees to pay the market maker $p_j := (L_f(t_{j-1}, t_j) - m - L_d(t_{j-1}, t_j))\delta$ units of domestic currency in 3 months time, at t_j, to continue in the deal.

Here, S denotes the prevailing exchange rate, m is a fixed margin and δ is an appropriate quarterly interest accrual factor.

The value $V(L_d, L_f, S, t)$ (per unit of notional principal) of the deal *to the market maker*, after imposing a particular functional form on the bond price volatility term structure $\sigma(t, T)$ [Babbs 1993], may be expressed as $V(X_d, X_f, X_S, t)$ in terms of three state variables X_d, X_f and X_S driven by three independent Wiener processes through a linear relationship involving parametrically specified functions [Babbs 1994b]. This volatility specification gives rise to an extended Vasicek-type short rate process in each economy. After division by a suitable numeraire $P_d(t, H)$, namely the domestic pure discount bond price maturing at the end-date H of the economy, the normalized value $V^*(t) := V(t)/P_d(t, H)$ must be a martingale under the risk-adjusted probabilities. It therefore follows from Ito's lemma that within a period $[t_{j-1}, t_j)$ i.e. between cash flows, V^* must satisfy a parabolic PDE in the three state variables of the form

$$\frac{1}{2}\nabla\Lambda(t)(\nabla V^*)' - \frac{\partial V^*}{\partial t} = 0, \tag{23}$$

where the gradient operator is given by

$$\nabla := \left(\frac{\partial}{\partial X_d}, \frac{\partial}{\partial X_f}, \frac{\partial}{\partial X_S}\right), \tag{24}$$

the coefficient (covariance) matrix

$$\Lambda(t) = \begin{pmatrix} \lambda_d^2 & H^{df} & H^{dS} \\ H^{df} & \lambda_f^2 & H^{fS} \\ H^{dS} & H^{fS} & (H^{SS})^2 \end{pmatrix}(t) \tag{25}$$

is a function of time only, and prime denotes transpose.

At the end of the penultimate period $[t_{N-2}, t_{N-1})$, the value function V satisfies the jump condition

$$V(t_{N-1}) := XS(t_{N-1}) \wedge P_d(t_{N-1}, t_N)p_N, \tag{26}$$

reflecting the counterparty's choice of the least cost option between making the swap payment and paying to terminate. Then (26) is a terminal condition for the value function PDE in the penultimate period. Solving this penultimate period problem provides terminal conditions for the the preceeding period via the general jump condition

$$V(t_{j-1}-) := XS(t_{j-1}) \wedge [P_d(t_{j-1}, t_j)p_j + V(t_j)], \tag{27}$$

which is the same as the penultimate period's condition (26) but with the additional continuation value of $V(t_j)$. These jump conditions enable us to solve for the entire discounted value surface V^* by a period-by-period dynamic programming backwards recursion similar to that described for discretely sampled lookback and Asian options in §4. For further model details see [Hutton 1995].

6 Numerical Methods and Results for Swaps

After choosing a suitable localisation of the spatial domain as ± 3 standard deviations of the three underlying state variables (illustrated by Figure 6) from the starting pont $(0, 0, 0)$ and the corresponding boundary values there, finite difference discretization of the parabolic PDE (23) allows us to solve the valuation problem numerically in each period $[t_{j-1}, t_j)$, by solving the linear system $Cu = \theta$ defined analogously to (12). Again, in practice this is solved in the time stepping form

$$\text{solve } Au^m = \phi^m - Bu^{m-1} \quad m = 1, \ldots, M, \tag{28}$$

where m is the discretization of time remaining to the end of the period. The matrices A and B are in general order $(I-1)^3$ nested tridiagonal matrices, with 19 bands of non-zero entries (see Figure 7).

Use of implicit-type finite differences such as Crank-Nicolson necessitates the solution of the linear system (28) at each time step, i.e. solution of an $(I-1)^3$ linear system. This was attempted initially via banded matrix LU decomposition but proved infeasibly slow — at each time step m the decomposition uses $O(I^7)$ operations, and the time-dependent PDE coefficients mean that one must recompute the LU decomposition at each time step. Furthermore, this must be repeated for each period. For this problem, the fastest of the standard finite difference methods is the explicit scheme, where the matrix A reduces to diagonal, and hence each time step requires only a matrix vector multiplication. The disadvantage of the explicit method, as described in §3, is that one must choose the number of time steps M proportional to the square of the number of space steps I^2 to give a stable scheme (the exact stability condition is not known here), but in three dimensions this is more than compensated for by the speed of each time step relative to implicit schemes, and overall results in an $O(I^5)$ algorithm.

Some of the data supplied to the model are plotted in Figures 8 and 9. Table 7 shows computed values of the normalized deal value at launch for termination payments $X = 1\%$ and $X = 10,000\%$ (i.e. effectively non-terminable) of nominal. Clearly accuracy of within a basis point (.01%) is achieved in both cases in a solution time of 47s. The choice of time steps M in Table 7 illustrates the afore-mentioned stability condition. Figures 10, 11 and 12 show comparable two-dimensional cross-sections through the value surface of the terminable deal in periods 39, 20 and 1 respectively. The termination option is clearly shown in the capping of the value surface at the termination cost at the end of the period, and the surface moves as one expects with respect to the underlying state variables. The trough-shaped nature of the projected value surface in the exchange rate canonical variable X_S in period 20 (Figure 11) is due to the values chosen for the *fixed* variables which evaluate the four dimensional value surface close to the counterparty termination point.

Further work will be directed towards speeding up the solution. A relatively cheap improvement could be obtained by an adaptive time step explicit method, where the time step varies according to the (as yet unknown) stability condition. In addition, a nested tridiagonal LU decomposition could be tried on the full implicit scheme. Ultimately, however, it seems that some form of multi-grid method on a parallel computer is necessary to achieve high accuracy in a reasonable time, or indeed reasonable accuracy in reasonable time for a higher dimensional model, such as two factor interest rate term structures. Furthermore, it would be interesting to extend the linear programming approach to American interest rate derivatives, such as an American swaption. Clearly, a full implicit method will run into the same problem as encountered in the European case, but an *Alternating Direction Implicit* (ADI) discretization method, with a simplex solver adapted for tridiagonal matrices used to solve the implicit steps, could prove a powerful approach.

7 Conclusions

This paper has investigated the application of novel direct numerical methods to the valuation of both vanilla and exotic American options — with both continuous and discrete sampling — as well as to a multi-period terminable differential swap with three stochastic state variables. Further numerical investigation of discretely sampled exotics is required, and there is much scope for speeding up all algorithms implemented here. In general numerical solution of PDEs by direct methods is fast, robust and flexible, with the added advantage of giving instant risk-management parameters using the appropriate difference approximation from the values computed on the discretization mesh. The use of these methods in real-time trading systems seems to us inevitable.

References

[1] Babbs, S.H. (1990). *The Term Structure of Interest Rates: Stochastic Processes and Contingent Claims.* PhD Thesis, Imperial College, London University.

[2] Babbs, S.H. (1992). Binomial Valuation of Lookback Options. Working Paper, Midland Global Markets, London, April 1992.

[3] Babbs, S.H. (1994a). The Valuation of Cross-Currency Interest-Sensitive Claims With Application to "Diff" Swaps. Working Paper, Midland Global Markets, London, February 1994.

[4] Babbs, S.H. (1994b). Valuation of Cross-Currency Interest-Sensitive Claims Under the Cross-Currency Extended Vasicek Model: A PDE approach. Research Note, First National Bank of Chicago, London, December 1994.

[5] Borwein, J.M. and Dempster, M.A.H. (1989). The Linear Order Complementarity Problem. *Maths of OR* **14** 534–558.

[6] Carr, P. (1993). Deriving Derivatives of Derivative securities. Working paper, Johnson Graduate School of Management, Cornell University.

[7] Cryer, C.W. and Dempster, M.A.H. (1980). Equivalence of Linear Complementarity Problems and Linear Programs in Vector Lattice Hilbert spaces. *SIAM J. Control Optim.* **18** 1 76–90.

[8] Dempster, M.A.H. and Hutton, J.P. (1995). Fast Numerical Valuation of American Options by Linear Programming. To be submitted to *Mathematical Finance.*

[9] Geske, R. and Johnson, H. (1984). The American Option Valued Analytically. *J. Finance* **39** 1511–1524.

[10] Hutton, J.P. (1993). Pricing American Stock Options. Working paper 93-6, Department of Mathematics, University of Essex.

[11] Hutton, J.P. (1995). *Fast Pricing of Derivative Securities.* Ph.D. Thesis, Department of Mathematics, University of Essex.

[12] IBM Corporation. (1992). *Optimization Subroutines Library Guide and Reference Release 2.* 4th Edition.

[13] Jaillet, P., Lamberton, D. and Lapeyre, B. (1990). Variational Inequalities and the Pricing of American Options. *Acta Appl. Math.* **21** 263–289.

[14] Wilmott, P., Dewynne, J. and Howison, S. (1993). *Option Pricing: Mathematical Models and Computation.* Oxford Financial Press.

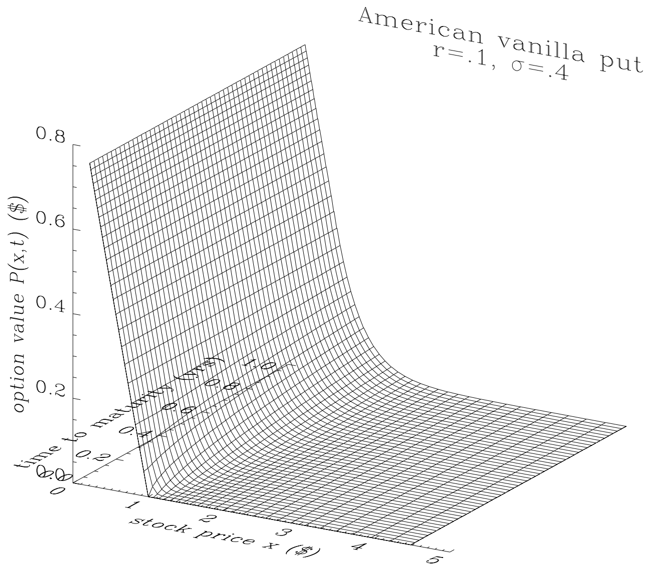

Figure 1: (LP′) solution surface with true stock price axis

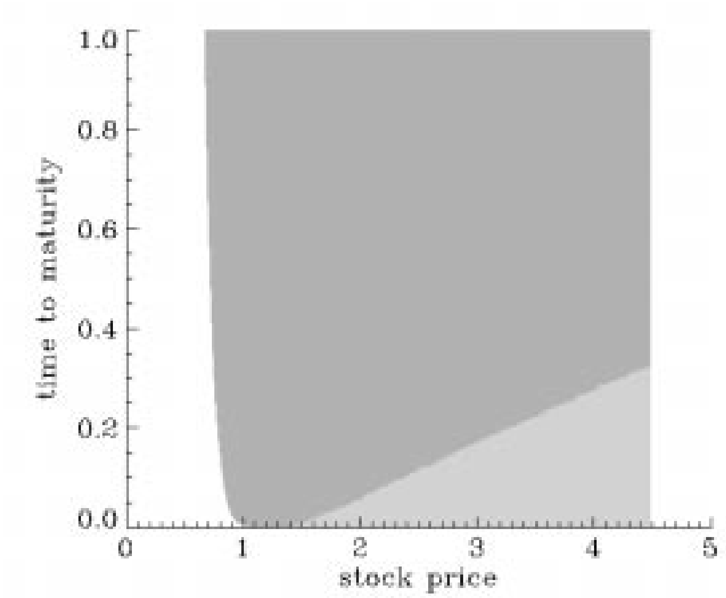

Figure 2: The computed optimal stopping boundary

Risk-free rate r	Volatility σ	Geske & Johnson $P_{\rm an}(1,0)$	Implicit $P_{LP}(1,0)$	Crank-Nicolson $P_{LP}(1,0)$	Implicit error $(\times 10^{-4})$	Crank-Nicolson error $(\times 10^{-4})$
.125	.5	.1476	.1475	.1479	-1	3
.080	.4	.1258	.1255	.1256	-2	1
.045	.3	.1005	.1001	.1004	-4	-1
.020	.2	.0712	.0708	.0710	-4	-2
.005	.1	.0377	.0374	.0375	-3	-2
.090	.3	.0859	.0858	.0861	-1	2
.040	.2	.0640	.0637	.0639	-3	-1
.010	.1	.0357	.0354	.0355	-3	-2
.080	.2	.0525	.0525	.0526	0	1
.020	.1	.0322	.0319	.0320	-3	-2
.120	.2	.0439	.0439	.0440	0	1
.030	.1	.0292	.0289	.0290	-3	-2
					$(\sum e_i^2)^{\frac{1}{2}} = 9$	$(\sum e_i^2)^{\frac{1}{2}} = 6$

Table 1: Accuracy of two American vanilla put finite difference schemes

Volatility $\sigma = .2$						
time steps $M = 1000$					Explicit $M = \max\{1000, M_{\min}\}$	
space steps I	PSOR time (s)	PSOR iterations	Simplex time (s)	Simplex iterations	M	time (s)
75	.83	17, 13	2.04	0, 3	1000	.05
150	1.56	17, 12	3.81	0, 6	1000	.1
300	2.69	17, 11	7.53	0, 13	1200	.2
600	3.50	16, 7	15.2	0, 27	4800	.61
1200	5.87	15, 6	31.3	1, 55	19200	4.9
2400	33.3	17, 16	66.2	7, 114	76800	37.0
4800	214	62, 47	144	17, 232	307200	317.0
9600	1270	214, 134	323	36, 468	1228800	5770
Volatility $\sigma = .4$						
time steps $M = 1000$					Explicit $M = \max\{1000, M_{\min}\}$	
space steps I	PSOR time (s)	PSOR iterations	Simplex time (s)	Simplex iterations	M	time (s)
75	.9	18,14	2.11	0, 9	1000	.05
150	1.55	18, 13	3.98	0, 18	1000	.1
300	1.99	18, 8	7.85	0, 38	1600	.32
600	3.29	18, 6	16.4	2, 78	6400	2.46
1200	19.1	20, 20	34.5	8, 59	25600	19.9
2400	122	72, 60	76.6	21, 323	102400	149
4800	807	250, 188	178	45, 650	409600	1280
9600	5080	831, 559	430	94, 1304	1638400	10500

Table 2: Comparative solution times for PSOR, simplex and explicit finite difference algorithms for varying space steps

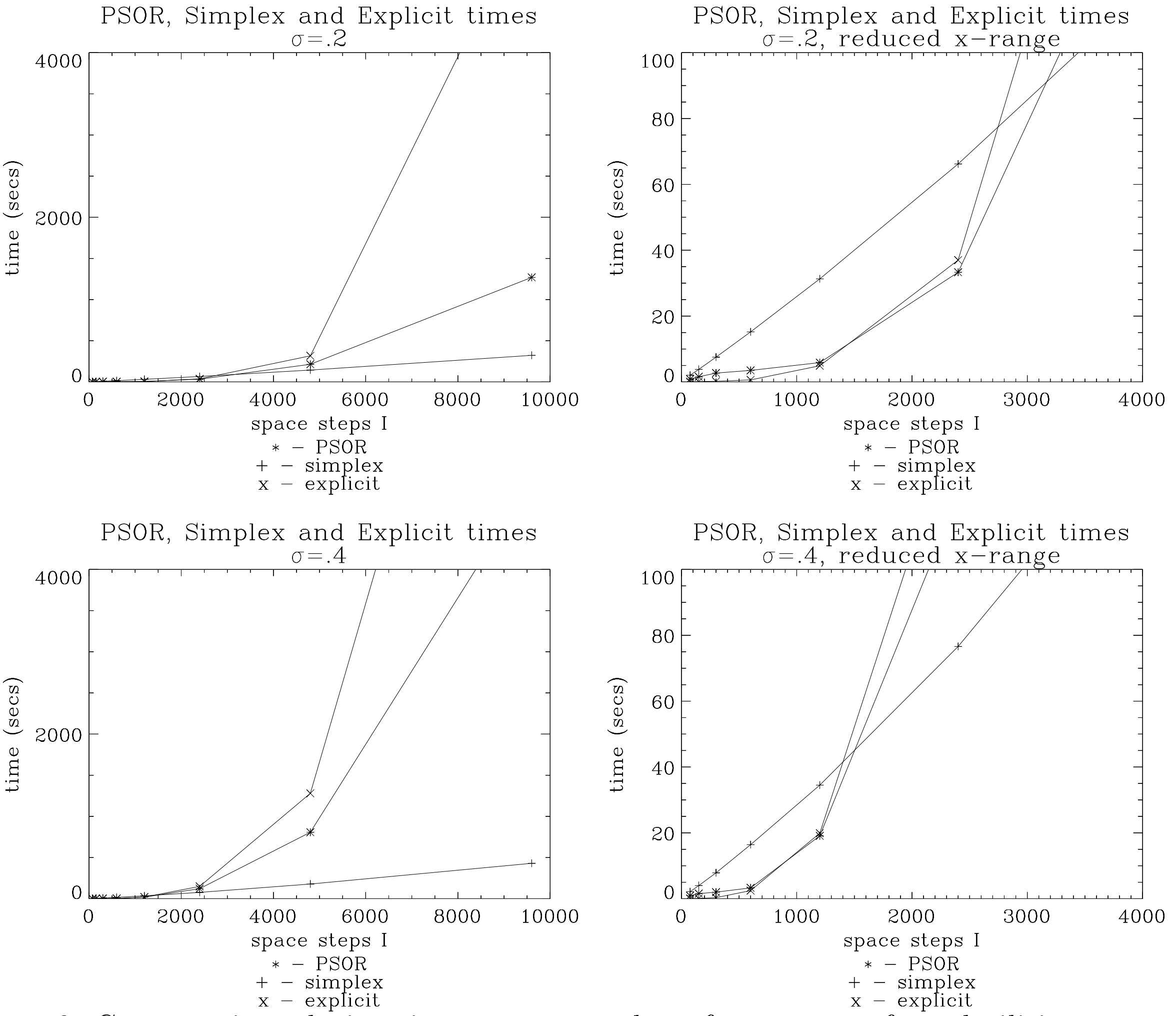

Figure 3: Comparative solution times versus number of space steps for volatilities $\sigma = 0.2$ and 0.4

Risk-free rate r	Volatility σ .05	.1	.2	.4	.8
.05	3.82	9.81	32.9	127	*
.1	3.26	9.15	32.6	122	*
.2	2.13	7.04	28.4	114	*
.4	1.64	3.80	21.1	101	*
.8	1.12	2.96	11.2	71.9	*

Risk-free rate r	Volatility σ .05	.1	.2	.4	.8
.05	24.7	26.6	31.0	41.7	46.1
.1	24.8	27.0	30.3	38.2	51.4
.2	24.8	25.3	25.9	32.8	44.9
.4	23.8	24.7	25.6	29.2	38.1
.8	23.4	24.3	25.6	26.8	33.1

Table 3: PSOR and Simplex times for varying riskless rate r and volatility σ (* $\Rightarrow$ failure to converge in 2000s)

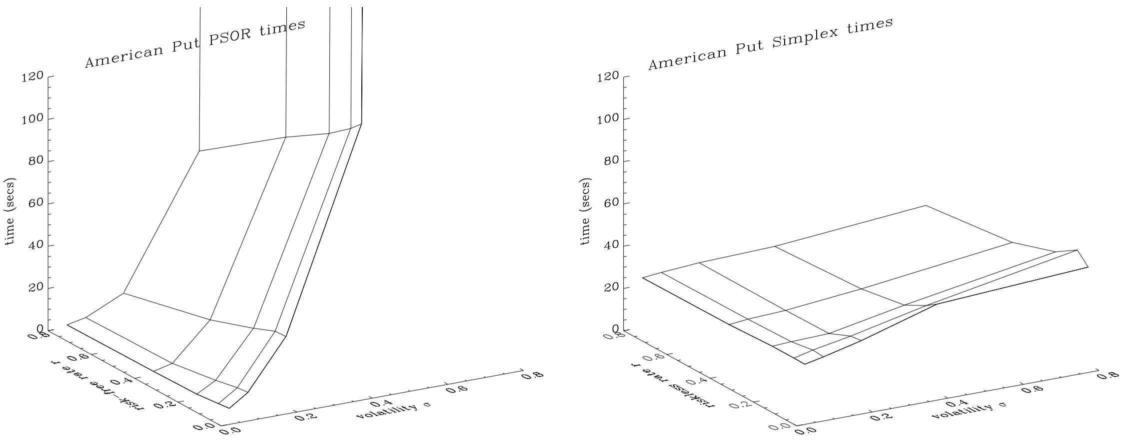

Figure 4: PSOR and simplex times for varying r and σ

Volatility $\sigma = .2$, $M = 1000$			
space	PSOR	Simplex	
steps I	time (s)	time (s)	value $P_{LP}(0, .5)$
75	.76	1.60	.1091
150	1.36	2.85	.1054
300	2.11	5.52	.1036
600	3.63	11.4	.1026
1200	17.0	24.4	.1022
2400	102	54.9	.1020
4800	632	131	.1018
9600	3330	324	.1018
Binomial value			.1017

Table 4: PSOR and Simplex results for the American lookback put with varying space steps

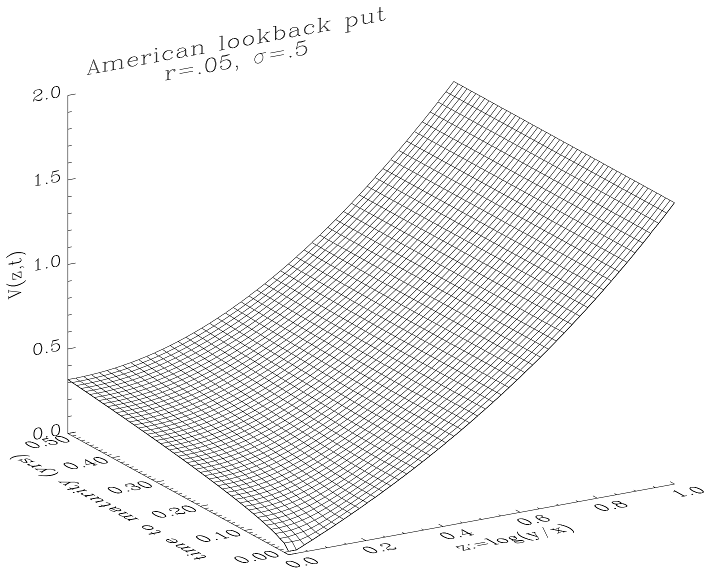

Figure 5: American lookback put value surface with exercise boundary

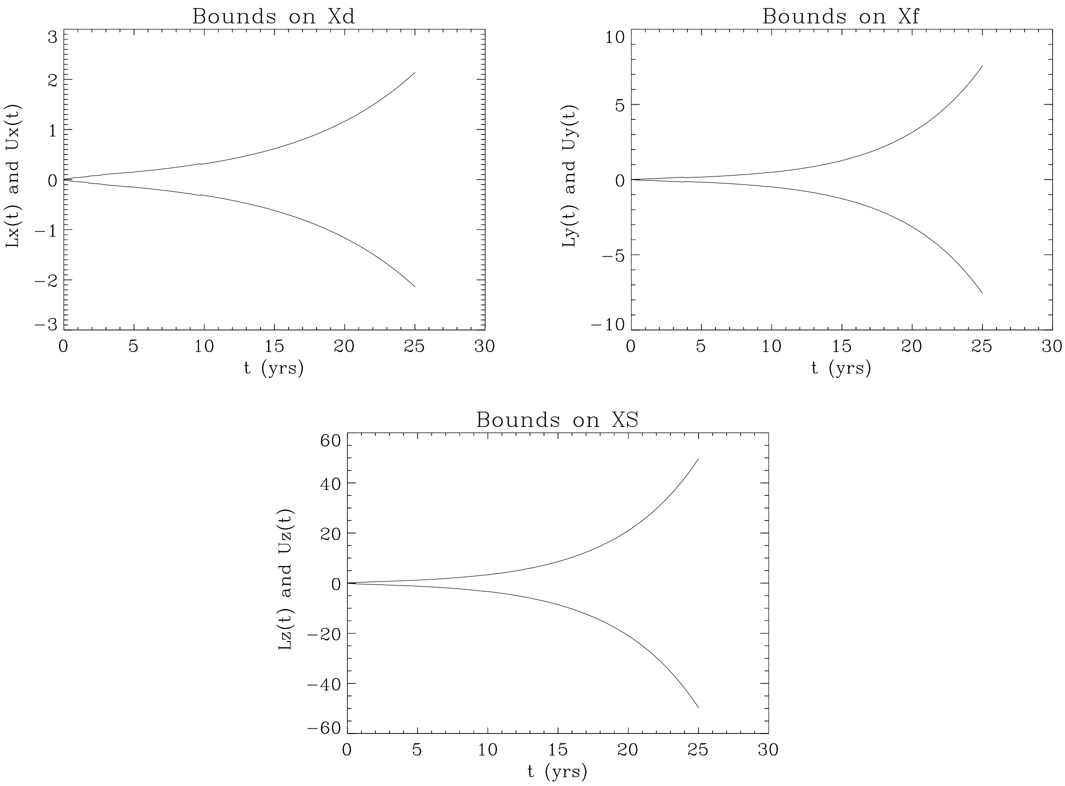

Figure 6: Bounds on Gaussian state variables $X_d(t)$, $X_f(t)$ and $X_S(t)$

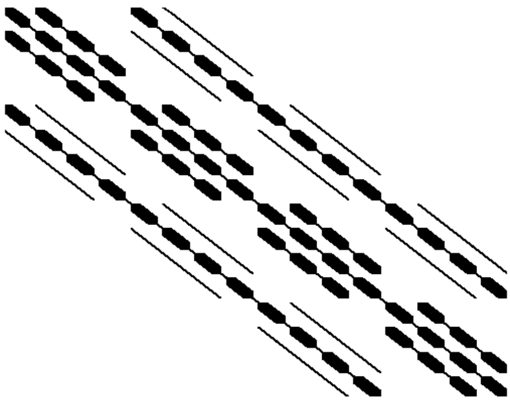

Figure 7: Bitmap of nested tridiagonal diff swap matrix A: shaded regions represent non-zero matrix elements

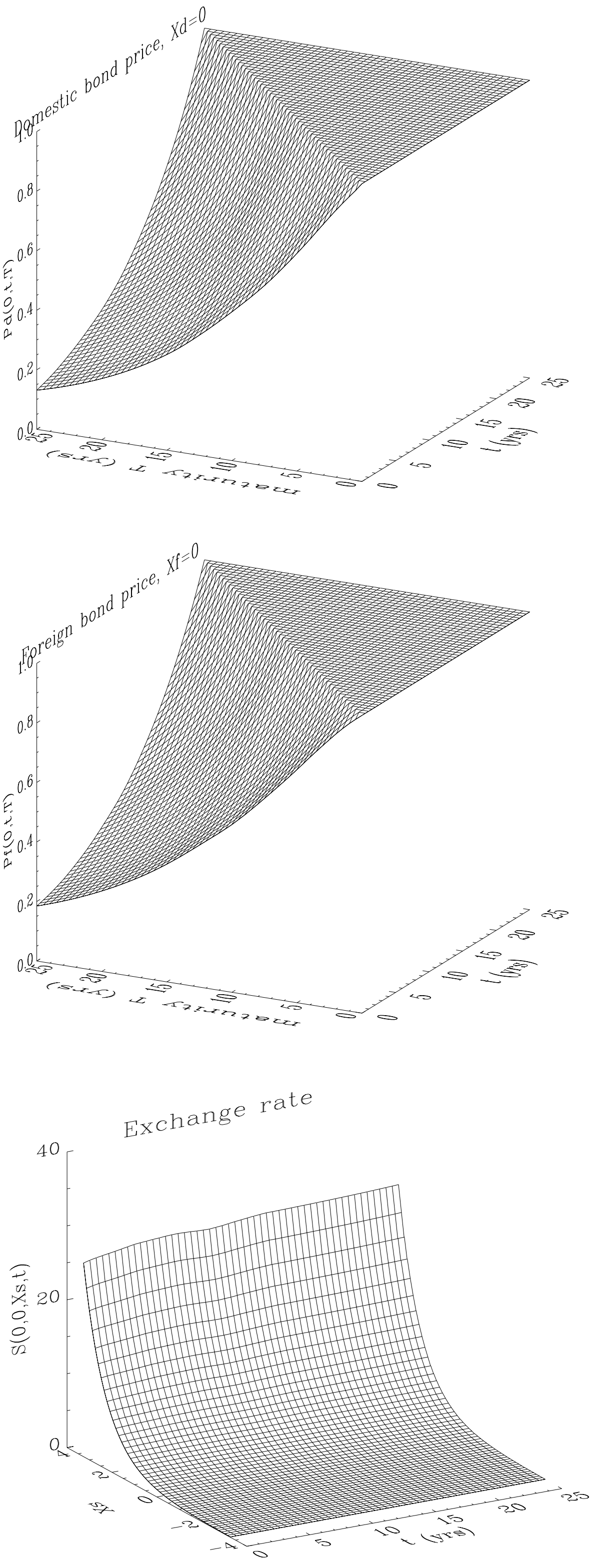

Figure 8: Bond prices $P_d(0,t,T)$ and $P_f(0,t,T)$ and exchange rate $S(0,0,X_S,t)$

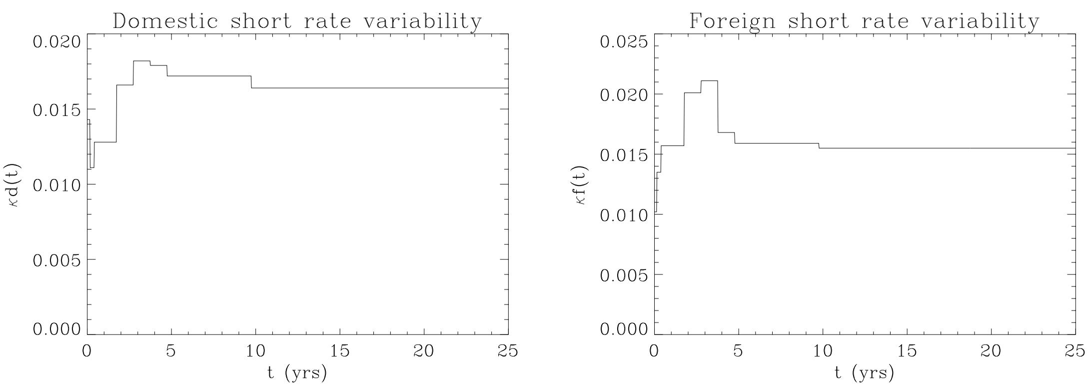

Figure 9: Prospective short rate variabilities $\kappa_d(t)$ and $\kappa_f(t)$

discretization $M \times I \times J \times K$	$X = 10000$ V	$X = .01$ V	time (s)
20×6^3	-.086798	-.124087	0.21
20×10^3	-.086293	-.129086	0.57
20×20^3	-.085919	-.123529	3.90
20×40^3	-.085815	-.123216	31.29
40×80^3	-.085750	-.123057	411.12
100×160^3	-.085721	-.122993	$\sim$ 7300.00
true value	-.085712		

Table 5: Diff swap deal value with varying discretization, just-stable explicit method.

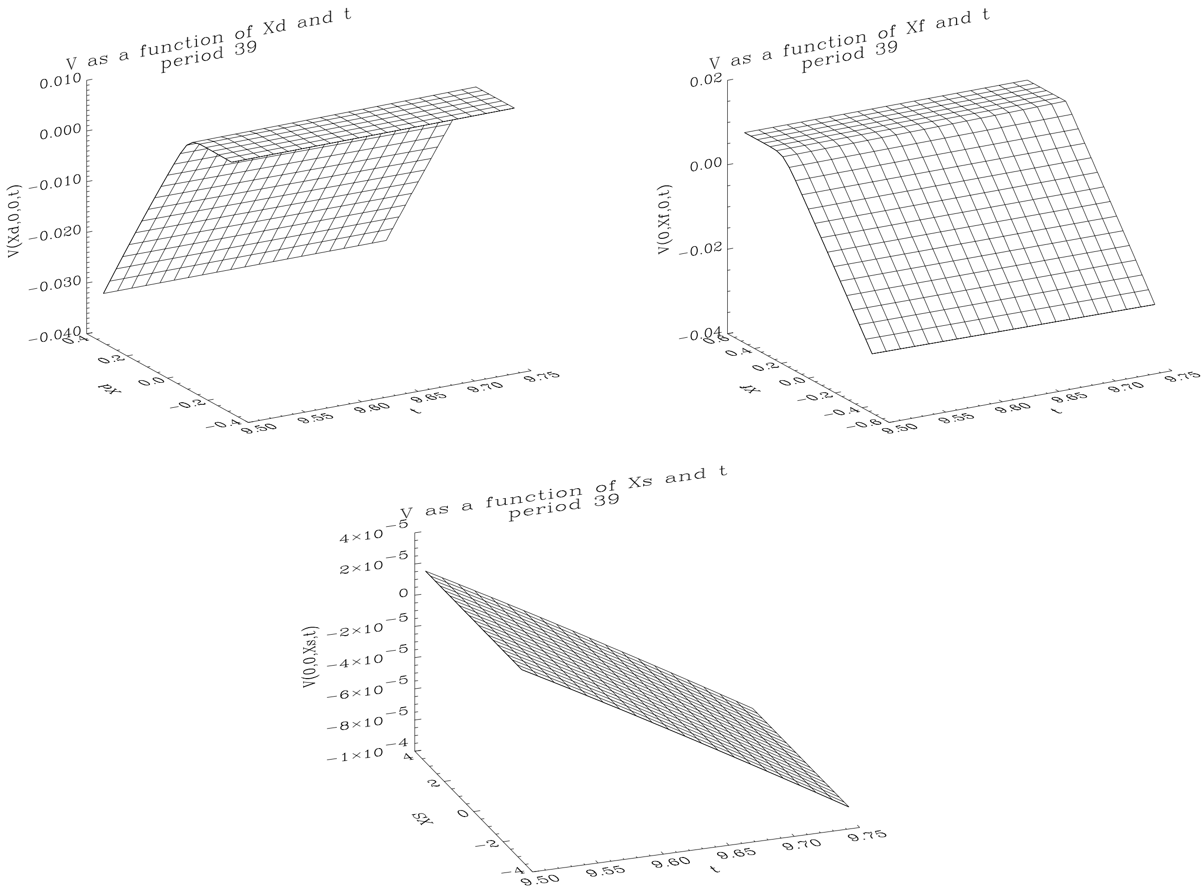

Figure 10: Solution for the penultimate period 39

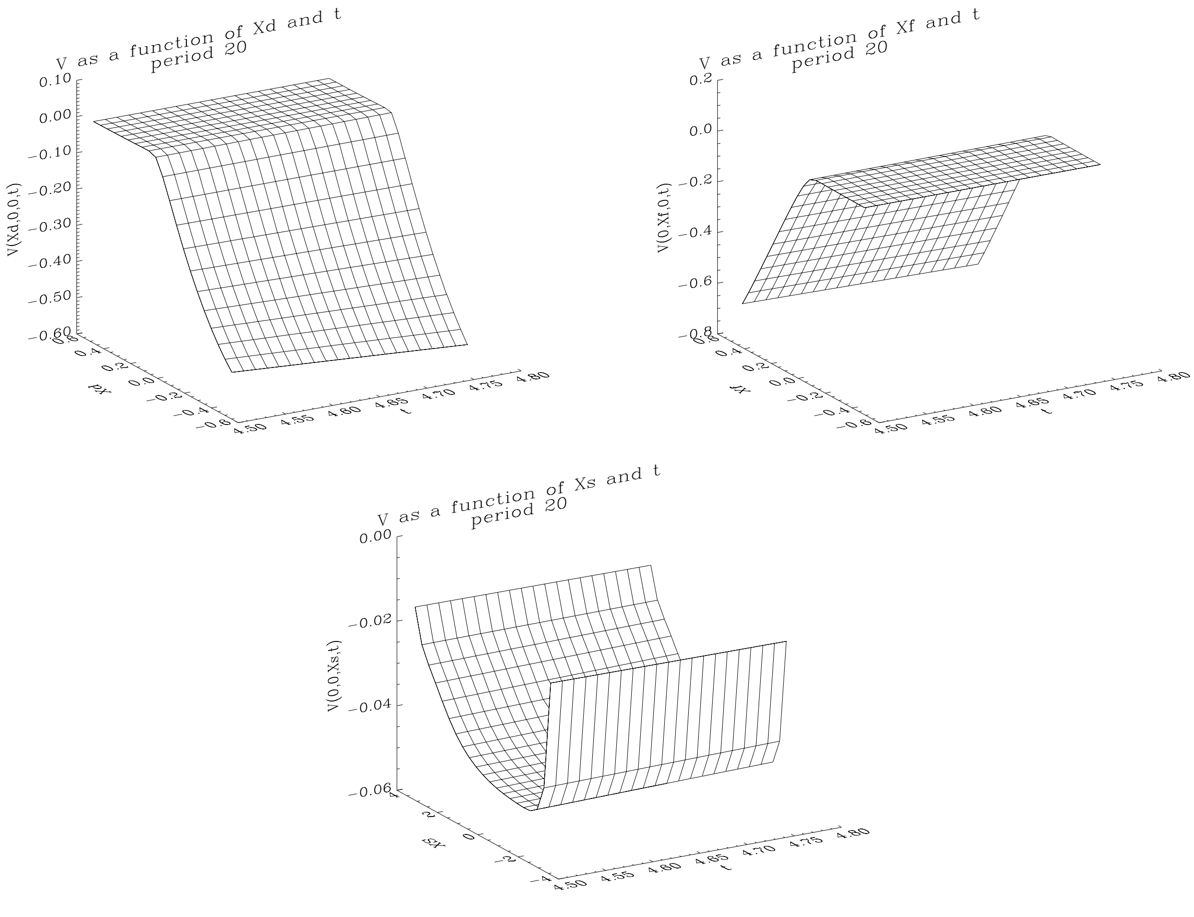

Figure 11: Solution for the middle period 20

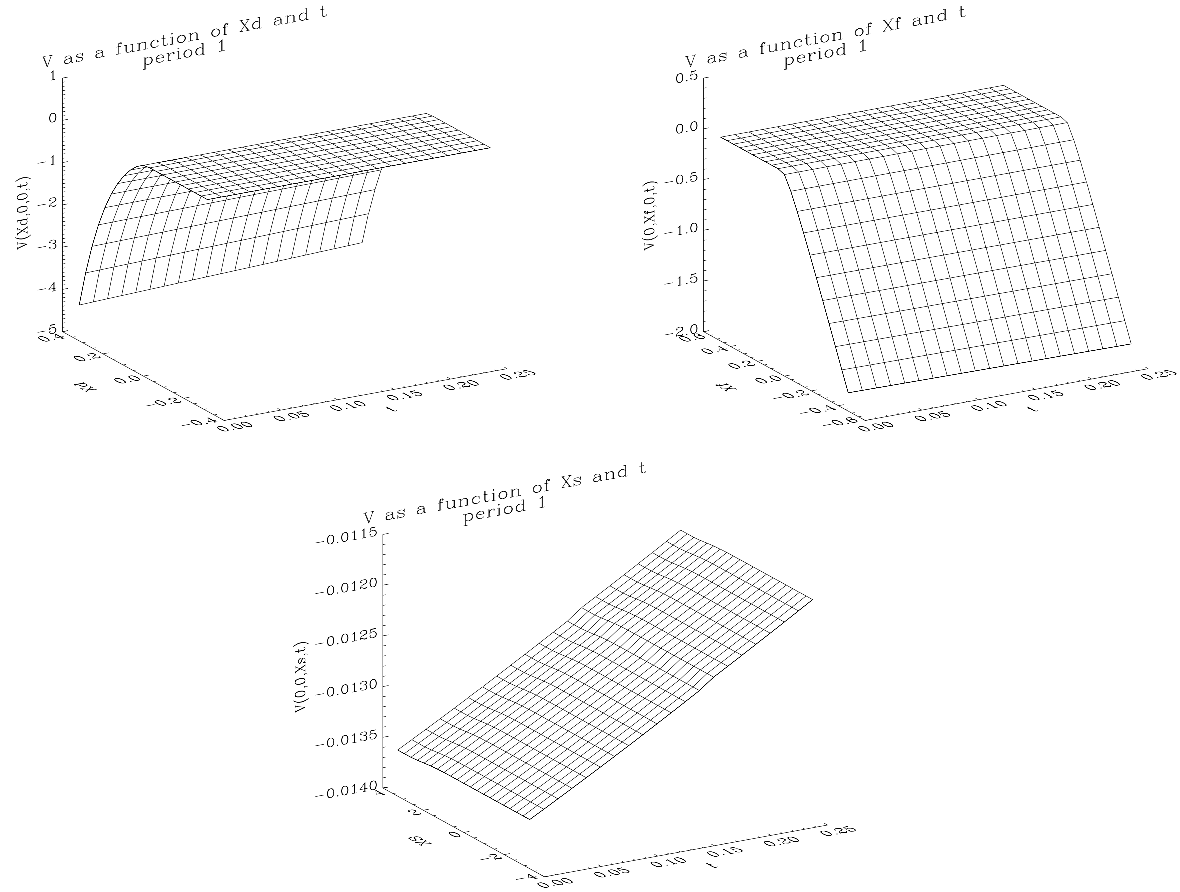

Figure 12: Solution for the first period 1