14. Cross-currency Derivatives

In this chapter, we deal with derivative securities related to at least two economies (a domestic market and a foreign market, say). Any such security will be referred to as a *cross-currency derivative*. In contrast to the model examined in Chap. 4, all interest rates and exchange rates are assumed to be random. It seems natural to expect that the fluctuations of interest rates and exchange rates will be highly correlated. This feature should be reflected in the valuation and hedging of foreign and cross-currency derivative securities in the domestic market. Feiger and Jacquillat (1979) (see also Grabbe (1983)) were probably the first to study, in a systematic way, the valuation of currency options within the framework of stochastic interest rates (they do not provide a closed-form solution for the price, however). More recently, Amin and Jarrow (1991) extended the HJM approach by incorporating foreign economies. Frachot (1995) examined a special case of the HJM model with stochastic volatilities, in which the bond price and the exchange rate are assumed to be deterministic functions of a single state variable.

The first section introduces the basic assumptions of the model along the same lines as in Amin and Jarrow (1991). In the next section, the model is further specified by postulating deterministic volatilities for all bond prices and exchange rates. We examine the arbitrage valuation of foreign market derivatives such as currency options, foreign equity options, cross-currency swaps and swaptions, and basket options (see Jamshidian (1993b, 1994b), Turnbull (1994), Frey and Sommer (1996), Brace and Musiela (1997), and Dempster and Hutton (1997)).

Let us explain briefly the last three contracts. A *cross-currency swap* is an interest rate swap agreement in which at least one of the reference interest rates is taken from a foreign market; the payments of a cross-currency swap can be denominated in units of any foreign currency, or in domestic currency. As one might guess, a *cross-currency swaption* is an option contract written on the value of a cross-currency swap. Finally, by a *basket option* we mean here an option written on a basket (i.e., weighted average) of foreign interest rates. Typical examples of such contracts are *basket caps* and *basket floors.* The final section is devoted to the valuation of foreign market interest rate derivatives in the framework of the lognormal model of forward LIBOR rates. It appears that closed-form expressions for the prices of such interest rate derivatives as quanto caps and cross-currency swaps are not easily available in this case, since the bond price volatilities follow stochastic processes with rather involved dynamics.

14.1 Arbitrage-free Cross-currency Markets

To analyze cross-currency derivatives within the HJM framework, or in a general stochastic interest rate model, we need to expand our model so that it includes foreign assets and indices. Generally speaking, the superscript i indicates that a given process represents a quantity (e.g., an exchange rate, interest rate, stock price) related to the i^{th} *foreign market.* The *exchange rate* Q^i_t of currency i, which is denominated in domestic currency per unit of the currency i, establishes the direct link between the spot domestic market and the i^{th} spot foreign market. As usual, we write $\mathbb{P}^*$ to denote the domestic martingale measure, and W^* stands for the d-dimensional standard Brownian motion under $\mathbb{P}^*$. Our aim is to construct an arbitrage-free model of foreign markets in a similar way to that of Chap. 4. In order to avoid rather standard Girsanov-type transformations, we prefer to start by postulating the "right" (that is, arbitrage-free) dynamics of all relevant processes. For instance, in order to prevent arbitrage between investments in domestic and foreign bonds, we assume that the dynamics of the i^{th} exchange rate Q^i under the measure $\mathbb{P}^*$ are

$$dQ^i_t = Q^i_t\big((r_t - r^i_t)\,dt + \nu^i_t \cdot dW^*_t\big), \quad Q^i_0 > 0, \tag{14.1}$$

where r_t and r^i_t stand for the spot interest rate in the domestic and the i^{th} foreign market, respectively. The rationale behind expression (14.1) is similar to that which leads to formula (4.14) of Chap. 4. In the case of generalized HJM methodology, the interest rate risk will be modelled by the domestic and foreign market instantaneous forward rates, denoted by $f(t,T)$ and $f^i(t,T)$ respectively. We postulate that for any maturity $T \leq T^*$, the dynamics under $\mathbb{P}^*$ of the *foreign forward rate* $f^i(t,T)$ are given by the following expression

$$df^i(t,T) = \sigma_i(t,T) \cdot (\sigma^*_i(t,T) - \nu^i_t)\,dt + \sigma_i(t,T) \cdot dW^*_t, \tag{14.2}$$

where

$$\sigma^*_i(t,T) = \int_t^T \sigma_i(t,u)\,du, \quad \forall\, t \in [0,T].$$

We assume also that for every i we are given an *initial foreign term structure* $f^i(0,T)$, $T \in [0,T^*]$, and that the foreign spot rates r^i satisfy $r^i_t = f^i(t,t)$ for every $t \in [0,T^*]$. The price $B^i(t,T)$ of a T-maturity *foreign zero-coupon bond,* denominated in foreign currency, is

$$B^i(t,T) = \exp\Big(-\int_t^T f^i(t,u)\,du\Big), \quad \forall\, t \in [0,T].$$

Consequently, the dynamics of $B^i(t,T)$ under the domestic martingale measure $\mathbb{P}^*$ are

$$dB^i(t,T) = B^i(t,T)\Big(\big(r_t^i + \nu_t^i \cdot \sigma_i^*(t,T)\big)\,dt - \sigma_i^*(t,T)\cdot dW_t^*\Big), \tag{14.3}$$

with $B^i(T,T) = 1$, or equivalently

$$dB^i(t,T) = B^i(t,T)\Big(\big(r_t^i - \nu_t^i \cdot b^i(t,T)\big)\,dt + b^i(t,T)\cdot dW_t^*\Big). \tag{14.4}$$

Similarly, we assume that the price of an arbitrary *foreign asset* Z^i that pays no dividend satisfies[1]

$$dZ_t^i = Z_t^i\Big(\big(r_t^i - \nu_t^i \cdot \xi_t^i\big)\,dt + \xi_t^i\cdot dW_t^*\Big), \quad Z_0^i > 0, \tag{14.5}$$

for some process ξ^i. For simplicity, the adapted volatility processes $\sigma_i(t,T)$, ν^i and ξ^i, taking values in $\mathbb{R}^d$, are assumed to be bounded.

Remarks. Let us denote $dW_t^i = dW_t^* - \nu_t^i\,dt$. Then (14.2) and (14.3) become

$$df^i(t,T) = \sigma_i(t,T)\cdot\sigma_i^*(t,T)\,dt + \sigma_i(t,T)\cdot dW_t^i \tag{14.6}$$

and

$$dB^i(t,T) = B^i(t,T)\Big(r_t^i\,dt - \sigma_i^*(t,T)\cdot dW_t^i\Big) \tag{14.7}$$

respectively, where W^i follows a Brownian motion under the spot probability measure $\mathbb{P}^i$ of the i^{th} market, and where the probability measure $\mathbb{P}^i$ is obtained from Girsanov's theorem (cf. (14.14)). It is instructive to compare (14.6)–(14.7) with formulas (11.14)–(11.15) of Corollary 11.1.1.

Let us verify that under (14.4)–(14.5), the combined market is arbitrage-free for both domestic and foreign-based investors. It is easily seen that processes $B^i(t,T)Q_t^i$ and $Z_t^iQ_t^i$, which represent prices of foreign assets expressed in domestic currency, satisfy

$$d\,(B^i(t,T)Q_t^i) = B^i(t,T)Q_t^i\left(r_t\,dt + \big(\nu_t^i + b^i(t,T)\big)\cdot dW_t^*\right) \tag{14.8}$$

and

$$d\,(Z_t^iQ_t^i) = Z_t^iQ_t^i\left(r_t\,dt + \big(\nu_t^i + \xi_t^i\big)\cdot dW_t^*\right). \tag{14.9}$$

Let B_t represent a domestic savings account. It follows immediately from (14.8)–(14.9) that the relative prices $B^i(t,T)Q_t^i/B_t$ and $Z_t^iQ_t^i/B_t$ of foreign assets, expressed in units of domestic currency, are local martingales under the domestic martingale measure $\mathbb{P}^*$. Because of this property, it is clear that by proceeding along the same lines as in Chap. 8, it is possible to construct an arbitrage-free model of the cross-currency market after making a judicious choice of the class of admissible trading strategies.

[1] Recall that the superscript i refers to the fact that Z_t^i is the price of a given asset at time t, expressed in units of the i^{th} foreign currency.

Remarks. The existence of short-term rates in all markets is not an essential condition if one wishes to construct an arbitrage-free model of a cross-currency market under uncertain interest rates. It is enough to postulate suitable dynamics for all zero-coupon bonds in all markets, as well as for the corresponding exchange rates. In such an approach, it is natural to make use of forward measures, rather than spot martingale measures. Assume, for instance, that $B(t,T^*)$ models the price of a domestic bond for the horizon date T^*, and $\mathbb{P}_{T^*}$ is the domestic forward measure for this date. For any fixed i, we need to specify the dynamics of the foreign bond price $B^i(t,T)$, expressed in units of the i^{th} currency, and the exchange rate process Q^i. In such an approach, it is sufficient to assume that for every T, the process

$$\tilde{F}_B^i(t,T,T^*) = \frac{B^i(t,T)Q_t^i}{B(t,T^*)}, \quad \forall\, t \in [0,T],$$

follows a local martingale under $\mathbb{P}_{T^*}$. One needs to impose the standard conditions that rule out arbitrage between foreign bonds, as seen from the perspective of a foreign-based investor (a similar remark applies to any foreign market asset).

14.1.1 Forward Price of a Foreign Asset

Let us start by analyzing the forward price of a foreign bond in the domestic market. It is not hard to check that for any maturities $T \le U$, the dynamics of the forward price $F_{B^i}(t,U,T) = B^i(t,U)/B^i(t,T)$, under the domestic martingale measure $\mathbb{P}^*$, expressed in the i^{th} foreign currency, satisfy

$$dF_{B^i}(t,U,T) = F_{B^i}(t,U,T)\gamma^i(t,U,T)\cdot\Big(dW_t^* - \big(\nu_t^i + b^i(t,T)\big)\,dt\Big), \quad (14.10)$$

where $\gamma^i(t,U,T) = b^i(t,U) - b^i(t,T)$. On the other hand, when expressed in units of the domestic currency, the forward price at time t for settlement at date T of the U-maturity zero-coupon bond of the i^{th} foreign market equals[2]

$$\tilde{F}_B^i(t,U,T) = B^i(t,U)Q_t^i/B(t,T), \quad \forall\, t \in [0,T]. \quad (14.11)$$

Relationship (14.11) is in fact a universal property, meaning that it can be derived by simple no-arbitrage arguments, independently of the model of term structure. Notice that

$$\tilde{F}_B^i(t,U,T) = \frac{B^i(t,U)Q_t^i}{B(t,T)} \neq \frac{B^i(t,U)Q_t^i}{B^i(t,T)} = Q_t^i F_{B^i}(t,U,T),$$

in general. This means that the domestic forward price of a foreign bond does not necessarily coincide with the foreign market forward price of the bond,

[2] It should be made clear that we consider here a forward contract in which a U-maturity foreign bond is delivered at time T, in exchange for $\tilde{F}_B^i(t,U,T)$ units of the domestic currency.

when the latter is converted into domestic currency at the current exchange rate. It is useful to observe that in the special case when $T = U$, the forward price $\tilde{F}^i_B(t,T,T)$ satisfies

$$\tilde{F}^i_B(t,T,T) = B^i(t,T)Q^i_t/B(t,T) = F_{Q^i}(t,T), \quad \forall\, t \in [0,T], \tag{14.12}$$

i.e., it agrees with the *forward exchange rate* for the settlement date T (cf. formula (4.16) in Chap. 4). More generally, we have the following result, which is valid for any foreign market asset Z^i (recall that the price Z^i_t is expressed in units of the $i^{\rm th}$ foreign currency).

Lemma 14.1.1. *The domestic forward price $\tilde{F}_{Z^i}(t,T)$ for the settlement at time T of the foreign market security Z^i (which pays no dividends) satisfies*

$$\tilde{F}_{Z^i}(t,T) = \frac{Z^i_t Q^i_t}{B(t,T)} = F_{Z^i}(t,T)F_{Q^i}(t,T). \tag{14.13}$$

Proof. The first equality follows by standard no-arbitrage arguments. For the second, notice that

$$\frac{Z^i_t Q^i_t}{B(t,T)} = \frac{Z^i_t}{B^i(t,T)}\frac{B^i(t,T)}{B(t,T)}\,Q^i_t = F_{Z^i}(t,T)F_{Q^i}(t,T),$$

where $F_{Z^i}(t,T)$ is the foreign forward price (in units of the $i^{\rm th}$ currency). □

For our further purposes, it is useful to examine the dynamics of the forward price of a foreign market asset. Let us start by analyzing the case of a foreign zero-coupon bond. It is easily seen that for any choice of maturities $T \le U \le T^*$, the dynamics of the forward price process $\tilde{F}^i_B(t,U,T)$ under the domestic martingale measure $\mathbb{P}^*$ are given by the expression

$$d\tilde{F}^i_B(t,U,T) = \tilde{F}^i_B(t,U,T)\Big(\nu^i_t + b^i(t,U) - b(t,T)\Big)\cdot\Big(dW^*_t - b(t,T)dt\Big),$$

or, in the standard HJM framework

$$d\tilde{F}^i_B(t,U,T) = \tilde{F}^i_B(t,U,T)\Big(\nu^i_t + \sigma^*(t,T) - \sigma^*_i(t,U)\Big)\cdot\Big(dW^*_t + \sigma^*(t,T)\,dt\Big),$$

since $b(t,T) = -\sigma^*(t,T)$. Similarly, the dynamics of the forward price $\tilde{F}_{Z^i}(t,T)$ under the domestic martingale measure $\mathbb{P}^*$ are

$$d\tilde{F}_{Z^i}(t,T) = \tilde{F}_{Z^i}(t,T)\Big(\nu^i_t + \xi^i_t - b(t,T)\Big)\cdot\Big(dW^*_t - b(t,T)dt\Big),$$

that is

$$d\tilde{F}_{Z^i}(t,T) = \tilde{F}_{Z^i}(t,T)\Big(\nu^i_t + \xi^i_t + \sigma^*(t,T)\Big)\cdot\Big(dW^*_t + \sigma^*(t,T)\,dt\Big).$$

Let $\mathbb{P}^i$ be the probability measure on $(\Omega,\mathcal{F}_{T^*})$ defined by the Doléans exponential

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^*} = \mathcal{E}_{T^*}(U^i) = \mathcal{E}_{T^*}\Big(\int_0^\cdot \nu_u^i \cdot dW_u^*\Big), \quad \mathbb{P}^*\text{-a.s.} \tag{14.14}$$

By virtue of Girsanov's theorem, the process W^i, which is given by the formula

$$W_t^i = W_t^* - \int_0^t \nu_u^i \, du, \quad \forall\, t \in [0, T^*],$$

follows a Brownian motion under the probability measure $\mathbb{P}^i$. Since

$$\begin{aligned} df^i(t,T) &= \sigma_i(t,T) \cdot \sigma_i^*(t,T)\, dt + \sigma_i(t,T) \cdot dW_t^i, \\ dB^i(t,T) &= B^i(t,T)\Big(r_t^i\, dt - \sigma_i^*(t,T) \cdot dW_t^i\Big), \end{aligned}$$

and

$$dZ_t^i = Z_t^i\Big(r_t^i\, dt + \xi_t^i \cdot dW_t^i\Big),$$

we conclude that the probability measure $\mathbb{P}^i$ is the (spot) martingale measure of the i^{th} foreign market (cf. formulas (4.7)–(4.8) of Chap. 4). Let us now examine the corresponding forward probability measures. Recall that the forward measure $\mathbb{P}_T$ in the domestic market is given on $(\Omega, \mathcal{F}_T)$ by means of the following expression (cf. formula (9.32) in Sect. 9.6)

$$\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \mathcal{E}_T(U^T) = \mathcal{E}_T\Big(\int_0^\cdot b(u,T) \cdot dW_u^*\Big), \quad \mathbb{P}^*\text{-a.s.}$$

Moreover, under the domestic forward measure $\mathbb{P}_T$, the process W^T, which equals

$$W_t^T = W_t^* - \int_0^t b(u,T)\, du = W_t^* + \int_0^t \sigma^*(u,T)\, du,$$

is a d-dimensional standard Brownian motion. Analogously, the forward measure for the i^{th} foreign market, denoted by $\mathbb{P}_T^i$, is defined on $(\Omega, \mathcal{F}_T)$ by the formula

$$\frac{d\mathbb{P}_T^i}{d\mathbb{P}^i} = \mathcal{E}_T(U^{T,i}) = \mathcal{E}_T\left(\int_0^\cdot b^i(u,T)\, dW_u^i\right), \quad \mathbb{P}^*\text{-a.s.} \tag{14.15}$$

The process $W^{T,i}$, which satisfies

$$W_t^{T,i} = W_t^i - \int_0^t b^i(u,T)\, du = W_t^* - \int_0^t \big(\nu_u^i + b^i(u,T)\big)\, du, \tag{14.16}$$

follows a d-dimensional standard Brownian motion under $\mathbb{P}_T^i$. Furthermore, the foreign market forward rate $f^i(\cdot, T)$ follows a local martingale under $\mathbb{P}_T^i$, more explicitly

$$df^i(t,T) = \sigma_i(t,T) \cdot dW_t^{T,i}.$$

The next result links the forward measure of a foreign market to the domestic spot martingale measure.

Lemma 14.1.2. *The Radon-Nikodým derivative on* $(\Omega, \mathcal{F}_T)$ *of the forward measure* $\mathbb{P}^i_T$ *of the* i^{th} *foreign market with respect to the domestic spot martingale measure* $\mathbb{P}^*$ *equals*

$$\frac{d\mathbb{P}^i_T}{d\mathbb{P}^*} = \mathcal{E}_T(V^{T,i}), \quad \mathbb{P}^*\text{-a.s.}, \tag{14.17}$$

where

$$V^{T,i}_t = \int_0^t \big(\nu^i_u + b^i(u,T)\big) \cdot dW^*_u, \quad \forall\, t \in [0,T].$$

Proof. For any two continuous semimartingales X, Y defined on a probability space $(\Omega, \mathbb{F}, \mathbb{Q})$, with $X_0 = Y_0 = 0$, we have (see Theorem II.37 in Protter (1990))

$$\mathcal{E}_t(X)\,\mathcal{E}_t(Y) = \mathcal{E}_t(X + Y + \langle X, Y\rangle), \quad \forall\, t \in [0,T].$$

Applying this equality to the density

$$\frac{d\mathbb{P}^i_T}{d\mathbb{P}^*} = \frac{d\mathbb{P}^i_T}{d\mathbb{P}^i}\,\frac{d\mathbb{P}^i}{d\mathbb{P}^*} = \mathcal{E}_T(U^{T,i})\,\mathcal{E}_T(U^i),$$

we obtain

$$\frac{d\mathbb{P}^i_T}{d\mathbb{P}^*} = \mathcal{E}_T(U^{T,i} + U^i + \langle U^{T,i}, U^i\rangle). \tag{14.18}$$

Furthermore, by virtue of (14.14) and (14.15), we find that

$$\langle U^{T,i}, U^i\rangle_t = \int_0^t b^i(u,T) \cdot \nu^i_u\, du,$$

and thus for every $t \in [0,T]$

$$U^{T,i}_t + U^i_t + \langle U^{T,i}, U^i\rangle_t = \int_0^t b^i(u,T) \cdot (dW^*_u - \nu^i_u\, du)$$
$$+ \int_0^t \nu^i_u \cdot dW^*_u + \int_0^t b^i(u,T) \cdot \nu^i_u\, du = \int_0^t (\nu^i_u + b^i(u,T)) \cdot dW^*_u = V^{T,i}_t.$$

Combining the last equality with (14.18), we obtain (14.17). □

The next auxiliary result, which gives the density of the foreign forward measure with respect to the domestic forward measure, can be proved along similar lines.

Lemma 14.1.3. *The following formula holds*

$$\frac{d\mathbb{P}^i_T}{d\mathbb{P}_T} = \mathcal{E}_T(Z^{T,i}), \quad \mathbb{P}_T\text{-a.s.},$$

where

$$Z^{T,i}_t = \int_0^t \big(\nu^i_u + b^i(u,T) - b(u,T)\big) \cdot dW^T_u, \quad \forall\, t \in [0,T].$$

14.1.2 Valuation of Foreign Contingent Claims

In this section, we deal with the valuation of general contingent claims denominated in foreign currency. Consider a time T *contingent claim* Y^i in the i^{th} foreign market – that is, a contingent claim denominated in the currency of market i. We assume as usual that Y^i is a random variable, measurable with respect to the σ-field $\mathcal{F}_T$. Under appropriate integrability conditions, its arbitrage price at time t, expressed in domestic currency, is

$$\pi_t(Y^i) = B_t \, \mathbb{E}_{\mathbb{P}^*}\big(Y^i Q^i_T / B_T \,|\, \mathcal{F}_t\big) = B(t,T)\, \mathbb{E}_{\mathbb{P}_T}(Y^i Q^i_T \,|\, \mathcal{F}_t),$$

where the second equality is a consequence of Lemma 9.6.3. Indeed, a claim $X_T = Y^i Q^i_T$, which is denominated in units of domestic currency, can be priced as any "usual" domestic contingent claim. An alternative way of valuing Y^i is to first determine the price $\pi^i_t(Y^i)$ in units of foreign currency, which is

$$\pi^i_t(Y^i) = B^i_t \, \mathbb{E}_{\mathbb{P}^i}\big(Y^i / B^i_T \,|\, \mathcal{F}_t\big) = B^i(t,T)\, \mathbb{E}_{\mathbb{P}^i_T}(Y^i \,|\, \mathcal{F}_t), \tag{14.19}$$

and then to convert it into domestic currency, using the current exchange rate. This means that we have

$$\pi_t(Y^i) = Q^i_t \pi^i_t(Y^i) = Q^i_t B^i(t,T)\, \mathbb{E}_{\mathbb{P}^i_T}(Y^i \,|\, \mathcal{F}_t). \tag{14.20}$$

The former method for the valuation of foreign market contingent claims is frequently referred to as the *domestic market method,* while the latter is known as the *foreign market method.* Since the arbitrage price is uniquely defined, both methods must necessarily give the same price for any given foreign claim. A comparison of (14.19) and (14.20) yields immediately an interesting equality

$$B_t \, \mathbb{E}_{\mathbb{P}^*}\big(Y^i Q^i_T B_T^{-1} \,|\, \mathcal{F}_t\big) = Q^i_t B^i_t \, \mathbb{E}_{\mathbb{P}^i}\big(Y^i (B^i_T)^{-1} \,|\, \mathcal{F}_t\big), \tag{14.21}$$

which can alternatively be proved by standard arguments. To show more directly that (14.21) holds, observe that

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^*} = \mathcal{E}_T(U^i) \stackrel{\text{def}}{=} \xi^i_T, \quad \mathbb{P}^*\text{-a.s.}$$

On the other hand, the exchange rate Q^i is easily seen to satisfy

$$Q^i_t = Q^i_0 B_t (B^i_t)^{-1}\, \mathcal{E}_t(U^i), \quad \forall\, t \in [0,T^*],$$

so that Q^i_T and Q^i_t satisfy the following relationship

$$Q^i_T = Q^i_t B_T B^i_t \xi^i_T \left(B_t B^i_T \xi_t\right)^{-1}. \tag{14.22}$$

Consequently, from Bayes rule we get

$$\mathbb{E}_{\mathbb{P}^i}\big(Y^i Q^i_t B^i_t (B^i_T)^{-1} \,|\, \mathcal{F}_t\big) = \frac{\mathbb{E}_{\mathbb{P}^*}(Y^i Q^i_t B^i_t (B^i_T)^{-1} \xi^i_T \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}^*}(\xi^i_T \,|\, \mathcal{F}_t)}.$$

Finally, taking (14.22) into account, we obtain

$$Q_t^i B_t^i \, \mathbb{E}_{\mathbb{P}^i}\big(Y^i/B_T^i \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{P}^*}\left(\frac{Y^i Q_t^i B_t^i \xi_T^i}{\xi_t^i B_T^i} \,\Big|\, \mathcal{F}_t\right) = B_t \, \mathbb{E}_{\mathbb{P}^*}(Y^i Q_T^i / B_T \,|\, \mathcal{F}_t),$$

as expected.

14.1.3 Cross-currency Rates

In some instances it will be convenient to consider a *cross-currency rate,* which is simply the exchange rate between two foreign currencies. Consider two foreign markets, say l and m, and denote by $Q^{m/l}$ the corresponding cross-currency rate. More specifically, we assume that the exchange rate $Q^{m/l}$ is the price of one unit of currency l denominated in currency m. In terms of our previous notation, we have

$$Q_t^{m/l} \overset{\text{def}}{=} Q_t^l / Q_t^m, \quad \forall\, t \in [0, T^*],$$

hence by the Itô formula

$$dQ_t^{m/l} = Q_t^{m/l}\Big(\big(r_t^m - r_t^l - \nu_t^m \cdot (\nu_t^l - \nu_t^m)\big)\, dt + (\nu_t^l - \nu_t^m) \cdot dW_t^*\Big),$$

which, after rearranging, gives

$$dQ_t^{m/l} = Q_t^{m/l}\Big(\big(r_t^m - r_t^l\big)\, dt + \big(\nu_t^l - \nu_t^m\big) \cdot dW_t^m\Big),$$

where W^m is a Brownian motion of the m^{th} foreign market under the spot martingale measure $\mathbb{P}^m$. Concluding, we can identify the volatility $\nu^{m/l}$ of the exchange rate $Q^{m/l}$ in terms of the volatilities ν^l and ν^m of the exchange rates Q^l and Q^m, respectively, as

$$\nu_t^{m/l} = \nu_t^l - \nu_t^m \tag{14.23}$$

for every $t \in [0, T^*]$.

14.2 Gaussian Model

We do not present here a systematic study of various option contracts based on foreign currencies, bonds and equities. We consider instead just a few typical examples of foreign market options (cf. Chap. 4). For simplicity, we assume throughout that the volatilities of all prices and exchange rates involved in a given contract follow deterministic functions. This assumption, which can be substantially weakened in some circumstances, leads to closed-form solutions for the prices of typical cross-currency options. Results obtained in this section are straightforward generalizations of option valuation formulas established in Chap. 4.

14.2.1 Currency Options

The first task is to examine the arbitrage valuation of European currency options in a stochastic interest rate framework. Recall that the forward exchange rate $F_{Q^i}(t,T)$ may be interpreted as the forward price for the settlement date T of one unit of foreign currency (i.e., of a foreign zero-coupon bond that matures at T). This implies the martingale property of F^i_Q under the domestic forward probability measure. More precisely, for any fixed T, we have under the domestic forward measure $\mathbb{P}_T$

$$dF_{Q^i}(t,T) = F_{Q^i}(t,T)\sigma_{Q^i}(t,T) \cdot dW^T_t \tag{14.24}$$

for a deterministic function $\sigma_{Q^i}(\cdot,T) : [0,T] \to \mathbb{R}$. In view of (14.12), the volatility σ_{Q^i} can be expressed in terms of bond price volatilities and the volatility of the exchange rate. For any maturity $T \in [0,T^*]$, we have

$$\sigma_{Q^i}(t,T) = \nu^i_t + b^i(t,T) - b(t,T), \quad \forall\, t \in [0,T]. \tag{14.25}$$

Our goal is to value a European currency call option with the payoff at expiry date T

$$C^{Q^i}_T \stackrel{\text{def}}{=} N(Q^i_T - K)^+ = N(F_{Q^i}(T,T) - K)^+,$$

where N is a preassigned number of units of foreign currency (we set $N = 1$ in what follows), K is the strike exchange rate, and T is the option expiry date. The arbitrage price of such an option under deterministic interest rates was found in Chap. 4 (see Proposition 4.2.2). Under the present assumption – that is, when $\sigma_{Q^i}(t,T)$ is deterministic – the closed-form expression for the price of a currency option can be established using the forward measure approach. Since $C^{Q^i}_T$ is expressed in domestic currency, it is enough to find the expected value of the option's payoff under the domestic forward probability measure $\mathbb{P}_T$ for the date T. Since this involves no difficulties, we prefer instead to apply a simple approach to the replication of currency options, based on the idea employed in Sect. 11.3.5. We claim that for every $t \in [0,T]$, we have

$$C^{Q^i}_t = B^d(t,T)\Big(F^i_t\, N\big(\tilde{d}_1(F^i_t,t,T)\big) - KN\big(\tilde{d}_2(F^i_t,t,T)\big)\Big), \tag{14.26}$$

where $F^i_t = F_{Q^i}(t,T)$ is the forward exchange rate,

$$\tilde{d}_{1,2}(F,t,T) = \frac{\ln(F/K) \pm \frac{1}{2}\, v^2_{Q^i}(t,T)}{v_{Q^i}(t,T)},$$

and $v_{Q^i}(t,T)$ represents the volatility of the forward exchange rate integrated over the time interval $[t,T]$ – that is

$$v^2_{Q^i}(t,T) = \int_t^T |\sigma_{Q^i}(u,T)|^2\, du = \int_t^T |\nu^i_u + b^i(u,T) - b(u,T)|^2\, du.$$

Formula (14.26) can be rewritten as follows

$$C_t^{Q^i} = B^i(t,T)Q_t^i N\big(h_1(Q_t^i,t,T)\big) - KB(t,T)\,N\big(h_2(Q_t^i,t,T)\big), \qquad (14.27)$$

where

$$h_{1,2}(Q_t^i,t,T) = \frac{\ln(Q_t^i/K) + \ln(B^i(t,T)/B(t,T)) \pm \frac{1}{2}\,v_{Q^i}^2(t,T)}{v_{Q^i}(t,T)}\,.$$

To check, in an intuitive way, the validity of (14.26) for $t = 0$, let us consider the following combined spot-forward trading strategy: at time 0 we purchase $F_C(0,T) = C_0^{Q^i}/B^d(0,T)$ zero-coupon domestic bonds maturing at T; in addition, at any time $t \in [0,T]$, we are long $\psi_t^1 = N(\tilde{d}_1(F_t^i,t,T))$ forward currency contracts. The wealth of this portfolio at expiry equals

$$F_C(0,T) + \int_0^T \psi_t^1\,dF_{Q^i}(t,T) = (Q_T^i - K)^+,$$

since direct calculations yield

$$N(\tilde{d}_1(F_t^i,t,T))\,dF_{Q^i}(t,T) = dF_C(t,T),$$

where $F_C(t,T) = C_t^{Q^i}/B^d(t,T)$ is the forward price of the option. We conclude that formula (14.26) is valid for $t = 0$. A general formula can be established by similar arguments. For the valuation formula (14.27) to hold, it is sufficient to assume that the volatility of the forward exchange rate follows a deterministic function.

14.2.2 Foreign Equity Options

The following examples deal with various kinds of European options written on a foreign market asset.

Option on a foreign asset struck in foreign currency. Let Z^i stand for the price of a foreign asset (for instance, a bond or a stock). We consider a European call option with the payoff at expiry

$$C_T^1 \stackrel{\text{def}}{=} Q_T^i(Z_T^i - K^i)^+,$$

where K^i is the strike price, denominated in the i^{th} foreign currency. To price this option, it is convenient to apply the foreign market method. It appears that it is sufficient to convert the foreign price of the option into domestic currency at the current exchange rate. Therefore, we get an intuitively obvious result (cf. Sect. 4.5 and Corollary 11.3.3 of Sect. 11.3)

$$C_t^1 = Q_t^i\Big(Z_t^i\,N\big(g_1(Z_t^i,t,T)\big) - K^iB^i(t,T)\,N\big(g_2(Z_t^i,t,T)\big)\Big), \qquad (14.28)$$

where

$$g_{1,2}(Z_t^i,t,T) = \frac{\ln(Z_t^i/K^i) - \ln B^i(t,T) \pm \frac{1}{2}\,v_{Z^i}^2(t,T)}{v_{Z^i}(t,T)}$$

and

$$v_{Z^i}^2(t,T) = \int_t^T |\xi_u^i - b^i(u,T)|^2 \, du.$$

Note that this result remains valid even if the volatility of the exchange rate is random, provided that the volatility function of the asset's foreign market forward price is deterministic.

Option on a foreign asset struck in domestic currency. Suppose now that the option on a foreign asset has its strike price expressed in domestic currency, so that the payoff from the option at expiry equals

$$C_T^2 \stackrel{\text{def}}{=} (Q_T^i Z_T^i - K)^+,$$

where K is expressed in units of the domestic currency. By applying the domestic market method to the synthetic domestic asset $\tilde{Z}_t^i = Q_t^i Z_t^i$, it is not hard to check that the arbitrage price of this option at time $t \in [0,T]$ is

$$C_t^2 = \tilde{Z}_t^i \, N\big(l_1(\tilde{Z}_t^i, t, T)\big) - KB(t,T)\, N\big(l_2(\tilde{Z}_t^i, t, T)\big), \tag{14.29}$$

where

$$l_{1,2}(\tilde{Z}_t^i, t, T) = \frac{\ln(\tilde{Z}_t^i/K) - \ln B(t,T) \pm \frac{1}{2}\tilde{v}_{Z^i}^2(t,T)}{\tilde{v}_{Z^i}(t,T)}$$

and

$$\tilde{v}_{Z^i}^2(t,T) = \int_t^T |\nu_u^i + \xi_u^i - b(u,T)|^2 \, du.$$

For instance, if the underlying asset of the option is a foreign zero-coupon bond with maturity $U \geq T$, we obtain

$$C_t^2 = Q_t^i B^i(t,U)\, N\big(\hat{l}_1(B^i(t,U),t,T)\big) - KB(t,T)\, N\big(\hat{l}_2(B^i(t,U),t,T)\big),$$

where

$$\hat{l}_{1,2}(B^i(t,U),t,T) = \frac{\ln(Q_t^i/K) + \ln(B^i(t,U)/B(t,T)) \pm \frac{1}{2}\tilde{v}_U^2(t,T)}{\tilde{v}_U(t,T)}$$

and

$$\tilde{v}_U^2(t,T) = \int_t^T |\nu_u^i + b^i(u,U) - b(u,T)|^2 \, du.$$

It is not difficult to check that if we choose the maturity date U equal to the expiry date T, then the formula above agrees, as expected, with the currency option valuation formula (14.27). Also, it is clear that to establish equality (14.29), it is sufficient to assume that the volatility $\tilde{v}_{Z^i}(t,T)$ of the domestic forward price of the foreign asset Z^i follows a deterministic function.

Quanto option. As usual, let Z^i denote the price process of a certain foreign asset. The payoff at expiry of a quanto call equals (in domestic currency)

$$C_T^3 \stackrel{\text{def}}{=} \bar{Q}^i (Z_T^i - K^i)^+, \tag{14.30}$$

where $\bar{Q}^i$ is the prescribed exchange rate that is used eventually to convert the terminal payoff into domestic currency. Therefore, $\bar{Q}^i$ is specified in domestic currency per unit of the i^{th} foreign currency. Moreover, the exercise price K^i is expressed in units of the i^{th} foreign currency. Let $F_{Z^i}(t,T)$ be the forward price of the asset Z^i in the foreign market. Recall that we write $F_{Q^i}(t,T)$ to denote the forward exchange rate for the i^{th} currency. Observe that the cross-variation of these processes satisfies

$$d\langle F_{Q^i}(\cdot,T), F_{Z^i}(\cdot,T)\rangle_t = F_{Q^i}(t,T)\, F_{Z^i}(t,T)\, \sigma_{Q^i,Z^i}(t,T)\, dt,$$

where $\sigma_{Q^i,Z^i}(\cdot,T)$ is a deterministic function. We find it convenient to denote

$$v_{Q^i,Z^i}(t,T) = \int_t^T \sigma_{Q^i,Z^i}(u,T)\, du, \quad \forall\, t \in [0,T].$$

Assume, in addition, that the volatility ξ^i of an underlying asset Z^i is also deterministic, and put

$$v_{Z^i}^2(t,T) = \int_t^T (\xi_u^i)^2\, du.$$

Then the arbitrage price of a quanto call option at time $t \in [0,T]$ equals

$$C_t^3 = \bar{Q}^i B(t,T)\Big(F_{Z^i}(t,T)\, e^{-v_{Q^i,Z^i}(t,T)}\, N\big(c_1(Z_t^i,t,T)\big) - K^i N\big(c_2(Z_t^i,t,T)\big)\Big),$$

where

$$c_{1,2}(z,t,T) = \frac{\ln(z/K^i) - \ln B^i(t,T) - v_{Q^i,Z^i}(t,T) \pm \frac{1}{2}\, v_{Z^i}^2(t,T)}{v_{Z^i}(t,T)}.$$

The reader may find it instructive to compare this result with the formula established in Proposition 4.5.1.

Equity-linked foreign exchange option. The payoff at expiry of an Elf-X option equals (see Sect. 4.5)

$$C_T^4 \stackrel{\text{def}}{=} (Q_T^i - K)^+ Z_T^i = (Q_T^i - K)^+ F_{Z^i}(T,T), \tag{14.31}$$

where K is a fixed level of the i^{th} exchange rate, and $F_{Z^i}(t,T)$ is the foreign market forward price of a foreign asset Z^i. The dynamics of the price of the foreign asset Z^i and foreign bond $B^i(t,T)$ under the domestic martingale measure $\mathbb{P}^*$ are (see (14.4)–(14.5))

$$dZ_t^i = Z_t^i\Big((r_t^i - \nu_t^i \cdot \xi_t^i)\, dt + \xi_t^i \cdot dW_t^*\Big)$$

and

$$dB^i(t,T) = B^i(t,T)\Big((r_t^i - \nu_t^i \cdot b^i(t,T))\, dt + b^i(t,T) \cdot dW_t^*\Big)$$

respectively.

Using Itô's formula, we find the dynamics of the foreign market forward price $F_{Z^i}(t,T)$ under the domestic martingale measure $\mathbb{P}^*$, namely

$$\begin{aligned} dF_{Z^i}(t,T) &= F_{Z^i}(t,T)\big(b^i(t,T)-\xi^i_t\big)\cdot\big(\nu^i_t+b^i(t,T)\big)\,dt \\ &\quad + F_{Z^i}(t,T)\big(\xi^i_t-b^i(t,T)\big)\cdot dW^*_t. \end{aligned}$$

Consequently, under the domestic forward measure $\mathbb{P}_T$, we have

$$dF_{Z^i}(t,T)=F_{Z^i}(t,T)\Big(\big(b^i(t,T)-\xi^i_t\big)\cdot\sigma_{Q^i}(t,T)\,dt+\big(\xi^i_t-b^i(t,T)\big)\cdot dW^T_t\Big),$$

where (cf. (14.25))

$$\sigma_{Q^i}(t,T)=\nu^i_t+b^i(t,T)-b(t,T),\quad \forall\, t\in[0,T].$$

For the sake of notational simplicity, we consider the case $t=0$. Let us define an auxiliary probability measure $\mathbb{Q}_T$ by setting

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}=\mathcal{E}_T\Big(\int_0^{\cdot}\zeta_u\cdot dW^T_u\Big)=\eta_T,\quad \mathbb{P}_T\text{-a.s.},$$

where $\zeta_t=\xi^i_t-b^i(t,T)$ for $t\in[0,T]$. It is easily seen that Z^i_T equals

$$Z^i_T=F_{Z^i}(T,T)=F_{Z^i}(0,T)\,\eta_T\,e^{\theta(0,T)},$$

where we write

$$\theta(0,T)=\int_0^T\big(b^i(u,T)-\xi^i_u\big)\cdot\sigma_{Q^i}(t,T)\,du.$$

The price of the option at time 0 equals

$$C^4_0=B(0,T)\mathbb{E}_{\mathbb{P}_T}\big((Q^i_T-K)^+Z^i_T\big),$$

or equivalently,

$$C^4_0=B(0,T)F_{Z^i}(0,T)e^{\theta(0,T)}\,\mathbb{E}_{\mathbb{Q}_T}\big(F^i_Q(T,T)-K\big)^+. \tag{14.32}$$

To evaluate the expectation in (14.32), we need to analyze the dynamics of the forward exchange rate $F_{Q^i}(t,T)$ under the auxiliary probability measure $\mathbb{Q}_T$. We know already that $F_{Q^i}(t,T)$ satisfies, under $\mathbb{P}_T$, the following SDE (cf. (14.24))

$$dF_{Q^i}(t,T)=F_{Q^i}(t,T)\,\sigma_{Q^i}(t,T)\cdot dW^T_t.$$

Therefore, under $\mathbb{Q}_T$ we have

$$dF_{Q^i}(t,T)=F_{Q^i}(t,T)\sigma_{Q^i}(t,T)\cdot\Big(\big(\xi^i_t-b^i(t,T)\big)\,dt+d\hat{W}_t\Big),$$

where the process $\hat{W}$, given by the formula

$$\hat{W}_t = W_t^T - \int_0^t \zeta_u \, du, \quad \forall \, t \in [0, T],$$

follows a Brownian motion under $\mathbb{Q}_T$. Consequently, the forward exchange rate $F_{Q^i}(t, T)$ can be represented as follows

$$F_{Q^i}(T, T) = F_{Q^i}(0, T) e^{-\theta(0,T)} \, \mathcal{E}_T\Big(\int_0^{\cdot} \sigma_{Q^i}(u, T) \, d\hat{W}_u \Big).$$

Putting the last equality into (14.32), we obtain

$$C_0^4 = B(0, T) F_{Z^i}(0, T) \, \mathbb{E}_{\mathbb{Q}_T}\big((F_{Q^i}(0, T) e^{\xi} - K e^{-\theta(0,T)})^+ \big), \tag{14.33}$$

where ξ is a Gaussian random variable, with zero mean and the variance under $\mathbb{Q}_T$

$$\mathrm{Var}_{\mathbb{Q}_T}(\xi) = v_{Q^i}^2(0, T) = \int_0^T |\sigma_{Q^i}(u, T)|^2 \, du.$$

Calculation of the expected value in (14.33) is standard. In general, we find that the price at time $t \in [0, T]$ of the Elf-X call option equals

$$C_t^4 = B(t, T) F_{Z^i}(t, T) \Big(F_t^i \, N\big(w_1(F_t^i, t, T)\big) - K e^{\theta(t,T)} \, N\big(w_2(F_t^i, t, T)\big) \Big),$$

where $F_t^i = F_{Q^i}(t, T)$,

$$w_{1,2}(F, t, T) = \frac{\ln(F/K) - \theta(t, T) \pm \frac{1}{2} v_{Q^i}^2(t, T)}{v_{Q^i}(t, T)},$$

and

$$\theta(t, T) = \int_t^T (b^i(u, T) - \xi_u^i) \cdot \sigma_{Q^i}(u, T) \, du, \quad \forall \, t \in [0, T].$$

After simple manipulations, we find that

$$C_t^4 = Z_t^i \Big(Q_t^i \, N\big(\tilde{w}_1(Q_t^i, t, T)\big) - K \, e^{\theta(t,T)} \frac{B(t, T)}{B^i(t, T)} \, N\big(\tilde{w}_1(Q_t^i, t, T)\big) \Big),$$

where

$$\tilde{w}_{1,2}(q, t, T) = \frac{\ln(q/K) + \ln(B^i(t, T)/B(t, T)) - \theta(t, T) \pm \frac{1}{2} v_{Q^i}^2(t, T)}{v_{Q^i}(t, T)}.$$

This ends the derivation of the option's pricing formula.

Remarks. Assume that the domestic and foreign interest rates r_t and r_t^i are deterministic for every $t \in [0, T]$; that is, $b(t, T) = b^i(t, T) = 0$. In this case, the value of C_t^4 given by the formula above agrees with the formula established in Proposition 4.5.2. Furthermore, if we take the foreign bond that pays one unit of the foreign currency at time T as the underlying foreign asset of the option, then θ vanishes identically, and we recover the currency option valuation formula (14.27).

14.2.3 Cross-currency Swaps

Cross-currency swaps are financial instruments that allow financial managers to capture existing and expected floating or money market rate spreads between alternative currencies without incurring foreign exchange exposure. Let us briefly describe a typical cross-currency swap. The party entering into such a swap will typically agree to receive payment in a particular currency on a specific principal amount, for a specific term, at the prevailing floating money market rate in that currency (such as, e.g., the LIBOR). In exchange, this party will make payments on the same principal amount, in the same currency, for the same term, based on the prevailing floating money market rate in another currency. Therefore, the major features of a typical cross-currency swap are that: (a) both payments and receipts (which are based on the same notional principal) are on a floating-rate basis, with the rate reset at specified intervals (usually quarterly or semi-annually); (b) all payments under the transaction are made in the preassigned currency, thereby eliminating foreign exchange exposure; and (c) consistent with the transaction's single-currency nature, no exchange of principal amounts is required. Our aim is to find valuation formulas for cross-currency swaps as well as for their derivatives, such as *cross-currency swaptions* – that is, options written on cross-currency swaps.

Formally, by a *cross-currency* (or *differential*) *swap* we mean an interest rate swap agreement in which at least one of the interest rates involved is related to a foreign market. In contrast to a classic fixed-for-floating (single-currency) swap agreement, in a typical cross-currency swap, both underlying interest rates are preassigned floating rates from two markets. To be more specific, a *floating-for-floating cross-currency* $(k, l; m)$ *swap* per unit of m^{th} currency consists of swapping the floating rates of another two currencies. At each of the payment dates T_j, $j = 1, \ldots, n$, the floating rate $L^k(T_{j-1})$ of currency k is received and the corresponding floating rate $L^l(T_{j-1})$ of currency l is paid. Let us emphasize that in the most general form of a swap, the payments are made in units of still another foreign currency, say m. Similarly, by a *fixed-for-floating cross-currency* $(k; m)$ *swap* we mean a cross-currency swap with payments in the m^{th} foreign currency, in which one of the underlying rates of interest is a prespecified fixed rate, while the other is a reference floating rate from currency k.

Floating-for-floating $(k, 0; 0)$ **swaps.** Let us first consider a floating-for-floating cross-currency $(k, 0; 0)$ swap between two parties in which, at each of the payment dates, the *buyer* pays the *seller* a U.S. dollar[3] amount equal to a fixed notional principal times the then level of a prespecified U.S. floating interest rate. The seller pays the buyer a U.S. dollar amount equal to the same principal times the then level of a prespecified foreign (e.g., Japanese,

[3] For ease of exposition, we assume hereafter that U.S. dollars play the role of the domestic currency.

German, Australian) floating interest rate. If foreign interest rates are higher than U.S. interest rates, one may expect that the buyer should pay the seller a positive up-front fee, negotiated between the counterparts at the time the contract is entered into. Our goal is to determine – following, in particular, Jamshidian (1993b, 1994a) and Brace and Musiela (1997) – this up-front cost, called the value of the cross-currency swap. We will also examine a corresponding hedging portfolio. It is clear that at each of the payment dates T_j, $j = 1, \dots, n$, the interest determined by the floating rate $L^k(T_{j-1})$ of the foreign currency k is received and the interest corresponding to the floating rate $L(T_{j-1})$ of the domestic currency is paid. In our framework, the rate levels $L^i(T_{j-1})$, $j = 1, \dots, n$, are set by reference to the zero-coupon bond prices; namely, we have

$$B^i(T_{j-1}, T_j)^{-1} = 1 + L^i(T_{j-1})(T_j - T_{j-1}) = 1 + \delta_j L^i(T_{j-1}) \tag{14.34}$$

for $i = 0, k$. The time t value, in units of the domestic currency, of a floating-for-floating $(k, 0; 0)$ cross-currency forward swap is

$$\mathbf{CCFS}_t(k, 0; 0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} \Big(L^k(T_{j-1}) - L(T_{j-1}) \Big) \delta_j \,\Big|\, \mathcal{F}_t \Big\},$$

or equivalently

$$\mathbf{CCFS}_t(k, 0; 0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} \Big(\frac{1}{B^k(T_{j-1}, T_j)} - \frac{1}{B(T_{j-1}, T_j)} \Big) \,\Big|\, \mathcal{F}_t \Big\}.$$

We define a (T, U) *roll bond* to be a dollar cash security that pays $1/B(T, U)$ dollars at its maturity U. Similarly, by a (T, U) *quanto roll bond* we mean a security that pays $1/B^k(T, U)$ dollars at time U. In view of the last equality, it is evident that a long position in a cross-currency swap is equivalent to being long a portfolio of (T_j, T_{j+1}) quanto roll bonds, and short a portfolio of (T_j, T_{j+1}) roll bonds. Therefore, we need to examine the following conditional expectation

$$\mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_j}} \Big(\frac{1}{B^k(T_{j-1}, T_j)} - \frac{1}{B(T_{j-1}, T_j)} \Big) \,\Big|\, \mathcal{F}_t \Big\}$$

for any $t \le T_{j-1}$. One can easily check that

$$\mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_{T_j}\, B(T_{j-1}, T_j)} \,\Big|\, \mathcal{F}_t \Big) = B(t, T_{j-1}).$$

Indeed, to replicate the payoff of a (T_{j-1}, T_j) roll bond it is sufficient to buy at time $t \le T_{j-1}$ one bond with maturity T_{j-1}, and then reinvest the principal received at time T_{j-1} by purchasing $1/B(T_{j-1}, T_j)$ units of bonds with maturity T_j. The problem of the replication of a cross-currency swap thus reduces to replication of a quanto roll bond for $t \le T_{j-1}$, supplemented by a simple netting of positions at payment dates. Observe that for $t \geq T_{j-1}$, we have simply

$$\mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_{T_j}\, B^k(T_{j-1},T_j)}\,\Big|\,\mathcal{F}_t\Big) = \frac{B(t,T_j)}{B^k(T_{j-1},T_j)}.$$

In particular, for $t = T_{j-1}$ this yields

$$\mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_{T_{j-1}}}{B_{T_j}\, B^k(T_{j-1},T_j)}\,\Big|\,\mathcal{F}_{T_{j-1}}\Big) = \frac{B(T_{j-1},T_j)}{B^k(T_{j-1},T_j)}.$$

Therefore, our goal is now to find a replicating strategy for the contingent claim X that settles at time T_{j-1} and whose value is

$$X = \frac{B(T_{j-1},T_j)}{B^k(T_{j-1},T_j)}. \tag{14.35}$$

To simplify the notation, we denote $T = T_{j-1}$ and $U = T_j$. Let us consider a dynamic portfolio composed at any time $t \leq T$ of ϕ^1_t units of U-maturity domestic bonds, ϕ^2_t units of U-maturity foreign bonds, and finally ϕ^3_t units of T-maturity foreign bonds. The wealth of such a portfolio at time $t \leq T$, expressed in domestic currency, equals

$$V_t(\phi) = \phi^1_t B(t,U) + Q^k_t(\phi^2_t B^k(t,U) + \phi^3_t B^k(t,T)),$$

or in short

$$V_t(\phi) = \phi^1_t B(t,U) + \phi^2_t \tilde{B}^k(t,U) + \phi^3_t \tilde{B}^k(t,T),$$

where for any maturity date T, we write

$$\tilde{B}^k(t,T) = Q^k_t B^k(t,T).$$

Note that $\tilde{B}^k(t,T)$ and $\tilde{B}^k(t,U)$ stand for the price at time t of the foreign market zero-coupon bond, expressed in units of domestic currency, with maturities T and U respectively. As usual, we say that a portfolio ϕ is self-financing when the relationship

$$dV_t(\phi) = \phi^1_t\, dB(t,U) + \phi^2_t\, d\tilde{B}^k(t,U) + \phi^3_t\, d\tilde{B}^k(t,T) \tag{14.36}$$

is valid. To provide an intuitive argument supporting (14.36), observe that in the case of a discretely adjusted portfolio we have

$$\begin{aligned}
&\phi^1_{t_1} B(t_2,U) + \phi^2_{t_1} \tilde{B}^k(t_2,U) + \phi^3_{t_1} \tilde{B}^k(t_2,T)\\
&\quad = \phi^1_{t_2} B(t_2,U) + \phi^2_{t_2} \tilde{B}^k(t_2,U) + \phi^3_{t_2} \tilde{B}^k(t_2,T)
\end{aligned}$$

if a portfolio is held fixed over the interval $[t_1,t_2)$, and revised at time t_2. This shows that processes $\tilde{B}^k(t,T)$ and $\tilde{B}^k(t,U)$ can be formally seen as prices of domestic securities. Recall that

$$dB(t,T) = B(t,T)\,\big(r_t\, dt + b(t,T)\cdot dW^*_t\big) \tag{14.37}$$

and

$$dB^k(t,T) = B^k(t,T)\left(r_t^k\,dt + b^k(t,T)\cdot dW_t^*\right),$$

where W^* follows a Brownian motion under $\mathbb{P}^*$, and the exchange rate Q^k satisfies (see formula (14.1))

$$dQ_t^k = Q_t^k\Big((r_t - r_t^k)\,dt + \nu_t^k \cdot dW_t^*\Big).$$

Finally, recall that the forward exchange rate for the settlement date U is

$$F_{Q^k}(t,U) = \frac{B^k(t,U)}{B(t,U)}\,Q_t^k, \quad \forall\, t \in [0,U],$$

and the forward price of a T-maturity foreign market bond for settlement at time U equals

$$F_{B^k}(t,T,U) = \frac{B^k(t,T)}{B^k(t,U)}, \quad \forall\, t \in [0,U],$$

where $U \le T$. Observe that we have the following expression for the forward price $F_{B^k}(t,T,U)$, under the domestic martingale measure $\mathbb{P}^*$,

$$dF_{B^k}(t,T,U) = F_{B^k}(t,T,U)\gamma^k(t,T,U)\cdot d\big(W_t^* - (\nu_t^k + b^k(t,U))\,dt\big),$$

where $\gamma^k(t,T,U) = b^k(t,T) - b^k(t,U)$. We will show that to replicate a short position in a cross-currency swap, it is enough to hold a continuously rebalanced portfolio involving domestic and foreign zero-coupon bonds with maturities corresponding to the payment dates of the underlying swap. The net value of positions in foreign bonds is assumed to be zero – that is, the instantaneous profits or losses from foreign market positions are immediately converted into domestic currency and invested in domestic bonds. We start with an auxiliary lemma.

Lemma 14.2.1. *Let $V^k(T,U)$ stand for the following process*

$$V_t^k(T,U) = \frac{B(t,U)B^k(t,T)G_t^k(T,U)}{B^k(t,U)}, \quad \forall\, t \in [0,T], \tag{14.38}$$

where

$$G_t^k(T,U) = \exp\left(\int_t^T g_u^k(T,U)\,du\right) \tag{14.39}$$

and

$$g_t^k(T,U) = \gamma^k(t,U,T)\cdot(\nu_t^k + b^k(t,U) - b(t,U)), \tag{14.40}$$

where $\gamma^k(t,U,T) = b^k(t,U) - b^k(t,T)$. Suppose that the process $G^k(T,U)$ is adapted. Then the Itô differential of $V^k(T,U)$ is given by the following expression

$$\frac{dV_t^k(T,U)}{V_t^k(T,U)} = \frac{dB(t,U)}{B(t,U)} - \frac{dB^k(t,U)}{B^k(t,U)} + \frac{dB^k(t,T)}{B^k(t,T)} + \gamma^k(t,T,U)\cdot\nu_t^k\,dt.$$

Proof. Since $G^k(T,U)$ is an adapted process of finite variation, Itô's formula yields

$$\begin{aligned} dV_t^k(T,U) &= G_t^k(T,U)\Big(F_t\,dB(t,U) + B(t,U)\,dF_t + d\langle B(\cdot,U),F\rangle_t\Big) \\ &\quad + \gamma^k(t,T,U)\cdot(\nu_t^k + b^k(t,U) - b(t,U))V_t^k(T,U)\,dt, \end{aligned}$$

where we write $F_t = F_{B^k}(t,T,U)$. The asserted formula now follows easily from equality (14.37), combined with the dynamics of the forward price $F_{B^k}(t,T,U)$ under the domestic martingale measure. □

Note that the adapted process $g^k(T,U)$ is linked to the instantaneous covariance between the U-delivery forward exchange rate and the T-delivery forward price of a U-maturity foreign market bond. More explicitly, the cross-variation equals

$$\langle F_{Q^k}(\cdot,U), F_{B^k}(\cdot,T,U)\rangle_t = \int_0^t g_u^k(T,U)F_{Q^k}(u,U)F_{B^k}(u,T,U)\,du.$$

Assumption. We assume from now on that $G^k(T,U)$ follows an adapted process of finite variation (in particular, it can be a deterministic function).

The above assumption is motivated by the following two arguments, each of a different nature. First, we can make use of Lemma 14.2.1. Second, it is evident that if $G^k(T,U)$, and consequently the process $V^k(T,U)$, were not adapted, then the process ϕ that is given by formula (14.41) below, would fail to satisfy the definition of a trading strategy.

Proposition 14.2.1. *Let us consider the portfolio $\phi = (\phi^1,\phi^2,\phi^3)$ that equals*

$$\phi_t^1 = \frac{V_t^k(T,U)}{B(t,U)}, \quad \phi_t^2 = -\frac{V_t^k(T,U)}{\tilde{B}^k(t,U)}, \quad \phi_t^3 = \frac{V_t^k(T,U)}{\tilde{B}^k(t,T)}. \tag{14.41}$$

Then the strategy ϕ is self-financing and the wealth process $V(\phi)$ equals $V^k(T,U)$.

Proof. For the last claim, it is enough to check that

$$V_t(\phi) = \phi_t^1 B(t,U) + Q_t^k(\phi_t^2 B^k(t,U) + \phi_t^3 B^k(t,T)) = V_t^k(T,U)$$

for every $t \in [0,T]$. It remains to verify that the trading strategy ϕ is self-financing. By virtue of (14.36) and (14.41), it is clear that we need to show the following equality

$$dV_t^k(T,U) = V_t^k(T,U)\Big(\frac{dB(t,U)}{B(t,U)} - \frac{d\tilde{B}(t,U)}{\tilde{B}^k(t,U)} + \frac{d\tilde{B}^k(t,T)}{\tilde{B}^k(t,T)}\Big). \tag{14.42}$$

For any maturity T, we have

$$d\tilde{B}^k(t,T) = Q_t^k \, dB^k(t,T) + B^k(t,T)\, dQ_t^k + \langle Q^k, B(\cdot,T)\rangle_t,$$

so that

$$d\tilde{B}^k(t,T) = \tilde{B}^k(t,T)\Big(\big(r_t + b^k(t,T)\cdot \nu_t^k\big)\, dt + \big(b^k(t,T) + \nu_t^k\big)\cdot dW_t^*\Big).$$

A substitution of this relationship into the right-hand side of (14.42) gives

$$\begin{aligned}\frac{dV_t^k(T,U)}{V_t^k(T,U)} &= \frac{dB(t,U)}{B(t,U)} - b^k(t,U)\cdot dW_t^* + b^k(t,T)\cdot dW_t^* + \gamma^k(t)\cdot \nu_t^k\, dt \\ &= \frac{dB(t,U)}{B(t,U)} - \frac{dB^k(t,U)}{B^k(t,U)} + \frac{dB^k(t,T)}{B^k(t,T)} + \gamma^k(t)\cdot \nu_t^k\, dt,\end{aligned}$$

where $\gamma^k(t) = \gamma^k(t,T,U)$. Comparing this with the formula established in Lemma 14.2.1, we conclude that ϕ is self-financing. □

Corollary 14.2.1. *The arbitrage price at time $t \in [0, T_{j-1}]$ of a contingent claim X that is given by formula* (14.35), *equals*

$$\pi_t(X) = V_t^k(T_{j-1}, T_j) = \frac{B(t,T_j)B^k(t,T_{j-1})G_t^k(T_{j-1},T_j)}{B^k(t,T_j)}, \tag{14.43}$$

where

$$G_t^k(T_{j-1},T_j) = \exp\Big(\int_t^{T_{j-1}} \gamma^k(u,T_j,T_{j-1})\cdot\big(\nu_u^k + b^k(u,T_j) - b(u,T_j)\big)\, du\Big)$$

and $\gamma^k(u,T_j,T_{j-1}) = b^k(u,T_j) - b^k(u,T_{j-1})$.

Proposition 14.2.2. *The arbitrage price of the floating-for-floating cross-currency $(k,0;0)$ swap at time $t \in [0,T_0]$ equals*

$$\mathbf{CCFS}_t(k,0;0) = \sum_{j=1}^n \Big(\frac{B(t,T_j)B^k(t,T_{j-1})G_t^k(T_{j-1},T_j)}{B^k(t,T_j)} - B(t,T_{j-1})\Big).$$

Proof. It is enough to observe that

$$\begin{aligned}\mathbf{CCFS}_t(k,0;0) &= \mathbb{E}_{\mathbb{P}^*}\Big\{\sum_{j=1}^n \frac{B_t}{B_{T_j}}\Big(\frac{1}{B^k(T_{j-1},T_j)} - \frac{1}{B(T_{j-1},T_j)}\Big)\,\Big|\,\mathcal{F}_t\Big\} \\ &= \sum_{j=1}^n \mathbb{E}_{\mathbb{P}^*}\Big\{\frac{B_t B(T_{j-1},T_j)}{B_{T_{j-1}}B^k(T_{j-1},T_j)}\,\Big|\,\mathcal{F}_t\Big\} - \sum_{j=1}^n B(t,T_{j-1}) \\ &= \sum_{j=1}^n \big(V_t^k(T_{j-1},T_j) - B(t,T_{j-1})\big)\end{aligned}$$

and to apply Corollary 14.2.1. □

Floating-for-floating $(k,l;0)$ **swaps.** The next step is to examine the slightly more general case of a floating-for-floating cross-currency $(k,l;0)$ swap. The contractual conditions of a $(k,l;0)$ swap agreement imply immediately that its arbitrage price $\mathbf{CCFS}_t(k,l;0)$ at time t satisfies

$$\mathbf{CCFS}_t(k,l;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} \Big(L^k(T_{j-1}) - L^l(T_{j-1})\Big)\,\delta_j \,\Big|\, \mathcal{F}_t \Big\}, \tag{14.44}$$

or equivalently

$$\mathbf{CCFS}_t(k,l;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} \Big(\frac{1}{B^k(T_{j-1},T_j)} - \frac{1}{B^l(T_{j-1},T_j)}\Big) \,\Big|\, \mathcal{F}_t \Big\}.$$

It is easily seen that the arbitrage price $\mathbf{CCFS}(k,l;0)$ also admits the following representation

$$\mathbf{CCFS}_t(k,l;0) = \mathbf{CCFS}_t(k,0;0) - \mathbf{CCFS}_t(l,0;0).$$

Since we wish to apply the results of the previous section, we assume that for every $j = 1,\ldots,n$, both $G^k(T_{j-1},T_j)$ and $G^l(T_{j-1},T_j)$ follow adapted processes of finite variation. Arguing along the same lines as in the proof of Proposition 14.2.2, we find the following equality, which holds for $t \in [0,T_0]$

$$\mathbf{CCFS}_t(k,l;0) = \sum_{j=1}^{n} \Big(V_t^k(T_{j-1},T_j) - V_t^l(T_{j-1},T_j)\Big), \tag{14.45}$$

where $V_t^k(T_{j-1},T_j)$ and $V_t^l(T_{j-1},T_j)$ are given by the expressions

$$V_t^k(T_{j-1},T_j) = \frac{B(t,T_j)B^k(t,T_{j-1})G_t^k(T_{j-1},T_j)}{B^k(t,T_j)}$$

and

$$V^l(T_{j-1},T_j) = \frac{B(t,T_j)B^l(t,T_{j-1})G_t^l(T_{j-1},T_j)}{B^l(t,T_j)}$$

respectively. To visualize the replicating portfolio of the $(k,l;0)$ swap agreement, let us consider an arbitrary payment date T_j. Then the portfolio $\phi = (\phi^1,\phi^2,\phi^3,\phi^4,\phi^5)$ that replicates a particular payoff of a swap that occurs at time T_j involves, at time t, ϕ_t^1 units of T_j-maturity domestic bonds, where

$$\phi_t^1 = \frac{V_t^k(T_{j-1},T_j) - V_t^l(T_{j-1},T_j)}{B(t,T_j)},$$

and the following positions in foreign bonds $B^k(t,T_j)$, $B^k(t,T_{j-1})$, $B^l(t,T_j)$ and $B^l(t,T_{j-1})$ respectively

$$\phi_t^2 = -\frac{V_t^k}{\tilde{B}^k(t,T_j)},\ \phi_t^3 = \frac{V_t^k}{\tilde{B}^k(t,T_{j-1})},\ \phi_t^4 = \frac{V_t^l}{\tilde{B}^l(t,T_j)},\ \phi_t^5 = -\frac{V_t^l}{\tilde{B}^l(t,T_{j-1})},$$

where $V_t^k = V_t^k(T_{j-1}, T_j)$ and $V_t^l = V_t^l(T_{j-1}, T_j)$. It is not hard to verify that the trading strategy ϕ given by the last formula is self-financing. Moreover, for every $t \in [0, T_{j-1}]$, the wealth of such a portfolio, expressed in units of domestic currency, equals

$$\begin{aligned} V_t(\phi) &= \phi_t^1 B(t, T_{j-1}) + Q_t^l\big(\phi_t^1 B^l(t, T_j) + \phi_t^2 B^l(t, T_{j-1})\big) \\ &+ Q_t^k\big(\phi_t^3 B^k(t, T_j) + \phi_t^4 B^k(t, T_{j-1})\big) = V_t^k(T_{j-1}, T_j) - V_t^l(T_{j-1}, T_j), \end{aligned}$$

as expected.

Fixed-for-floating $(k;m)$ swaps. Before studying the general case of floating-for-floating cross-currency $(k,l;m)$ swaps, we find it convenient to examine the case of a *fixed-for-floating cross-currency* $(k;m)$ *swap.* In such a contract, a floating rate $L^k(T_{j-1})$ is received at each payment date, and a prescribed fixed interest rate κ is paid. Let us stress that the payments are made in units of the m^{th} foreign currency. Consequently the price of the fixed-for-floating swap, expressed in units of domestic currency, equals (cf. (14.34))

$$\mathbf{CCFS}_t^\kappa(k;m) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^n \frac{B_t}{B_{T_j}} \Big(L^k(T_{j-1}) - \kappa\Big)\, Q_{T_j}^m\, \delta_j \,\Big|\, \mathcal{F}_t \Big\}$$

for every $t \in [0, T_0]$. Equivalently,

$$\mathbf{CCFS}_t^\kappa(k;m) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^n \frac{B_t}{B_{T_j}} \Big(B^k(T_{j-1}, T_j)^{-1} - \tilde{\delta}_j\Big)\, Q_{T_j}^m \,\Big|\, \mathcal{F}_t \Big\}, \tag{14.46}$$

where $\tilde{\delta}_j = 1 + \kappa\delta_j$. Let us write the last representation as $\mathbf{CCFS}_t^\kappa(k;m) = I_t - J_t$, where the meaning of I_t and J_t is apparent from the context. We define

$$G_t^{km}(T,U) = \exp\Big(\int_t^T g_u^{km}(T,U)\, du \Big), \tag{14.47}$$

where

$$g_t^{km}(T,U) = \gamma^k(t,U,T) \cdot (\nu_t^{m/k} + b^k(t,U) - b^m(t,U)).$$

Observe that in the special case when $m = 0$, the process $G^{k0}(T,U)$ given by (14.47) coincides with the process $G^k(T,U)$ introduced in the preceding section (see formula (14.39)). For notational simplicity, we write

$$g_t^{kmj} = g_t^{km}(T_{j-1}, T_j), \quad \forall\, t \in [0, T_0],$$

and

$$\delta_t^*(k,m,j) = b^k(t,T_j) - b^m(t,T_j), \quad \forall\, t \in [0, T_j], \tag{14.48}$$

in what follows. Notice that

$$g_t^{kmj} = \gamma^k(u, T_j, T_{j-1}) \cdot (\nu_t^{m/k} + \delta_t^*(k,m,j)).$$

Lemma 14.2.2. *The following equalities hold for every $t \in [0, T_0]$*

$$J_t = \sum_{j=1}^{n} \tilde{\delta}_j Q_t^m B^m(t, T_j) \tag{14.49}$$

and

$$I_t = \sum_{j=1}^{n} \frac{Q_t^m B^m(t, T_j) B^k(t, T_{j-1}) G_t^{km}(T_{j-1}, T_j)}{B^k(t, T_j)}. \tag{14.50}$$

Proof. For the first formula, it is enough to observe that the equality

$$\mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_j}} Q_{T_j}^m \,\Big|\, \mathcal{F}_t \Big\} = Q_t^m B^m(t, T_j)$$

is valid for every $t \in [0, T_j]$. To establish (14.50), observe first that

$$\begin{aligned} I_t &= \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_j}} B^k(T_{j-1}, T_j)^{-1} Q_{T_j}^m \,\Big|\, \mathcal{F}_t \Big\} \\ &= \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_{j-1}}} B^k(T_{j-1}, T_j)^{-1} \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_{T_{j-1}}}{B_{T_j}} Q_{T_j}^m \,\Big|\, \mathcal{F}_{T_{j-1}} \Big) \,\Big|\, \mathcal{F}_t \Big\} \\ &= \mathbb{E}_{\mathbb{P}^*}\Big\{ \sum_{j=1}^{n} \frac{B_t}{B_{T_{j-1}}} B^k(T_{j-1}, T_j)^{-1} B^m(T_{j-1}, T_j) Q_{T_{j-1}}^m \,\Big|\, \mathcal{F}_t \Big\} \\ &= \sum_{j=1}^{n} Q_t^m \, \mathbb{E}_{\mathbb{P}^m}\Big\{ \frac{B_t^m}{B_{T_{j-1}}^m} B^k(T_{j-1}, T_j)^{-1} B^m(T_{j-1}, T_j) \,\Big|\, \mathcal{F}_t \Big\}. \end{aligned}$$

Consequently, we obtain the following equality

$$I_t = \sum_{j=1}^{n} Q_t^m B^m(t, T_{j-1}) \, \mathbb{E}_{\mathbb{P}_{T_{j-1}}^m}\Big\{ \frac{B^m(T_{j-1}, T_j)}{B^k(T_{j-1}, T_j)} \,\Big|\, \mathcal{F}_t \Big\}, \tag{14.51}$$

which expresses I_t in terms of the forward measure $\mathbb{P}_{T_{j-1}}^m$ for the market m. In order to evaluate this conditional expectation, observe that the dynamics of $B^m(t, T_j)$ and $B^k(t, T_j)$, $k \neq m$, as seen in the market m, are given by the following expressions

$$dB^m(t, T_j) = B^m(t, T_j)\big(r_t^m \, dt + b^m(t, T_j) \cdot dW_t^m\big)$$

and

$$dB^k(t, T_j) = B^k(t, T_j) \Big(\big(r_t^k - b^k(t, T_j) \cdot \nu_t^{m/k}\big) \, dt + b^k(t, T_j) \cdot dW_t^m \Big),$$

since $dW_t^k = dW_t^m - \nu_t^{m/k} \, dt$. Let us denote

$$H_t^1 = \frac{B^m(t, T_j)}{B^k(t, T_j)}, \quad H_t^2 = \frac{B^k(t, T_{j-1})}{B^m(t, T_{j-1})}, \quad \forall \, t \in [0, T_0].$$

Using Itô's formula, we get

$$dH_t^1 = H_t^1\Big(r_t^m - r_t^k + b^k(t,T_j)\cdot\big(\nu_t^{m/k} + \delta_t^*(k,m,j)\big)\Big)\,dt - H_t^1\delta_t^*(k,m,j)\cdot dW_t^m$$

and

$$\begin{aligned} dH_t^2 &= H_t^2\big(r_t^k - r_t^m - b^k(t,T_{j-1})\cdot\nu_t^{m/k} - b^m(t,T_{j-1})\cdot\delta_t^*(k,m,j-1)\big)\,dt \\ &\quad + H_t^2\delta_t^*(k,m,j-1)\cdot dW_t^m. \end{aligned}$$

Consequently, for the process H, which equals

$$H_t = H_t^1 H_t^2 = \frac{B^m(t,T_j)B^k(t,T_{j-1})}{B^k(t,T_j)B^m(t,T_{j-1})},$$

we obtain

$$dH_t = H_t\Big(g_t^{kmj}\,dt + \big(\gamma^k(t,T_{j-1},T_j) + \gamma^m(t,T_j,T_{j-1})\big)\cdot dW_t^{T_{j-1},m}\Big)$$

for every $j = 1,\dots,n$. Since the quantities g^{kmj} are assumed to follow deterministic functions, taking into account the relationship

$$\mathbb{E}_{\mathbb{P}^m_{T_{j-1}}}\Big\{\frac{B^m(T_{j-1},T_j)}{B^k(T_{j-1},T_j)}\,\Big|\,\mathcal{F}_t\Big\} = \mathbb{E}_{\mathbb{P}^m_{T_{j-1}}}\Big\{\frac{B^m(T_{j-1},T_j)B^k(T_{j-1},T_{j-1})}{B^k(T_{j-1},T_j)B^m(T_{j-1},T_{j-1})}\,\Big|\,\mathcal{F}_t\Big\}$$

and using the just-established expression for the differential of H, we obtain

$$\mathbb{E}_{\mathbb{P}^m_{T_{j-1}}}\Big\{\frac{B^m(T_{j-1},T_j)}{B^k(T_{j-1},T_j)}\,\Big|\,\mathcal{F}_t\Big\} = \frac{B^m(t,T_j)B^k(t,T_{j-1})}{B^k(t,T_j)B^m(t,T_{j-1})}\,e^{\int_t^{T_{j-1}} g_u^{kmj}\,du}.$$

Note that the last equality is valid for every $j = 1,\dots,n$. Combining it with (14.51), we arrive at equality (14.50). □

The next result follows from Lemma 14.2.2 and formula (14.46).

Proposition 14.2.3. *The arbitrage price at time $t \in [0,T_0]$ of a fixed-for-floating cross-currency $(k;m)$ swap with the underlying fixed interest rate equal to κ is given by the expression (recall that $\tilde\delta_j = 1 + \kappa\delta_j$)*

$$\mathbf{CCFS}_t^\kappa(k;m) = \sum_{j=1}^n Q_t^m B^m(t,T_j)\Big(\frac{B^k(t,T_{j-1})G_t^{km}(T_{j-1},T_j)}{B^k(t,T_j)} - \tilde\delta_j\Big).$$

Consider a fixed-for-floating cross-currency $(k;k)$ swap – that is, a swap in which the floating rate and the currency used for payments are those of the market k. Proposition 14.2.3 yields the following price of such a swap, expressed in units of domestic currency (cf. Sect. 13.1)

$$\mathbf{CCFS}_t^\kappa(k;k) = \sum_{j=1}^n Q_t^k\Big(B^k(t,T_{j-1}) - \tilde\delta_j B^k(t,T_j)\Big). \tag{14.52}$$

Similarly, for the $(k;0)$ fixed-for-floating swap (that is, an agreement in which the underlying floating rate is that of the market k, but where payments are made in domestic currency), we get

$$\mathbf{CCFS}_t^{\kappa}(k;0)=\sum_{j=1}^{n}B(t,T_j)\Big(\frac{B^k(t,T_{j-1})G_t^k(T_{j-1},T_j)}{B^k(t,T_j)}-\tilde{\delta}_j\Big). \tag{14.53}$$

Floating-for-floating $(k,l;m)$ swaps. Let us now examine the case of a general floating-for-floating cross-currency $(k,l;m)$ swap per unit of the currency m, which consists of swapping the floating rates of another two currencies, say k and l. At each of the payment dates T_j, $j=1,\ldots,n$, the floating rate $L^k(T_{j-1})$ of currency k is received and the corresponding floating rate $L^l(T_{j-1})$ of the currency l is paid. As usual, the underlying floating rates $L^i(T_{j-1})$, $i=k,l$; $j=1,\ldots,n$ are set by reference to the price of a zero-coupon bond, so that

$$B^i(T_{j-1},T_j)^{-1}=1+L^i(T_{j-1})(T_j-T_{j-1})=1+\delta_j L^i(T_{j-1})$$

for $j=1,\ldots,n$. The value at time t, in the domestic currency, of the floating-for-floating forward $(k,l;m)$ swap equals

$$\mathbf{CCFS}_t(k,l;m)=\mathbb{E}_{\mathbb{P}^*}\Big\{\sum_{j=1}^{n}\frac{B_t}{B_{T_j}}\Big(L^k(T_{j-1})-L^l(T_{j-1})\Big)\,Q^m_{T_j}\,\delta_j\,\Big|\,\mathcal{F}_t\Big\}. \tag{14.54}$$

Therefore, the following useful relationship

$$\mathbf{CCFS}_t(k,l;m)=\mathbf{CCFS}_t^{\kappa}(k;m)-\mathbf{CCFS}_t^{\kappa}(l;m) \tag{14.55}$$

holds for any fixed level κ. The next result follows immediately from formula (14.55) and Proposition 14.2.3.

Proposition 14.2.4. *The price* $\mathbf{CCFS}_t=\mathbf{CCFS}_t(k,l;m)$ *at time* $t\in[0,T_0]$ *of the floating-for-floating cross-currency forward swap of type* $(k,l;m)$ *in a Gaussian HJM model equals*

$$\mathbf{CCFS}_t=\sum_{j=1}^{n}Q_t^m B^m(t,T_j)\Big(\frac{B^k(t,T_{j-1})}{B^k(t,T_j)}\,e_t^{kmj}-\frac{B^l(t,T_{j-1})}{B^l(t,T_j)}\,e_t^{lmj}\Big),$$

where

$$g_t^{kmj}=\gamma^k(t,T_j,T_{j-1})\cdot\big(\nu_t^k-\nu_t^m+\delta_t^*(k,m,j)\big)$$

for every $k,m=0,\ldots,N$, $j=1,\ldots,n$, *and*

$$e_t^{pmj}=\exp\Big(\int_t^{T_{j-1}}g_u^{pmj}\,du\Big)$$

for $p=k,l$.

For the special case of a floating-for-floating cross-currency $(k,l;0)$ swap (i.e., a swap agreement with payments in domestic currency), the above proposition yields the pricing result (14.45), which was previously derived by means of a replicating portfolio.

14.2.4 Cross-currency Swaptions

By a *cross-currency swaption* (or *differential swaption*) we mean an option contract with a cross-currency forward swap being the option's underlying asset. Note that the option expiry date, T, precedes the initial date (i.e., the first reset date) of the underlying swap agreement – that is, $T \leq T_0$. We first examine the case of a fixed-for-floating swaption, subsequently we turn our attention to a floating-for-floating swaption.

A *fixed-for-floating* $(k;0)$ *cross-currency swaption* is an option, with expiry date $T \leq T_0$, whose holder has the right to decide whether he wishes to pay at some future dates $T_1, \dots, T_n$ a fixed interest rate, say κ, on some notional principal, and receive simultaneously a floating rate L^k of currency k. Note that we assume here that all payments are made in units of the domestic currency. Using our terminological conventions, a cross-currency fixed-for-floating $(k;0)$ swaption may be seen as a call option (whose exercise price equals zero) with a fixed-for-floating $(k;0)$ cross-currency swap being the underlying financial instrument. Therefore, it is clear that the value at time $t \in [0,T]$ of a fixed-for-floating $(k;0)$ cross-currency swaption, denoted by $\mathbf{CCS}_t^{\kappa,\lambda}(k;0)$ or shortly $\mathbf{CCS}_t(k;0)$, equals

$$\mathbf{CCS}_t^{\kappa,\lambda}(k;0) = \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_T}\big(\mathbf{CCFS}_T^{\kappa}(k;0) - 0\big)^+ \,\Big|\, \mathcal{F}_t\Big). \tag{14.56}$$

More explicitly, we have

$$\mathbf{CCS}_t(k;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T}\Big[\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^n \frac{B_t}{B_{T_j}}(L^k(T_{j-1}) - \kappa)\delta_j \,\Big|\, \mathcal{F}_T\Big)\Big]^+ \,\Big|\, \mathcal{F}_t\Big\}.$$

In view of (14.53), the formula above can also be rewritten in the following way

$$\mathbf{CCS}_t(k;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T}\Big[\sum_{j=1}^n B(T,T_j)\Big(\frac{B^k(T,T_{j-1})G_T^{kj}}{B^k(T,T_j)} - \tilde{\delta}_j\Big)\Big]^+ \,\Big|\, \mathcal{F}_t\Big\},$$

where $\tilde{\delta}_j = 1 + \kappa\delta_j$ and (cf. (14.39)–(14.40))

$$G_t^{kj} \stackrel{\text{def}}{=} G_t^k(T_{j-1},T_j) = \exp\Big(\int_t^{T_{j-1}} g_u^k(T_{j-1},T_j)\,du\Big) \tag{14.57}$$

for $t \in [0,T]$. Under the forward measure $\mathbb{P}_T$, we have

$$\mathbf{CCS}_t(k;0) = B(t,T)\,\mathbb{E}_{\mathbb{P}_T}\Big\{\Big[\sum_{j=1}^n \frac{B(T,T_j)}{B(T,T)}\Big(\frac{B^k(T,T_{j-1})G_T^{kj}}{B^k(T,T_j)} - \tilde{\delta}_j\Big)\Big]^+ \,\Big|\, \mathcal{F}_t\Big\},$$

or equivalently $\mathbf{CCS}_t(k;0) = B(t,T)\,\mathbb{E}_{\mathbb{P}_T}(X \,|\, \mathcal{F}_t)$, where

$$X = \Big(\sum_{j=1}^{n} F_B(T, T_j, T) \left(F_{B^k}(T, T_{j-1}, T_j)\, G_T^{kj} - \tilde{\delta}_j \right) \Big)^+.$$

Recall that the dynamics of the process $F_t = F_B(t, T_j, T)$ under the forward measure $\mathbb{P}_T$ of the domestic market are (cf. (11.40))

$$dF_t = F_t \,\gamma(t, T_j, T) \cdot dW_t^T,$$

so that for any $t \in [0, T]$, we have

$$F_T = F_t \exp\Big(\int_t^T \gamma(u, T_j, T) \cdot dW_u^T - \frac{1}{2} \int_t^T |\gamma(u, T_j, T)|^2\, du \Big).$$

Furthermore, by virtue of (14.10), we get the following expression for the dynamics under $\mathbb{P}_T$ of the process $F_t^{kj} = F_{B^k}(t, T_{j-1}, T_j)$

$$dF_t^{kj} = F_t^{kj} \gamma^k(t, T_{j-1}, T_j) \cdot \Big(dW_t^T + \big(b(t, T) - \nu_t^k - b^k(t, T_j) \big)\, dt \Big).$$

Consequently,

$$F_T^{kj} = L_t F_t^{kj} \exp\Big(\int_t^T \gamma_k(u, T_{j-1}, T_j) \cdot dW_u^T - \frac{1}{2} \int_t^T |\gamma^k(u, T_{j-1}, T_j)|^2\, du \Big),$$

where L is given by the following expression

$$L_t = \exp\Big(\int_t^T \gamma^k(u, T_{j-1}, T_j) \cdot \big(b(u, T) - b^k(u, T_j) - \nu_u^k \big)\, du \Big).$$

By straightforward calculations, we obtain

$$G_T^{kj} L_t = G_t^{kj} \exp\Big(\int_t^T \gamma^k(u, T_{j-1}, T_j) \cdot \gamma(u, T, T_j)\, du \Big).$$

We conclude that

$$\mathbf{CCS}_t(k; 0) = \mathbb{E}_{\mathbb{P}_T} \Big\{ \Big(\sum_{j=1}^{n} B(t, T_j) \big(G_t^{kj} F_t^{kj} N_t^T - \tilde{\delta}_j\, M_t^T \big) \Big)^+ \,\Big|\, \mathcal{F}_t \Big\},$$

where

$$N_t^T = \exp\Big(- \int_t^T \Gamma_u^{kj} \cdot dW_u^T - \frac{1}{2} \int_t^T |\Gamma_u^{kj}|^2 du \Big)$$

with

$$\Gamma_u^{kj} = \gamma(u, T_j, T) + \gamma^k(u, T_{j-1}, T_j), \tag{14.58}$$

and M_t^T equals

$$M_t^T = \exp\Big(\int_t^T \gamma(u, T_j, T) \cdot dW_u^T - \frac{1}{2} \int_t^T |\gamma(u, T_j, T)|^2\, du \Big).$$

We are in a position to state the following result, whose proof presents no difficulties.

Proposition 14.2.5. *The arbitrage price* $\mathbf{CCS}_t^{\kappa,\lambda}(k;0) = \mathbf{CCS}_t(k;0)$ *of a fixed-for-floating cross-currency swaption at time* $t \in [0,T]$ *equals*

$$\mathbf{CCS}_t(k;0) = \int_{\mathbb{R}^d} \Big(\sum_{j=1}^n B(t,T_j)\big(G_t^{kj} F_t^{kj} n_d(x+y_j^k) - \tilde{\delta}_j n_d(x+z_j)\big)\Big)^+ dx,$$

where the vectors $y_1^k, \dots, y_n^k, z_1, \dots, z_n \in \mathbb{R}^d$ *satisfy for* $i,j = 1,\dots,n$

$$y_i^k \cdot y_j^k = \int_t^T \Gamma_u^{ki} \cdot \Gamma_u^{kj}\, du, \quad y_i^k \cdot z_j = \int_t^T \Gamma_u^{ki} \cdot \gamma(u,T_j,T)\, du,$$

with Γ^{ki} *given by* (14.58), *and* $z_i \cdot z_j = \int_t^T \gamma(u,T_i,T)\cdot\gamma(u,T_j,T)\,du.$

By a *floating-for-floating cross-currency swaption* we mean a call option, with strike price equal to zero, to receive the floating rate L^k of currency k plus margin μ, and to pay simultaneously the floating rate L^l of currency l. At any time t before the swaption's expiry date T, the value of a floating-for-floating cross-currency swaption is

$$\mathbf{CCS}_t^{\mu}(k,l;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T} \Big[\mathbb{E}_{\mathbb{P}^*}\Big(\sum_{j=1}^n \frac{B_T}{B_{T_j}} \Delta_{j-1} \,\Big|\, \mathcal{F}_T \Big)\Big]^+ \,\Big|\, \mathcal{F}_t \Big\},$$

where $\Delta_{j-1} = \delta_j\big(L^k(T_{j-1}) + \mu - L^l(T_{j-1})\big)$. From the definition of a floating-for-floating cross-currency swap, it is easily seen that the value $\mathbf{CCS}_t^{\mu}(k,l;0)$ also equals

$$\mathbf{CCS}_t^{\mu}(k,l;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T} \Big(\mathbf{CCFS}_T(k,l;0) + \mu \sum_{j=1}^n \delta_j B(T,T_j)\Big)^+ \,\Big|\, \mathcal{F}_t \Big\}.$$

Therefore, using representation (14.45), we get

$$\mathbf{CCS}_t^{\mu}(k,l;0) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_T} \Big(\sum_{j=1}^n B(T,T_j)\big(\tilde{\Delta}_T^{j-1} + \mu\delta_j\big)\Big)^+ \,\Big|\, \mathcal{F}_t \Big\},$$

where

$$\tilde{\Delta}_T^{j-1} = \frac{B^k(T,T_{j-1})}{B^k(T,T_j)} G_T^{kj} - \frac{B^l(T,T_{j-1})}{B^l(T,T_j)} G_T^{lj} = F_T^{kj} G_T^{kj} - F_T^{lj} G_T^{lj}.$$

Proposition 14.2.6. *The arbitrage price of a floating-for-floating cross-currency swaption at time* $t \in [0,T]$ *equals*

$$\mathbf{CCS}_t^{\mu}(k,l;0) = \int_{\mathbb{R}^d} \Big(\sum_{j=1}^n B(t,T_j) f_j(x)\Big)^+ dx,$$

where for every $x \in \mathbb{R}^d$

$$f_j(x) = F_t^{kj} G_t^{kj} \, n_d(x + y_j^k) - F_t^{lj} G_t^{lj} \, n_d(x + y_j^l) + \mu \delta_j n_d(x + z_j)$$

and n_d *is the standard d-dimensional Gaussian probability density function. Moreover, the vectors* $y_1^k, \dots, y_n^k, y_1^l, \dots, y_n^l, z_1, \dots, z_n \in \mathbb{R}^d$ *satisfy for* $h = k, l$ *and* $i, j = 1, \dots, n$

$$y_i^h \cdot y_j^h = \int_t^T \Gamma_u^{hi} \cdot \Gamma_u^{hj} \, du, \quad y_i^h \cdot z_j = \int_t^T \Gamma_u^{hi} \cdot \gamma(u, T_j, T) \, du,$$

with Γ^{hi} *given by* (14.58), *and* $z_i \cdot z_j = \int_t^T \gamma(u, T_i, T) \cdot \gamma(u, T_j, T) \, du$.

Proof. The proof goes along the same lines as in the case of a cross-currency fixed-for-floating swaption. □

14.2.5 Basket Caps

As the next example of a foreign market interest rate derivative, we shall now consider a cap (settled in arrears) on a basket of floating rates L^k, $k = 1, \dots, N$ of foreign markets; such an option is usually referred to as a *basket cap*. In this agreement, the cash flows received at times $T_j, j = 1, \dots, n$, are

$$c_j = \Big(\sum_{k=1}^N w_k L^k(T_{j-1}) - \kappa \Big)^+ \delta_j,$$

where the weights w_k, $k = 1, \dots, N$ are assumed to be strictly positive constants, and κ is a preassigned rate of interest. The value of each particular basket caplet at time t equals

$$\begin{aligned} \mathbf{BCpl}_t^\kappa &= \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_j}} \Big(\sum_{k=1}^N w_k L^k(T_{j-1}) - \kappa \Big)^+ \delta_j \, \Big| \, \mathcal{F}_t \Big\} \\ &= \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_{j-1}}} \Big(\sum_{k=1}^N w_k \frac{B(T_{j-1}, T_j)}{B^k(T_{j-1}, T_j)} - \tilde{\delta}_j B(T_{j-1}, T_j) \Big)^+ \, \Big| \, \mathcal{F}_t \Big\}, \end{aligned}$$

where $\tilde{\delta}_j = \kappa \delta_j + \sum_{k=1}^N w_k$. Therefore,

$$\mathbf{BCpl}_t^\kappa = B(t, T_{j-1}) \, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big\{ \Big(\sum_{k=1}^N w_k \frac{B(T_{j-1}, T_j)}{B^k(T_{j-1}, T_j)} - \tilde{\delta}_j B(T_{j-1}, T_j) \Big)^+ \, \Big| \, \mathcal{F}_t \Big\}.$$

It is convenient to denote $F_t^{0j} = F_B(t, T_{j-1}, T_j)$ and $F_t^{kj} = F_{B^k}(t, T_{j-1}, T_j)$. Reasoning along the same lines as in Sect. 14.2.4, we find that (recall that G_t^{kj} is defined by (14.39)–(14.40), see also (14.57))

$$\mathbf{BCpl}_t^\kappa = B(t, T_{j-1}) \, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big\{ \Big(\sum_{k=1}^N w_k \, \frac{F_t^{kj} G_t^{kj} M_t^k}{F_t^{0j}} - \tilde{\delta}_j \, \frac{M_t^0}{F_t^{kj}} \Big)^+ \Big| \, \mathcal{F}_t \Big\},$$

where

$$M_t^k = \exp\Big(- \int_t^{T_{j-1}} \zeta_u^{kj} \cdot dW_u^{T_{j-1}} - \frac{1}{2} \int_t^{T_{j-1}} |\zeta_u^{kj}|^2 \, du \Big),$$

$\zeta_u^{kj} = \gamma^k(u, T_j, T_{j-1}) - \gamma(u, T_j, T_{j-1})$, and

$$M_t^0 = \exp\Big(\int_t^{T_{j-1}} \gamma(u, T_{j-1}, T_j) \cdot dW_u^{T_{j-1}} - \frac{1}{2} \int_t^{T_{j-1}} |\gamma(u, T_{j-1}, T_j)|^2 \, du \Big).$$

The following result can thus be proved by standard arguments.

Proposition 14.2.7. *The arbitrage price of a basket cap at time t equals*

$$\mathbf{BC}_t^\kappa = \sum_{j=1}^n B(t, T_j) \int_{\mathbb{R}^d} \Big(\sum_{k=1}^N w_k F_t^{kj} G_t^{kj} n_d(x + y_k) - \tilde{\delta}_j n_d(x + y_0) \Big)^+ dx,$$

where $\tilde{\delta}_j = \kappa \delta_j + \sum_{k=1}^N w_k$, the vectors $y_j^0, \dots, y_j^N \in \mathbb{R}^d$ are such that for every $k, l = 1, \dots, N$ and $j = 1, \dots, n$

$$y_j^k \cdot y_j^l = \int_t^{T_{j-1}} \zeta_u^{k,j} \cdot \zeta_u^{l,j} \, du, \quad y_j^0 \cdot y_j^k = \int_t^{T_{j-1}} \zeta_u^{kj} \cdot \gamma(u, T_{j-1}, T_j) \, du,$$

and

$$|y_j^0|^2 = \int_t^{T_{j-1}} |\gamma(u, T_{j-1}, T_j)|^2 \, du.$$

14.3 Model of Forward LIBOR Rates

A cross-currency extension of a market model of forward LIBOR rates can be constructed by proceeding along the same lines as in Sect. 14.1 – the details are omitted (see Mikkelsen (2002) and Schlögl (2002)). Let us only mention that the main notions we shall employ in what follows are the forward measures and the corresponding Brownian motions. For each date $T_0, T_1, \dots, T_n$, the dynamics of the forward LIBOR rate $L^i(t, T_j)$ of the i^{th} market are governed by the SDE

$$dL^i(t, T_{j-1}) = L^i(t, T_{j-1}) \, \lambda^i(t, T_{j-1}) \cdot dW_t^{T_j, i} \tag{14.59}$$

under the corresponding forward probability measure $\mathbb{P}_{T_j}^i$, where the volatilities $\lambda^i(t, T_{j-1})$ are assumed to follow deterministic functions. Furthermore, the process $W^{T_j, i}$ is related to the domestic forward Brownian motion W^{T_j} through the formula (cf. (14.16))

$$W_t^{T_j, i} = W_t^{T_j} - \int_0^t \big(\nu_u^i + b^i(u, T) - b(u, T) \big) \, du. \tag{14.60}$$

14.3.1 Quanto Cap

For simplicity, we consider only two dates, $T_j = T$ and $T_{j+1} = T + \delta$; that is, we deal with a single caplet with reset date T and settlement date $T + \delta$ (note that we consider here the case of a fixed-length accrual period). A *quanto caplet* with expiry date T pays to its holder at time $T + \delta$ the amount (expressed in domestic currency)

$$\mathbf{Cplq}_{\,T+\delta} = \delta\bar{Q}(L^i(T) - \kappa)^+,$$

where $\bar{Q}$ is the preassigned level of the exchange rate and $L^i(T)$ is the (spot) LIBOR rate at time T in the i^{th} market. Consequently, the domestic arbitrage price of a quanto caplet at time t equals

$$\mathbf{Cplq}_{\,t} = \delta\bar{Q}B(t, T + \delta)\,\mathbb{E}_{\mathbb{P}_{T+\delta}}\big((L^i(T) - \kappa)^+ \,|\, \mathcal{F}_t\big)$$

per one unit of nominal value (the nominal value of a quanto cap is expressed in foreign currency). It is thus clear that to value such a contract, we need to examine the dynamics of the foreign LIBOR rate under the domestic forward measure. The payoff of a quanto caplet, expressed in foreign currency, equals $\delta\bar{Q}(L^i(T) - \kappa)^+/Q^i_{T+\delta}$. Consequently, its price at time t, in domestic currency, admits the following representation

$$\mathbf{Cplq}_{\,t} = \delta\bar{Q}B(t, T + \delta)Q^i_t\,\mathbb{E}_{\mathbb{P}^i_{T+\delta}}\left(\frac{(L^i(T) - \kappa)^+}{F_{Q^i}(T + \delta, T + \delta)}\Big|\mathcal{F}_t\right),$$

where $\mathbb{P}^i_{T+\delta}$ is the foreign market forward measure. The last representation makes clear that the volatility of the forward exchange rate $F_{Q^i}(t, T)$, and thus also bond price volatilities $b^i(t, T+\delta)$ and $b(t, T+\delta)$ (cf. (14.24)–(14.25)), will enter the valuation formula. To uniquely specify these volatilities, we may adopt, for instance, the approach of Brace et al. (1997). In their approach, bond price volatilities are linked to forward LIBOR rates by means of the formula (cf. (12.29))

$$b^k(t, T + \delta) = -\sum_{m=0}^{n(t)} \frac{\delta L^k(t, T - m\delta)}{1 + \delta L^k(t, T - m\delta)}\,\lambda^k(t, T - m\delta), \tag{14.61}$$

where $n(t) = [\delta^{-1}(T - t)]$. It is thus clear that bond price volatilities $b^i(t, T+\delta)$ follow necessarily stochastic processes, and thus the Gaussian methodology examined in preceding sections is no longer applicable, even under deterministic volatilities of (spot) exchange rates. Therefore, we will examine an approximation of the caplet's price. Combining the dynamics (14.59), which read

$$dL^i(t, T) = L^i(t, T)\,\lambda^i(t, T) \cdot dW^{T+\delta, i}_t,$$

with (14.60), we obtain (recall that $L^i(T) = L^i(T, T)$)

$$L^i(T) = L^i(t,T)e^{-\eta^i(t,T)} \exp\Big(\int_t^T \lambda^i(u,T)\cdot dW_u^{T+\delta} - \frac{1}{2}\int_t^T |\lambda^i(u,T)|^2 du\Big),$$

where

$$\eta^i(t,T) = \int_t^T \lambda^i(u,T)\cdot \big(\nu_u^i + b^i(u,T+\delta) - b(u,T+\delta)\big)\, du.$$

To derive a simple approximate formula for the price of a quanto caplet at time t, it is convenient to substitute into (14.62) the term $\eta^i(t,T)$ with $\tilde{\eta}^i(t,T)$, where

$$\tilde{\eta}^i(t,T) = \int_t^T \lambda^i(u,T)\cdot \big(\nu_u^i + \tilde{b}^i(u,T+\delta) - \tilde{b}(u,T+\delta)\big)\, du \tag{14.62}$$

and $\tilde{b}^i(u,T+\delta)$ and $\tilde{b}(u,T+\delta)$ are $\mathcal{F}_t$-measurable random variables, namely (recall that $\lambda^k(t,T)$ is a deterministic function)

$$\tilde{b}^k(u,T+\delta) = \sum_{m=0}^{n(u)} \frac{\delta L^k(t,T-m\delta)}{1+\delta L^k(t,T-m\delta)}\, \lambda^k(u,T-m\delta) \tag{14.63}$$

for $k = 0, i$ and $u \in [t,T]$. The following result provides an approximate valuation formula for a quanto caplet in a lognormal model of forward LIBOR rates.

Proposition 14.3.1. *The arbitrage price of a quanto caplet written on a LIBOR rate of the $i^{\rm th}$ market satisfies*

$$\mathbf{Cplq}_t \approx \bar{Q}\delta B(t,T+\delta)\Big(L^i(t,T)e^{-\tilde{\eta}^i(t,T)}\, N\big(\tilde{e}_1(t,T)\big) - \kappa\, N\big(\tilde{e}_2(t,T)\big)\Big),$$

where

$$\tilde{e}^i_{1,2}(t,T) = \frac{\ln(L^i(t,T)/\tilde{\kappa}) \pm \frac{1}{2}\,\tilde{v}_i^2(t,T)}{\tilde{v}_i(t,T)}$$

and

$$\tilde{v}_i^2(t,T) = \int_t^T |\lambda^i(u,T)|^2\, du.$$

Proof. It is enough to observe that

$$\mathbf{Cplq}_t \approx \bar{Q}\delta B(t,T+\delta)e^{-\tilde{\eta}^i(t,T)}\, \mathbb{E}_{\mathbb{P}_{T+\delta}}\Big\{\Big(L^i(t,T)\Lambda(t,T) - \tilde{\kappa}\Big)^+ \Big|\, \mathcal{F}_t\Big\},$$

where

$$\Lambda(t,T) = \exp\Big(\int_t^T \lambda^i(u,T)\cdot dW_u^{T+\delta} - \frac{1}{2}\int_t^T |\lambda^i(u,T)|^2\, du\Big)$$

and $\tilde{\kappa} = \kappa e^{\tilde{\eta}^i(0,T)}$. The expected value can be evaluated in exactly the same way as in the case of a domestic caplet (see Proposition 12.6.1). □

14.3.2 Cross-currency Swap

As the next example, let us consider a floating-for-floating $(k, l; 0)$ swap. Recall that the payoffs of such a swap are made in domestic currency. Therefore, its price at time t, denoted by $\mathbf{CCFS}_t(k, l; 0)$, equals (cf. (14.44))

$$\mathbf{CCFS}_t(k,l;0) = \sum_{j=1}^{n} \delta\, B(t,T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}} \big(L^k(T_{j-1}) - L^l(T_{j-1}) \,\big|\, \mathcal{F}_t \big)$$

per one unit of the nominal value (in domestic currency), where $\mathbb{P}_{T_j}$ is the forward measure of the domestic market for the date T_j. Notice that

$$L^i(T_j) = L^i(t,T_j)\, e^{-\eta^i(t,T_j)}\, M^i(t,T_j)$$

for $i = k, l$ and $j = 0, \ldots, n-1$, where

$$M^i(t,T_j) = \exp\Big(\int_t^{T_j} \lambda^i(u,T_j) \cdot dW_u^{T_{j+1}} - \frac{1}{2} \int_t^{T_j} |\lambda^i(u,T_j)|^2\, du \Big)$$

and

$$\eta^i(t,T_j) = \int_t^{T_j} \lambda^i(u,T_j) \cdot \big(\nu_u^i + b^i(u,T_j) - b(u,T_j) \big)\, du.$$

Applying an approximation similar to that of the previous section, we find that

$$\mathbf{CCFS}_t(k,l;0) \approx \sum_{j=0}^{n-1} \delta\, B(t,T_{j+1}) \Big(L^k(t,T_j) e^{-\tilde{\eta}^k(t,T_j)} - L^l(t,T_j) e^{-\tilde{\eta}^l(t,T_j)} \Big),$$

where $\tilde{\eta}^k(t,T_j)$ and $\tilde{\eta}^l(t,T_j)$ are given by (14.62)–(14.63). Let us end this section by commenting on the valuation of a floating-for-floating $(k, l; m)$ swap, which is denominated in units of the m^{th} currency. The price of such a swap at time t, in domestic currency, equals (cf. (14.54))

$$\mathbf{CCFS}_t(k,l;m) = \sum_{j=0}^{n-1} \delta\, B(t,T_{j+1})\, \mathbb{E}_{\mathbb{P}_{T_j}} \Big\{ Q^m_{T_{j+1}} \big(L^k(T_j) - L^l(T_j) \big) \,\Big|\, \mathcal{F}_t \Big\}.$$

From the results of Sect. 14.2.1, we know that the forward exchange rate satisfies $F_{Q^m}(T_{j+1}, T_{j+1}) = Q^m_{T_{j+1}}$ and (see (14.24)–(14.25))

$$dF_{Q^m}(t,T_{j+1}) = F_{Q^m}(t,T_{j+1}) \sigma_{Q^m}(t,T_{j+1}) \cdot dW_t^{T_{j+1}},$$

where $\sigma_{Q^m}(t,T_{j+1}) = \nu_t^m + b^m(t,T_{j+1}) - b(t,T_{j+1})$. Replacing the term $\sigma_{Q^m}(t,T_{j+1})$ with

$$\tilde{\sigma}_{Q^m}(t,T_{j+1}) = \nu_t^m + \tilde{b}^m(t,T_{j+1}) - \tilde{b}(t,T_{j+1}),$$

it is possible to derive an approximate formula for the price of a floating-for-floating $(k, l; m)$ swap in a lognormal model of forward LIBOR rates.

14.4 Concluding Remarks

Let us acknowledge that term structure models examined in this text do not cover all kinds of risks occurring in actual fixed-income markets. Indeed, we assumed throughout that all primary fixed-income securities (and derivative financial contracts) are default-free; that is, we concentrated on the *market risk* due to the uncertain future behavior of asset prices, as opposed to the *credit risk* (or *default risk*) that is known to play a non-negligible role in some sectors of financial markets. The latter kind of risk relates to the possibility of default by one party to a contract. If default risk is accounted for, one has to deal with such financial contracts as, for instance, defaultable (or credit-risky) bonds, vulnerable options, or options on credit spreads.

In this regard, let us mention that in recent years, term structure models that take explicit account of the default risk has attracted growing interest. Generally speaking, this involves the study of the impact of credit ratings on the yield spread – that is, modelling the term structure of defaultable debt – as well as the valuation of other contingent claims that are subject to default risk. Mathematical tools that are used in this context have attracted attention of several researchers and, as a result, a considerable progress in the theory of default risk was achieved in recent years. From the practical perspective, the demand for more sophisticated mathematical models was further enhanced by a rapid growth of trading in *credit derivatives*, that is, financial contracts that are capable of transferring the credit (default) risk of some reference entity between the two counterparties.

Since neither the modelling of defaultable term structure nor the valuation and hedging of credit derivatives were covered in this text, we refer the reader to the recent monographs by Ammann (1999), Cossin and Pirotte (2000), Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Schönbucher (2003), and Lando (2004) for an introduction to this field. The interested readers may also consult original papers by Merton (1974), Geske (1977), Jonkhart (1979), Brennan and Schwartz (1980b), Ho and Singer (1982, 1984), Titman and Torous (1989), Chance (1990), Artzner and Delbaen (1992, 1995), Kim et al. (1993), Lando (1994), Hull and White (1995), Jarrow and Turnbull (1995), Cooper and Martin (1996), Duffee (1996, 1998), Longstaff and Schwartz (1995, 1997), Duffie (1996b), Duffie and Huang (1996), Duffie and Singleton (1997, 1999), Jarrow et al. (1997), Duffie et al. (1997), Huge and Lando (1999), Kusuoka (1999), Elliott et al. (2000), Jeanblanc and Rutkowski (2002), Bélanger et al. (2004), and Bielecki et al. (2004a, 2004b).

We have also implicitly assumed that the inflation plays no role in the valuation of interest rate derivatives. In fact, the distinction between *nominal* and *real* interest rates is not important for most interest rate derivatives, but, obviously, this is not true for the so-called *inflation-based derivatives*, such as *inflation-linked* bonds or options. Let us only mention that a few models aiming the valuation and hedging of inflation-based derivatives were recently developed (see, for instance, Jarrow and Yildirim (2000)).