Efficient analytic price approximation for American Options. Discrete time-dependent parameters

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The problem formulation

Given

- risk-neutral probability measure $\mathbb{P}$;
- discrete set of times $t_0, \ldots, t_{n+1}$ with $t_0 = 0, t_{n+1} = T$;
- interest rate $r(t) = r_i, t_i \leq t < t_{i+1}$;
- fixed strike $K > 0$.

For a stochastic underlying $S_t$ at $t = t_0$ find an approximation for the American Option price

$$V_A = \max_{\tau \in T(t,T)} \mathbb{E}_\mathbb{P} \left( R(t, \tau) \max (\phi (U_\tau - K), 0) \right), \quad (1)$$

where $\tau$ is a stopping time with values in $[t, T]$, $\phi = \pm 1$ for a call or put, $R(s, t) = \exp \left( - \int_s^t r(u) \, du \right)$. 
The underlying process

Time-dependent parameters.

1. **Proportional dividends**, \( U_t = S_t \) with \( S_t \) being a BS log-normal diffusion

\[
dS_t = S_t \left[ (r(t) - d(t)) \, dt + \sigma(t) \, dw_t \right],
\]
\[
\sigma(t) = \sigma_i > 0, \quad d(t) = d_i \geq 0, \quad t_i \leq t \leq t_{i+1}.
\]  

2. **Discrete dividends** – Strike convention, see [1]

\( U_t = S_t - D(t) \) with \( S_t \) same as in (2) and

\[
D(t) = \sum_{t_j < t} \frac{D_j}{R(t_j, t)}.
\]  

\( D_j > 0 \) at some \( 0 < j < n + 1 \). Usually all \( d_i = 0 \).

For Comparison, the conventional standard setting is

\( U_t = S_t, \ \sigma(t) = \text{const}, \ r(t) = \text{const}, \ d(t) = \text{const}, \ D_i = 0. \)
The Pricing Problem

The early exercise representation

The price definition (1) implies, with \( \text{par} = \{r, \sigma, d, D\} \),

\[
V_A(t, U_t) = \begin{cases} 
\phi(S_t - K(t)) , \phi(S^*_t - K(t)) \geq 0 , \\
V_{AI\text{ Intr}}(t, T, S_t, \text{par}) , \text{otherwise}, 
\end{cases}
\]

where \( V_{AI\text{ Intr}}(t, T, S_t, \text{par}) \) is the intrinsic value and

- \( K(t) = K \), proportional dividends;
- \( K(t) = K + D(t) \), discrete dividends – Strike convention.

If spot and settlement adjustments are required, \( r, d, D \) are modified and \( \phi(S_t - K(t)) \) from (3) is multiplied by \( \beta > 0 \).

The optimal exercise boundary \( S^*_t \) is the value of \( S_t \) for which \( \phi(S_t - K(t)) = V_{AI\text{ Intr}}(t, T, S_t, \text{par}) \) and, as follows from the no arbitrage condition, \( \phi = \frac{\partial}{\partial S_t} V_{AI\text{ Intr}}(t, T, S_t, \text{par}) \).
The intrinsic value decomposition

Definition (1) can be reformulated to

$$\max_{\tau \in T(t,T)} \left[ \mathbb{E}_P \left( R(t,\tau) (\phi(S_\tau - K))_+ \right) , \mathbb{E}_P \left( R(t,T) (\phi(S_\tau - K))_+ \right) \right]$$

which leads to

$$V_{AIntr}(t,T,S_t,\text{par}) = V_{E,[t,T]} + V_t,$$  \hspace{1cm} (5)

where

- $V_{E,[t,T]}(S_t,\text{par})$ is a European contract with strike $K(T)$ maturing at time $T$ and priced using the BS formula;
- $V_t(t,T,S_t,\text{par})$ is the early exercise premium (EEP) to be approximated.
DVM=Decoupled Volatility Method

For a non-BS underlying $U_t$ decomposition (4) is viewed as a sum of

- $V_t(t, T, S_t, \text{par})$, a function of a log-normal diffusion $S_t$ with parameters $\text{par}$ satisfying the BS equation;
- $V_{E,[t,T]}(S_t, \widehat{\text{par}})$, a black box European option price function of $S_t$, returning Greeks, possibly with a different parameter set $\widehat{\text{par}}$, see [4].

Following the practice of quoting the European implied volatility at fixed strikes $K_k, k = 0, \ldots, m - 1$, it is possible to view $\widehat{\text{par}}$ as $\{r, \{\widehat{\sigma}_k\}, \text{d}, \text{D}\}$, where each $\widehat{\sigma}_k$ is a vector of implied volatilities $\widehat{\sigma}_{k,i}$ of a chain of European options at strike $K_k$ maturing at times $t_i$. In case of BS, $\widehat{\text{par}} = \text{par}$. 
DVM pros

The DVM approach

- allows for efficient approximation of $V_t(t, T, S_t, \text{par})$;
- for given vectors $\sigma$ and $\hat{\sigma}_k$ allows for very fast pricing OTC and repricing listed American options with integrated stable $\Delta, \Gamma, \Theta$;
- works with discrete dividends;
- allows for efficient extraction of underlying volatility (UV) $\sigma$ and surface of untraded European implied volatilities (IV) $\hat{\sigma}_k$ from listed American options;
- provides vectors of extracted IV $\hat{\sigma}_k$ for models expecting European input for calibration;
- can be applied on its own for pricing European/American vanillas at unlisted strikes or OTC options using just-in-time market correction (Vanna-Volga, etc.).
**DVM contras**

The major disadvantages of DVM are

- it is not a consistent underlying model, for $\hat{\sigma}_k \neq \sigma$ a stochastic process $U_t$ does not exist;
- the extracted European prices are not uniquely determined.

It should be noted however that more advanced local and multi-factor models (Local Volatility, Heston, Local Stochastic Volatility) do not presently offer

- fast algorithmic solutions, returning reliable price and Greeks, for American options;
- accurate calibration directly from American options on a timely basis.
The stepping expression

The option is priced backwards at times $t_{n-j}, j = 0, \ldots, n$. At $t = t_{n-j}, S_t = S_{n-j}$,

$$V_{AI\text{Intr},n-j}(T, S_{n-j}, \text{par}, \hat{\text{par}}) = V_{E,n-j}(T, S_{n-j}, \text{par}, \hat{\text{par}}) + V_{n-j}(T, S_{n-j}, \text{par})$$ (6)

where, see [5],

- the **forward start option**

$$V_{E,n-j} = R(t_{n-j}, t_{n-j+1}) \mathbb{E}(V_{A,n-j+1}(T, S_{n-j+1}) | S_{n-j})$$ (7)

is a European type option (not vanilla) on $[t_{n-j}, t_{n-j+1}]$ with payoff $V_{AI\text{Intr},n-j+1}(T, S_{n-j+1}, \text{par}, \hat{\text{par}})$

- $V_{n-j}$ is the EEP on $[t_{n-j}, t_{n-j+1}]$, expressed according to the Ju–Zhong approximation, [3]
Expanding $V_{A,n-j+1}$ following (3), we get

$$V_{E,n-j} = R(t_{n-j}, t_{n-j+1}) \mathbb{E}(V_{E,n-j+1} | S_{n-j}) + \frac{R(t_{n-j}, t_{n-j+1}) \mathbb{E}(F_{n-j+1} - G_{n-j+1} | S_{n-j})}{(8)}$$

where,

$$F_{n-j+1} = \phi(S_{n-j+1} - K_{n-j+1}) \mathbb{I}_{n-j+1} + V_{n-j+1} (1 - \mathbb{I}_{n-j+1})$$
$$G_{n-j+1} = V_{E,n-j+1} \mathbb{I}_{n-j+1}$$
$$\mathbb{I}_{n-j+1} = \mathbb{I}_{\phi(S_{n-j+1} - K_{n-j+1}) \geq 0}.$$ 

Then (3) can be applied again to $V_{E,n-j+1}$. The expression for $V_{E,n-j+1}$ can be unwound to $j = 0$. 

The price formula, **parallelizable**

Making use of the martingale property we end up with

\[
V_{AI\text{Intr},n-j} = V_{n-j} + V_{E,[t_{n-j},T]} + \sum_{k=n-j+1}^{n} R(t_{n-j}, t_k) E \left( F_k - G_k | S_{n-j} \right),
\]

in which,

- \( V_{E,[t_{n-j},T]} \) is computed by the BS formula as a function of \( S_{n-j} \) and \( \hat{\text{par}} \), according to DVM;
- \( E \left( F_k | S_{n-j} \right) = I_{\text{ex}} + I_c \) with \( I_{\text{ex}} \) expressed in a closed form, \( I_c \) computed efficiently, using quadratures.
Expressible in a closed form
1 and 2 steps

Case \( j = n \). The last step for \( n > 0 \), the only step for \( n = 0 \).
With \( F_{n-j+1} = 0, G_{n-j+1} = 0 \) we get

\[
V_{AIntr,n-j} (T, S_{n-j}) = V_E, [t_{n-j}, T] (S_{n-j}, \hat{\sigma}_{n-j}) + V_{n-j} (T, S_{n-j}, \sigma_{n-j}),
\]

epressed in the closed form, equivalent to

\[
\sigma (t) = \text{const}, \hat{\sigma} (t) = \text{const}, r (t) = \text{const}, d (t) = \text{const}.
\]

Case \( j = n - 1 \), the 2-step option. \( \mathbb{E} (G_{n-j+1} | S_{n-j}) \) is expressed in a closed form, using \( N_1, N_2 \).
Approximated by expressible in a closed form
3 or more steps, general settings

Case $n \geq 2$. $V_{AIntr,n-j+2}$ is approximated by

$$V_{AIntr,n-j+2} = V_{E,[t_{n-j+2},T]} \left( S_{n-j+2}, \hat{\text{par}} \right) + V_{MBAW,[t_{n-j+2},T]} \left( S_{n-j+2}, \text{par} \right),$$

where $V_{MBAW,[t_{n-j+2},T]} \left( S_{n-j+2}, \text{par} \right)$ is constructed using the inverse MBAW interpolation, [3], with following settings:

- single curve, faster, appropriate most of times;
- multiple curve segments, slower, more precise.

$\tilde{G}_{n-j+1} = \mathbb{E} \left( V_{AIntr,n-j+2} \mid S_{n-j+1} \right) \mathbb{I}_{n-j+1}$ is used to approximate $G_{n-j+1}$. Finally, $\mathbb{E} \left( G_{n-j+1} \mid S_{n-j} \right)$ is replaced with $\mathbb{E} \left( \tilde{G}_{n-j+1} \mid S_{n-j} \right)$, expressible in a closed form in terms of $N_1, N_2, N_3$, see [5]. Available efficient algorithms for $N_m, m > 3$ would increase the maximal number of steps computed in a closed form from 2 to $m$. 
The Optimal Exercise Boundary

The Optimal Exercise Boundary \( S^*_{n-j} \) is computed at every time \( t_{n-j} \). In case of proportional dividends it is positive and finite and can be computed iteratively, except for cases of \( r = 0 \) for puts or \( d = 0 \) for calls.

In case of discrete dividends, see below, it is possible to have no exercise at some \( t_{n-j} \). Every no-exercise time point \( t_{n-j} \) can be omitted with pricing continued on \([t_{n-j-1}, t_{n-j+1}]\) with discount \( R(t_{n-j-1}, t_{n-j+1}) \) and

\[
\begin{align*}
\tilde{D}_{[t_{n-j-1},t_{n-j+1}]} &= D_{n-j-1} + D_{n-j}; \\
\hat{\sigma}_{[t_{n-j-1},t_{n-j+1}]}^2 (t_{n-j-1}, t_{n-j+1}) &= \\
\sigma_{n-j-1}^2 \Delta t_{n-j-1} + \sigma_{n-j}^2 \Delta t_{n-j}, \\
\hat{\sigma}_{[t_{n-j-1},t_{n-j+1}]}^2 (t_{n-j-1}, t_{n-j+1}) &= \\
\hat{\sigma}_{n-j-1}^2 \Delta t_{n-j-1} + \hat{\sigma}_{n-j}^2 \Delta t_{n-j}.
\end{align*}
\]
Assumptions.

In order to consistently account for possible discontinuities in $K(t)$ at times $t_{n-j}$, the priced option, without losing the generality, will be considered on $[t_0+, t_{n+1}-]$ (post-dividend start, pre-dividend end).

The present approximation makes a simplifying assumption of discrete accrual of interest on dividends. With $D_j r_i \Delta t_i \leq \epsilon_r K$, $\epsilon_r << 1$, strikes $K(t)$ can be assumed changing discretely, $K_{n-j} = K(t_{n-j})$ at $[t_{n-j}+, t_{n-j+1}-], j = 0, \ldots, n$. Inserting more points $t_{n-j}$ leads into longer inter-dividend intervals $\Delta t$ to a finer time partition, better supporting this assumption.

Another typical (but not necessary) simplifying assumption is $d = 0$. 
Pricing calls.

A call can be exercised only at times $t_{n-j}$— in the Bermudan style if

$$K_{n-j} < K_{n-j+1} R(t_{n-j}, t_{n-j+1}),$$

(10)

with $S_{n-j} - K_{n-j+1} R(t_{n-j}, t_{n-j+1})$ being the asymptote for $V_{E,n-j+1} (t_{n-j+1}, T, S_{n-j+1})$. With (10),

- if true, the early exercise boundary $S_{n-j}^*$ exists and can be computed, pricing follows (2) with $V_{n-j} = 0$;
- otherwise, the exercise does not happen and time $t_{n-j}$ can be omitted.
(10) false means the put can be exercised at \( t_{n-j} \), the regular case, same EEP term \( V_{n-j} \).

(10) true means no exercise in \([t_{n-j} - \delta t_{n-j,n-j}, t_{n-j}]\) with

\[
\delta t_{n-j,l} = \frac{1}{r_{l-1}} \ln \frac{K_{n-j+1} R(t_l, t_{n-j+1})}{K_l}.
\]

If \( \delta t_{n-j,n-j} \ll \Delta t_{n-j-1} \), the \( \delta t_{n-j} \) can be ignored and \( V_{n-j-1} \) considered over \([t_{n-j-1}, t_{n-j}]\) in a regular way.

If \( \delta t_{n-j,n-j} \leq \Delta t_{n-j-1} \), a new point \( \tilde{t}_{n-j} = t_{n-j} - \delta t_{n-j} \) is inserted with exercise in \([t_{n-j-1}, \tilde{t}_{n-j}]\) (with the EEP term \( V_{n-j-1} \)) and no exercise in \([\tilde{t}_{n-j}, t_{n-j}]\) (no EEP term).

If \( \delta t_{n-j,n-j} > \Delta t_{n-j-1} \), the no-exercise iteratively expands left into \([t_{n-j-k-1}, t_{n-j-k}]\), \( k > 1 \), with \( \delta t_{n-j,n-j-k} \) considered.
Description.

Selected results of comparison of computed American option prices, their Greeks and computational times, using the approximation and 10000-node trinomial tree, for calls and puts at various strikes are shown below. The underlying $S$ is log normal, proportional and discrete dividends – strike convention are considered. The columns of output tables show the Strike ($K$), Price, Delta ($\Delta$), Gamma ($\Gamma$), computed using the approximation. Last three column show relative difference with the tree:

$$RelPrice = \left| 1 - \frac{Price_T}{Price_A} \right|,\ Rel\Delta = \left| 1 - \frac{\Delta_T}{\Delta_A} \right|,\ Rel\Gamma = \left| 1 - \frac{\Gamma_T}{\Gamma_A} \right|.$$ 

A permutation in a volatility vector was chosen intentionally to show the inapplicability of quadratic averaging and differencing in volatility to American options versus European ones.
Calls, \( S_0 = 100 \)

\[\sigma = (0.08, 0.06, 0.04, 0.05), \quad t = (0, 0.02, 0.1, 0.27, 0.52),\]
\[d = (0.03, 0.03, 0.03, 0.03), \quad r = (0.035, 0.04, 0.045, 0.05)\]

Calls, speed up factor range 5600-7500

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<th>(\Delta)</th>
<th>(\Gamma)</th>
<th>RelPrice</th>
<th>Rel(\Delta)</th>
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Testing Proportional dividends

**Calls, \( S_0 = 100 \)**

\[ \sigma = (0.08, 0.05, 0.04, 0.06) \]

Calls, permuted volatility vector, speed up factor range 4400 - 6400,

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<th>Price</th>
<th>( \Delta )</th>
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Puts, permuted volatility vector.

\( \sigma = (0.08, 0.06, 0.04, 0.05) \), speed up factor range 5900-7900.

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<th>( \Delta )</th>
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\( \sigma = (0.08, 0.05, 0.04, 0.06) \), speed up factor range 5700-7600.

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<td>4.2e-4</td>
<td>8.3e-3</td>
</tr>
</tbody>
</table>
Input

\[ S_0 = 100, \]
\[ t = (0, 0.0194, 0.103, 0.269, 0.519, 1.02, 3.02), \]
\[ \sigma = (0.08, 0.075, 0.04, 0.05, 0.06, 0.082), \]
\[ D = (0, 3, 0, 3, 3, 3, 0), \]
\[ r = (0.035, 0.04, 0.045, 0.05, 0.06, 0.065) \]
# Calls

Calls, speed up factor range **1600-2100**

<table>
<thead>
<tr>
<th>K</th>
<th>Price</th>
<th>Δ</th>
<th>Γ</th>
<th>RelPrice</th>
<th>RelΔ</th>
<th>RelΓ</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>25.910</td>
<td>0.991</td>
<td>0.002</td>
<td>1.053e-5</td>
<td>9.148e-6</td>
<td>4.278e-2</td>
</tr>
<tr>
<td>80</td>
<td>21.840</td>
<td>0.975</td>
<td>0.005</td>
<td>1.281e-5</td>
<td>3.888e-5</td>
<td>6.324e-3</td>
</tr>
<tr>
<td>85</td>
<td>17.902</td>
<td>0.940</td>
<td>0.009</td>
<td>1.652e-5</td>
<td>7.407e-6</td>
<td>3.907e-2</td>
</tr>
<tr>
<td>90</td>
<td>14.198</td>
<td>0.880</td>
<td>0.015</td>
<td>2.147e-5</td>
<td>4.534e-5</td>
<td>5.636e-2</td>
</tr>
<tr>
<td>95</td>
<td>10.850</td>
<td>0.792</td>
<td>0.022</td>
<td>1.960e-5</td>
<td>1.479e-4</td>
<td>2.502e-2</td>
</tr>
<tr>
<td>100</td>
<td>7.964</td>
<td>0.680</td>
<td>0.028</td>
<td>2.872e-5</td>
<td>6.992e-5</td>
<td>2.589e-2</td>
</tr>
<tr>
<td>105</td>
<td>5.603</td>
<td>0.554</td>
<td>0.031</td>
<td>2.112e-5</td>
<td>1.382e-4</td>
<td>2.544e-2</td>
</tr>
<tr>
<td>110</td>
<td>3.776</td>
<td>0.428</td>
<td>0.030</td>
<td>4.994e-5</td>
<td>5.554e-5</td>
<td>5.030e-2</td>
</tr>
<tr>
<td>115</td>
<td>2.439</td>
<td>0.313</td>
<td>0.027</td>
<td>2.619e-5</td>
<td>3.174e-4</td>
<td>2.602e-2</td>
</tr>
<tr>
<td>120</td>
<td>1.511</td>
<td>0.217</td>
<td>0.023</td>
<td>2.955e-5</td>
<td>3.525e-4</td>
<td>2.466e-2</td>
</tr>
<tr>
<td>125</td>
<td>0.900</td>
<td>0.144</td>
<td>0.018</td>
<td>3.769e-5</td>
<td>4.198e-4</td>
<td>2.457e-2</td>
</tr>
</tbody>
</table>
## Puts

Puts, speed up factor range 1700-2000

<table>
<thead>
<tr>
<th>K</th>
<th>Price</th>
<th>$\Delta$</th>
<th>$\Gamma$</th>
<th>RelPrice</th>
<th>Rel$\Delta$</th>
<th>Rel$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>0.054</td>
<td>-0.013</td>
<td>0.003</td>
<td>7.488e-2</td>
<td>8.564e-2</td>
<td>9.845e-2</td>
</tr>
<tr>
<td>80</td>
<td>0.200</td>
<td>-0.043</td>
<td>0.009</td>
<td>7.367e-2</td>
<td>6.968e-2</td>
<td>4.998e-2</td>
</tr>
<tr>
<td>85</td>
<td>0.631</td>
<td>-0.126</td>
<td>0.024</td>
<td>5.638e-2</td>
<td>3.739e-2</td>
<td>6.124e-3</td>
</tr>
<tr>
<td>90</td>
<td>1.708</td>
<td>-0.301</td>
<td>0.046</td>
<td>3.261e-2</td>
<td>7.995e-3</td>
<td>1.791e-2</td>
</tr>
<tr>
<td>95</td>
<td>3.855</td>
<td>-0.556</td>
<td>0.057</td>
<td>1.407e-2</td>
<td>5.922e-3</td>
<td>2.195e-2</td>
</tr>
<tr>
<td>100</td>
<td>7.199</td>
<td>-0.795</td>
<td>0.044</td>
<td>4.535e-3</td>
<td>7.027e-3</td>
<td>9.935e-4</td>
</tr>
<tr>
<td>105</td>
<td>11.499</td>
<td>-0.951</td>
<td>0.020</td>
<td>9.727e-4</td>
<td>3.956e-3</td>
<td>2.626e-2</td>
</tr>
<tr>
<td>110</td>
<td>16.288</td>
<td>-0.996</td>
<td>0.003</td>
<td>5.427e-5</td>
<td>7.784e-4</td>
<td>1.256e-1</td>
</tr>
<tr>
<td>115</td>
<td>21.164</td>
<td>-1</td>
<td>1.0e-4</td>
<td>1.354e-5</td>
<td>6.631e-6</td>
<td>1.484e-1</td>
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<tr>
<td>120</td>
<td>26.045</td>
<td>-1</td>
<td>6.8e-6</td>
<td>9.547e-6</td>
<td>1.040e-5</td>
<td>3.069e+0</td>
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<tr>
<td>125</td>
<td>30.926</td>
<td>-1</td>
<td>8.4e-7</td>
<td>7.989e-6</td>
<td>1.266e-6</td>
<td>9.537e+1</td>
</tr>
</tbody>
</table>
Given

- discrete set of times $t_0, \ldots, t_n$ with $t_0 = 0$;
- interest rate $r(t) = r_i, t_i \leq t \leq t_{i+1}$;
- a log-normal diffusion underlying $S_t$ with dividends, proportional $d(t) = d_i$ or discrete strike convention $D_i, t_i \leq t \leq t_{i+1}$;
- fixed strike $K > 0$;
- intrinsic prices $O_i$ at time $t_0$, strike $K$ of American puts or calls maturing at times $t_i, i \geq 1$.

Find the vector $\bar{\sigma}$ of BS implied volatilities $\bar{\sigma}_i$ on intervals $[t_i, t_{i+1}]$ for which $V_A(t_0, S_0, T = t_i, \text{par}) = O_i$. 
Solution procedure, parallelizable

BS implied volatilities are extracted sequentially by solving iteratively the inverse problem to

- scalar pricing to find $\bar{\sigma}_0$ from $O_1$;
- multi-step pricing from the above to find $\bar{\sigma}_i$ from $O_i$, $i > 1$, known $\bar{\sigma}_0, \ldots, \bar{\sigma}_{i-1}$.

It is not guaranteed that

- in real-time markets $\bar{\sigma}_{\text{call}} = \bar{\sigma}_{\text{put}}$;
- for a non-BS underlying at strikes $K, L$, $\bar{\sigma}_K = \bar{\sigma}_L$. 
Given

- a general underlying $S_t$ with proportional dividends $d_i$ or discrete strike convention dividends $D_i, t_i \leq t < t_{i+1}$;
- $m$ fixed strikes $K_k > 0$;
- $m$ pairs of vectors $C_{Mkt}^k, P_{Mkt}^k$ of prices $C_{Mkt}^k, P_{Mkt}^k$ at time $t_0$ and strikes $K_k$ of American calls and puts maturing at times $t_i, i \geq 1$;

Find the vector $\hat{\sigma}$ of the underlying volatility (UV) and $m$ vectors $\hat{\sigma}_k$ of European implied volatilities (IV) on intervals $[t_i, t_{i+1}]$ for which, following DVM,

- $V_A(t_0, S_0, T = t_i, K = K_k, \phi = -1, \hat{\text{par}}, \hat{\text{par}}) = P_{Mkt}^{k_i}$,
- $V_A(t_0, S_0, T = t_i, K = K_k, \phi = 1, \hat{\text{par}}, \hat{\text{par}}) = C_{Mkt}^k$.

The problem, as formulated, has $m + 1$ unknowns and $2m$ equations and cannot be solved. Solvable alternatives are offered below.
The weighted LS formulation for UV

Assuming the underlying close to log-normal, we look for volatility $\sigma(t)$ providing the weighted least square minimum for calls and puts at every time $t_i$.

Given $u_{\text{call},ki}, u_{\text{put},ki} \geq 0$, $\sum_{i,k} (u_{\text{call},ki} + u_{\text{put},ki}) = 1$, minimize

$$
\sum_{i,k} \left[ u_{\text{call},ki} \left( C_{ki}(\sigma) - C_{Mkt}^{\text{Mkt}} \right)^2 + u_{\text{put},ki} \left( P_{ki}(\sigma) - P_{Mkt}^{\text{Mkt}} \right)^2 \right],
$$

where $C_{ki}(\sigma), P_{ki}(\sigma)$ just abbreviate

$V_A(t_0, S_0, T = t_i, K = K_k, \phi, \text{par}, \hat{\text{par}} = \text{par})$.

Under market conditions $C_{k,j}^{\text{Mkt}} \geq C_{ki}^{\text{Mkt}}, P_{k,j}^{\text{Mkt}} \geq P_{ki}^{\text{Mkt}}$ for $j > i$ and $C_{k,l}^{\text{Mkt}} \leq C_{ki}^{\text{Mkt}}, P_{l,i}^{\text{Mkt}} \geq P_{ki}^{\text{Mkt}}$ for $l > k$ the solution exists within bounds $\min_k \bar{\sigma}_{\text{call,put},ki} \leq \sigma_i \leq \max_k \bar{\sigma}_{\text{call,put},ki}$, where $\bar{\sigma}_{ki}$ are raw implied volatilities computed above.
The 1st order term equation for UV

Time saver

Assuming 2nd order $\sigma$-terms $u_{ki}(\sigma_i - \bar{\sigma}_i)(\sigma_j - \bar{\sigma}_j)$ small (in line with the assumption for the underlying of being close to log-normal) we arrive at

$$
\sum_{i,k} [u_{call,ki} CT_{ki}^2 + u_{put,ki} PT_{ki}^2] \rightarrow \text{min},
$$

$$
CT_{ki} = \sum_{j \leq i} \nu_{call,ki} j (\sigma_j - \bar{\sigma}_{call,kj}),
$$

$$
PT_{ki} = \sum_{j \leq i} \nu_{put,ki} j (\sigma_j - \bar{\sigma}_{put,kj}),
$$

leading to a linear system after differentiation by every $\sigma_i$. Because of positive $\nu_{ki} = \frac{\partial C_{ki}^{Mkt}}{\partial \bar{\sigma}_j} (\bar{\sigma}_{kj})$, under market assumptions above the solution is contained within same bounds. A time saving trick is to use, instead of $\nu_{ki}$, the derivatives of European parts of $CT_{ki}, PT_{ki}$, easily obtainable analytically.
Another, even faster LS criterion matches directly $\sigma$ and $\bar{\sigma}_k$

$$\sum_{i,k} \left[ u_{call,ki} (\sigma_i - \bar{\sigma}_{call,ki})^2 + u_{put,ki} (\sigma_i - \bar{\sigma}_{put,ki})^2 \right] \rightarrow \min,$$

leading immediately to $\sigma_i = \frac{\sum_k (u_{call,ki} \bar{\sigma}_{call,ki} + u_{putl,ki} \bar{\sigma}_{put,ki})}{\sum_k (u_{call,ki} + u_{putl,ki})}$, more straightforwardly selecting heavier weighted $\bar{\sigma}_i$ but ignoring vegas as sensitivities.
More about weights

Every weight $u_{\text{call}/\text{put},ki}$ is a function $u(K_k, t_i, \phi)$.

A common, but not the only possible, choice is $u_{ki} = \theta_i w_{ki}$.

Most common choices for strike weights $w_{ki}$ are

- **uniform** $w_{\text{call},ki} = w_{\text{put},ki} = \frac{1}{mn}$;

- **volume-weighted** $w_{\text{call},ki} = \frac{1}{n} \frac{Vol_{\text{call},ki}}{\sum_k (Vol_{\text{call},ki} + Vol_{\text{put},ki})}$,

  $w_{\text{put},ki} = \frac{1}{n} \frac{Vol_{\text{put},ki}}{\sum_k (Vol_{\text{call},ki} + Vol_{\text{put},ki})}$.

The $Vol_{ki}$ above are end of day trade volumes, however it is also possible to consider (in any combination)

- intraday volumes traded within a fixed time $\Delta t$
- bid/ask or combined bid-ask volumes;
- call or put volumes only or one of two at any strike or time by setting the opposite side weight to 0.
Weights, continued

Strike $K_k$ at time $t_i$ on a call/put side can be excluded by setting $w_{\text{call/put}, ki} = 0$.

The most common choice for time weights $\theta_i$ are

- **deka-y**, $\theta_i = \frac{\lambda_i}{\sum_i \lambda_i}$, with $\lambda_i = e^{-\lambda \frac{\Delta t_i}{\Delta t}}$ for some fixed $\lambda, \Delta t$.
  
  $\lambda = 0$ gives uniform weights;

- **linear**, $\theta_i = \frac{\Delta t_{0,i}}{\Delta t_{0,n}}$. 
The Inverse Problem

The LS parity criterion for IV, parallelizable

It is now possible to find the vector of IV at strikes $K_k$ on call and put sides. A call is shown, a put is similar,

Given $\sigma, \hat{\sigma}_{\text{call}, j < i, k}$ and price $C_{ki}^{Mkt}$ at given $i, k$, find $\hat{\sigma}_{\text{call}, ki}$ for which $V_A(S_0, t_0, t_i, \text{par}, \text{par}_{\text{call}}) = C_{ki}^{Mkt}$.

Solved iteratively, using the direct pricing formula.

However, without the put-call parity for restored European contracts. To guarantee it we need $\hat{\sigma}_{\text{call}, k} = \hat{\sigma}_{\text{put}, k}$.

Given $\sigma$, prices $C_{ki}, P_{ki}$, weights $u_{\text{call}, ki}, u_{\text{put}, ki}$, find $\hat{\sigma}_k$, minimizing for kiven $k$

$$\sum_i \left[ u_{\text{call}, ki} \left( C_{ki}(\hat{\sigma}_k, \sigma) - C_{ki}^{Mkt} \right)^2 + u_{\text{put}, ki} \left( P_{ki}(\hat{\sigma}_k, \sigma) - P_{ki}^{Mkt} \right)^2 \right].$$

The bounds are $\min_k \hat{\sigma}_{\text{call}, put, ki} \leq \hat{\sigma}_{ki} \leq \min_k \hat{\sigma}_{\text{putl}, put, ki}$. 
The 1st order term equation for IV
Time saver and supersaver

Leaving just the 1st order Taylor term in $\hat{\sigma}_{ki}$ we arrive at

$$\sum_{i} \left[ u_{call,ki} C T_{ki}^2 + u_{put,ki} P T_{ki}^2 \right] \rightarrow \text{min},$$

$$C T_{ki} = \sum_{j \leq i} \hat{\nu}_{call,ki,j} (\hat{\sigma}_{kj} - \hat{\sigma}_{call,kj}), \quad P T_{ki} = \sum_{j \leq i} \hat{\nu}_{put,ki,j} (\hat{\sigma}_{kj} - \hat{\sigma}_{put,kj}),$$

differentiated to a linear system. A time saving simplification, computing $\hat{\nu}_{call/put,ki,j} = \frac{\partial C_{Mkt}^{Mkt}}{\partial \hat{\sigma}_{kj}} (\hat{\sigma}_{call/put,kj})$ from European parts of $C_{ki}^{Mkt}, P_{ki}^{Mkt}$ can be used here too.

And a faster LS criterion matches directly $\hat{\sigma}_{k}$ and $\hat{\sigma}_{call/put,k}$

$$\sum_{i} \left[ u_{call,ki} (\hat{\sigma}_{ki} - \hat{\sigma}_{call,ki})^2 + u_{put,ki} (\hat{\sigma}_{ki} - \hat{\sigma}_{put,ki})^2 \right] \rightarrow \text{min},$$

leading to $\hat{\sigma}_{i} = \frac{u_{call,ki} \hat{\sigma}_{call,ki} + u_{putl,ki} \hat{\sigma}_{put,ki}}{u_{call,ki} + u_{putl,ki}}$. 
Conclusions

Presented:

- an efficient analytic approximation for American options on log-normal underlyings with time-dependent parameters, proportional or discrete dividends – strike convention;

- the DVM framework, designed to price efficiently American options on a general underlying, proportional or discrete dividends;

- the comparison of results and computational times for the presented approximation and trinomial tree for a log-normal underlying, proportional or discrete dividends;

- the inverse problem of extraction of the time-dependent IV curve and UV surface for an general underlying, proportional or discrete dividends.
References


Y. Shkolnikov, *Decoupled American option pricing method, computation of implied volatilities, further applications*, SSRN, 2009, Abstract 1371930