# Numerix Model Calibration: The Multiple Curve Approach

*October 2013*

*This document describes the multiple-curve approach implemented to support Numerix model calibration.*

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# 1 Introduction

The liquidity crisis that occurred in August 2007 has changed the way fixed income derivatives are priced. The classic non-arbitrage framework, with its assumption that cash flows must be discounted at the same rate as money is lent, has necessarily been abandoned.[1] Rates now embed the borrower default risk; this leads to calculations that may appear inconsistent from a non-arbitrage point of view.

For instance, when a financial entity has to pay a coupon that depends on a reference index (e.g., the Libor rate), the coupon depends on the credit risk given by the market to that reference asset (here, the average credit risk of the Libor contributors), which is not necessarily the credit risk of this financial entity. The present value of that coupon is then calculated using a discount curve that depends on the way the derivative is funded. Usually, the discount curve is an OIS curve if the derivative is collateralized; otherwise, the discount curve could be the financial entity funding rate.

This simple example shows that even the simplest fixed income products are not simple anymore and that the usual pricing formulas have to be updated. However, updating the usual pricing formulas is just one part of the problem; all the stochastic fixed income models have to be updated, too. This document summarizes the changes done to take into account this new framework by recalling in Section 2 the classic single-curve approach, then developing forward rate formulas in the multi-curve framework in Section 3. In Section 4, we describe requirements for the extension of the stochastic fixed income models by tackling the calibration problem and in Section 6 we discuss the application of this framework to the case where the curve spread is deterministic.

# 2 Classic Approach (Single Curve)

In a classic interest rate (IR) model, the principal element is a zero-coupon bond. The price of a zero-coupon bond with maturity $T$ and observation date $t$ is denoted $P(t,T)$. In an economy without risk, the zero-coupon bond returns 1 for $t = T$. In the present crisis, however, this is no longer true, so the classic IR model must be modified. Let us recall some facts about the classic model.

A model constructed using a single (discounting) curve $\gamma(T)$ determines the stochastic evolution

$$P(t,T) = N(t)\mathbb{E}\left[\frac{1}{N(T)}\middle| F_t\right] \tag{2.1}$$

for the (risk-free) zero-coupon bond $P(t,T)$ , where $N(t)$ is a model numeraire and $\mathbb{E}$ is the corresponding expectation operator. The zero-coupon bond is linked with the model curve as

$$P(0,T) = e^{-T\gamma(T)}. \tag{2.2}$$

Indices are deterministic functions of zero-coupon bonds. For example, the risk-free (theoreti-

[1] For a precise description of the behavior of the financial markets during this period, see http://ssrn.com/abstract=1334356.

cal) Libor index $L_{th}(T_1, T_2)$ is defined as

$$L_{th}(T_1, T_2) = \frac{1}{\delta_l T} \left( \frac{1}{P(T_1, T_2)} - 1 \right), \tag{2.3}$$

where $\delta_l T = T_2 - T_1$ using the Libor day-count basis. We use the subscript "th" to assign the "theoretical value" in the single-curve setup. A theoretical forward Libor defined as

$$L_{th}(t; T_1, T_2) = \mathbb{E}_{T_2} \left[ L_{th}(T_1, T_2) \mid F_t \right] \tag{2.4}$$

can be also presented in terms of zero-coupon bonds,

$$L_{th}(t; T_1, T_2) = \frac{1}{\delta_l T} \left( \frac{P(t, T_1)}{P(t, T_2)} - 1 \right). \tag{2.5}$$

Indeed, rewriting the $T_2$-forward measure expression into the model measure, we obtain

$$L_{th}(t; T_1, T_2) = \frac{N(t)\mathbb{E}\left[ \frac{L(T_1,T_2)}{N(T_2)} \,\middle|\, F_t \right]}{P(t, T_2)} = \frac{P(t, T_1) - P(t, T_2)}{\delta_l T \, P(t, T_2)}.$$

# 3 Current Approach (Multiple Curves)

In reality, we should step out of the classic single-curve theory and treat the Libor index differently from its risk-free expression in Eq. (2.3):

$$L(T_1, T_2) \neq \frac{1}{\delta_l T} \left( \frac{1}{P(T_1, T_2)} - 1 \right). \tag{3.1}$$

As in the single-curve modeling, the (real) forward Libor is defined as the expectation of the Libor in $T_2$-forward measure,

$$L(t; T_1, T_2) = \mathbb{E}_{T_2} \left[ L(T_1, T_2) \mid F_t \right], \tag{3.2}$$

or, in the model measure,

$$L(t; T_1, T_2) = \frac{N(t)\mathbb{E}\left[ \frac{L(T_1,T_2)}{N(T_2)} \,\middle|\, F_t \right]}{P(t, T_2)}. \tag{3.3}$$

The market curve stripping gives us initial values of the forward Libor rates $L(0; T_1, T_2)$, which, in general, do not coincide with that of the risk-free (theoretical) forward Libor obtained from the discounting curve,

$$L(0; T_1, T_2) \neq L_{th}(t; T_1, T_2) = \frac{1}{\delta_l T} \left( \frac{P(0, T_1)}{P(0, T_2)} - 1 \right).$$

Consider now a fixed-floating swap with the following schedules:

- Floating payments: each payment period $i = 1, \ldots, M$ is composed of
  - Payment dates $T_i^p$,
  - Day-count fraction $\delta T_i$ with some basis $b$,

    - Libor rate $L(T_i^f, T_i^e)$ info:
      fixing date $T_i^f$, end date $T_i^e$ and Libor day-count fraction $\delta_l T_i$.

- Fixed payments: each payment period $i = 1, \cdots, \widetilde{M}$ is composed of

    - Payment dates $\widetilde{T}_i^p$,
    - Day-count fraction $\delta\widetilde{T}_i$ with some basis $\widetilde{b}$,
    - Fixed rate $K$.

Note that the floating payment date and the Libor end can be slightly different.

The floating leg, as seen from time $t$ before the swap dates, can be presented as

$$Fl(t) = \sum_{i=1}^{M} \delta T_i N(t) \mathbb{E}\left[ \frac{L(T_i^f, T_i^e)}{N(T_i^p)} \middle| F_t \right]. \tag{3.4}$$

Note that due to the eventual mismatch between the dates $T_i^p$ and $T_i^e$, the expression above cannot be written in terms of forward rates given by Eq. (3.3). However, we can come up with the approximation

$$N(t)\mathbb{E}\left[ \frac{L(T_i^f, T_i^e)}{N(T_i^p)} \middle| F_t \right] \simeq \frac{P(0,T_i^p)}{P(0,T_i^e)} N(t)\mathbb{E}\left[ \frac{L(T_i^f, T_i^e)}{N(T_i^e)} \middle| F_t \right],$$

where we have used

$$\frac{P(t,T_i^p)}{P(t,T_i^e)} \simeq \frac{P(0,T_i^p)}{P(0,T_i^e)}.$$

Thus, an accurate approximation of the floating payment in terms of the Libor fowards is

$$Fl(t) \simeq \sum_{i=1}^{M} \delta T_i \frac{P(0,T_i^p)}{P(0,T_i^e)} P(t,T_i^e)\, L(t;T_i^f, T_i^e). \tag{3.5}$$

Actually, we can define an *adjusted* dcf to absorb the bonds ratio

$$\delta_F T_i = \delta T_i \frac{P(0,T_i^p)}{P(0,T_i^e)} \tag{3.6}$$

to get the standard form (in which we replace the approximation sign by the equality keeping in mind the small magnitude of the correction)

$$Fl(t) = \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, L(t;T_i^f, T_i^e). \tag{3.7}$$

The fixed leg as seen from date $t$ is represented as in the single-curve theory,

$$Fi(t) = KA(t), \tag{3.8}$$

with annuity

$$A(t) = \sum_{i=1}^{\tilde{M}} \delta\tilde{T}_i\, P(t, \tilde{T}_i^p). \tag{3.9}$$

 

The swap price as seen at $t$ is thus

$$S(t) = Fl(t) - Fi(t) = \sum_{i=1}^{M} \delta_F T_i \, P(t, T_i^e) \, L(t; T_i^f, T_i^e) - K \sum_{i=1}^{\tilde{M}} \delta \tilde{T}_i \, P(t, \tilde{T}_i^p). \tag{3.10}$$

Its par value at the origin permits stripping out initial values of the forward Libor $L(0; T_i, T_{i+1})$.

The swap rate as seen at $t$ equals

$$R(t) = \frac{\sum_{i=1}^{M} \delta_F T_i \, P(t, T_i^e) \, L(t; T_i^f, T_i^e)}{\sum_{i=1}^{\tilde{M}} \delta \tilde{T}_i \, P(t, \tilde{T}_i^p)}. \tag{3.11}$$

# 4 Multiple-Curve Calibration

The forward rate modification in the presence of multiple curves leads to changes in the swaptions we use for the model calibration. Indeed, a European swaption is a right to enter at some date $t$ into a fixed-floating swap. The swaption price at origin is given by (see Eq. (3.10))

$$Sw(0) = \mathbb{E}\left[\frac{(S(t))^+}{N(t)}\right] = \mathbb{E}\left[\frac{\left(\sum_{i=1}^{M} \delta_F T_i \, P(t, T_i^e) \, L(t; T_i^f, T_i^e) - K \, A(t)\right)^+}{N(t)}\right]. \tag{4.1}$$

In a measure associated with annuity $A(t)$ as numeraire, we have

$$Sw(0) = A(0) \mathbb{E}_A \left[\left(R(t) - K\right)^+\right], \tag{4.2}$$

where the swap rate $R(t)$ is defined in Eq. (3.11). The swaption price is quoted in terms of the Black-Scholes volatility $\sigma$ of the lognormally distributed swap rate

$$Sw(0) = A(0) \, \text{BS}(t, R(0), K, \sigma), \tag{4.3}$$

where the swap rate at the origin

$$R(0) = \frac{\sum_{i=1}^{M} \delta_F T_i \, P(0, T_i^e) \, L(0; T_i^f, T_i^e)}{\sum_{i=1}^{\tilde{M}} \delta \tilde{T}_i \, P(0, \tilde{T}_i^p)} \tag{4.4}$$

is known, given the discounting curve $P(0, T)$ and forward Libor rates at origin $L(0; T_i^f, T_i^e)$ obtained in the curve stripping procedure.

Applying the BS formula to our swaption, we can obtain its price for the model calibration.

# 5 Multiple Curves: Market Input and Modeling

In this section, we sum up the market information in order to proceed with concrete modeling. The market input consists of the following:

- A discounting (e.g., OIS) curve at origin $P(0, T)$,
- Forward Libor values at origin $L(0; T_i^f, T_i^e)$,

 

- Forward swap rate values at origin $R(0)$ for any swaption using Eq. (4.4),
- European swaption prices obtained by the BS formula Eq. (4.3).

Given the market information, we should come up with an adequate model satisfying this market in order to price other securities. In the next section, we describe a deterministic approach to multiple-curve modeling.

# 6 Deterministic Spread

Here we work out a deterministic spread approach implemented in Numerix. It consists in treatment of a forward Libor rate as a sum of the theoretical Libor rate and some deterministic spread

$$L(t,T_1,T_2) = \frac{1}{\delta_l T}\left(\frac{P(t,T_1)}{P(t,T_2)} - 1\right) + s(T_1,T_2), \tag{6.1}$$

where the spread is calculated given market information at the origin,

$$s(T_1,T_2) = L(0,T_1,T_2) - \frac{1}{\delta_l T}\left(\frac{P(0,T_1)}{P(0,T_2)} - 1\right). \tag{6.2}$$

This approach does not require deep changes in the single-rate modeling and remains model independent: For any IR model operating with the single-curve risk-free rate, we define the real forward Libor rate as in Eq. (6.1).

The model calibration reduces to a swaption payment modification with respect to the single-curve modeling. Namely, the swap price (3.10) can be rewritten as

$$S(t) = \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, \left(L_{th}(t;T_i^f,T_i^e) + s(T_i^f,T_i^e)\right) - KA(t),$$

leading to

$$S(t) = \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, L_{th}(t;T_i^f,T_i^e) + \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, s(T_i^f,T_i^e) - KA(t). \tag{6.3}$$

Denoting the theoretical, or single-curve, swap part by

$$S_{th}(t) = \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, L_{th}(t;T_i^f,T_i^e) - K\,A(t), \tag{6.4}$$

we present the two-curve swap as

$$S(t) = S_{th}(t) + \sum_{i=1}^{M} \delta_F T_i\, P(t,T_i^e)\, s(T_i^f,T_i^e);$$

i.e., the only difference from the single-curve approach reduces to a spread fixed payment stream.

Thus, the multiple-curve model calibration with deterministic spread is equivalent to calibration of the single-curve model to *bespoke* swaption prices

$$\mathbb{E}\left[\frac{\left(\sum_{i=1}^{M}\delta_F T_i\, P(t,T_i^e)\, L_{th}(t;T_i^f,T_i^e) + \sum_{i=1}^{M}\delta_F T_i\, P(t,T_i^e)\, s(T_i^f,T_i^e) - K\, A(t)\right)^+}{N(t)}\right]. \tag{6.5}$$

Note that the swaptions for calibration are coded in terms of *fixed* payments streams; i.e., for dates $\tau_i$ and amounts $A_i$, the calibrator is able to handle the prices

$$\mathbb{E}\left[\frac{\left(\sum_{i=1}^{K} A_i P(t,\tau_i)\right)^+}{N(t)}\right]. \tag{6.6}$$

Obviously, the payment amounts should have different signs. For example, $A_1$ can be close to one, $A_K$ close to -1, and intermediate amounts slightly negative.

Finally, we will address a presentation of the theoretical Libor payments

$$Fl_{th}(t) = \sum_{i=1}^{M}\delta_F T_i\, P(t,T_i^e)\, L_{th}(t;T_i^f,T_i^e) \tag{6.7}$$

in terms of the fixed payment stream suitable for the calibration. Indeed, by definition we have

$$Fl_{th}(t) = \sum_{i=1}^{M}\frac{\delta_F T_i}{\delta_l T_i}\left(P(t,T_i^f) - P(t,T_i^e)\right). \tag{6.8}$$

So this form is already a fixed payment stream. If $T_{i+1}^f = T_i^e$, we denote tenor $\{\tau_i\}$ as $\tau_i = T_i^f$ for $i = 1,\ldots,M$ and $\tau_{M+1} = T_M^e$ as well as the ratio $\alpha_i = \frac{\delta_F T_i}{\delta_l T_i}$. Then, it is easy to see that

$$Fl_{th}(t) = \sum_{i=1}^{M}\alpha_i\left(P(t,\tau_i) - P(t,\tau_{i+1})\right) \tag{6.9}$$

$$= \alpha_1\, P(t,\tau_1) + \sum_{i=2}^{M}(\alpha_i - \alpha_{i-1})\, P(t,\tau_i) - \alpha_M\, P(t,\tau_{M+1}). \tag{6.10}$$

Finally, if we set $\alpha_i = 1$, we get the usual expression for the floating leg,

$$Fl_{th}(t) = P(t,\tau_1) - P(t,\tau_{M+1}). \tag{6.11}$$

The model construction works as follows:

- Get a discounting curve and pick up a single-curve IR model corresponding to it.
- Get initial forward Libor curve and calculate deterministic spread by Eq. (6.2).
- Convert calibrating swaptions into bespoke ones (Eq. (6.5)) depending solely on the single-curve model.
- Calibrate the single-curve model to the bespoke swaptions.
- During the pricing, calculate the real rate using a shift of the theoretical one simulated using the single-curve model in Eq. (6.1).

A similar reasoning applies for caps; a caplet can be seen as an option on a swap (i.e., a swaption) that pays a single Libor payment. Therefore, calibrating to a caplet in the two-curve framework with a deterministic spread is identical to calibrating to a caplet in the classical framework with a shifted strike.

## 6.1 Swap Rate for Deterministic Spread

We have already computed the swap rate for a general fixed-floating exchange in Eq. (3.11). If we know the underlying Libor curve and treat the spread in the deterministic way, the swap rate is

$$R(t) = \frac{Fl_{th}(t) + \sum_{i=1}^{M} \delta_F T_i \, P(t, T_i^e) \, s(T_i^f, T_i^e)}{\sum_{i=1}^{\tilde{M}} \delta \tilde{T}_i \, P(t, \tilde{T}_i^p)}, \tag{6.12}$$

where theoretical floating payment stream in Eq. (6.7) can be presented as in Eq. (6.8); see also formulas (6.10)–(6.11) for special cases of the schedule.

## 6.2 Calibration to CMS products

In [1], formulas were provided for LMM framework pricing of CMS products that could be used for calibration. These formulas assumed that the forward rate curve was identical to the discounting curve. The previous discussion shows that the extension of the calibration procedure to the two-curve framework is straightforward given a simple approximation.

For a CMS swap, we have

$$\begin{aligned}
PV(T_0) &= \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} R(T_j)\right] \\
&= \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} \frac{\sum_i \delta_i \left(L_{th}(T_j; T_i^f, T_i^e + s(T_j, T_i^f, T_i^e)\right) P(T_j, T_i^e)}{\sum_i \delta_i P(T_j, T_i^{e'})}\right] \\
&= \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} \frac{\sum_i \delta_i L_{th}(T_j; T_i^f, T_i^e) P(T_j, T_i^e)}{\sum_i \delta_i P(T_j, T_i^{e'})}\right] + \mathbb{E}_{T_j}\left[\frac{\sum_i \delta_i s(T_j, T_i^f, T_i^e) P(T_j, T_i^e)}{\sum_i \delta_i P(T_j, T_i^{e'})}\right] \\
&\approx \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} R_{th}(T_j)\right] + \left(R(T_0) - R_{th}(T_0)\right) P(T_0, T_j).
\end{aligned}$$

That is, the PV of the CMS swap in the two-curve pricing is equal to the PV of the same instrument in a one-curve pricing plus the discounted value of the swap spread.

A similar reasoning can be applied for CMS options:

$$\begin{aligned}
PV(T_0) &= \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} \left(R(T_j) - K\right)^+\right] \\
&\approx \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} \left(R_t h(T_j) + R(T_0) - R_{th}(T_0) - K\right)^+\right] \\
&\approx \sum_j \mathbb{E}_{T_j}\left[\delta_{T_j} \left(R_t h(T_j) - (K - R(T_0) + R_{th}(T_0))\right)^+\right].
\end{aligned}$$

Like swaption pricing, CMS option pricing in the two-curve framework can be approximated with the standard one-curve formula and a shifted strike (and a similar reasoning applies for CMS spread options).

## References

[1] Alexander Antonov, Matthieu Arneguy, 2009. Analytical Formulas for Pricing CMS Products in the Libor Market Model. Available at SSRN: http://ssrn.com/abstract=1352606.

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