Algorithmic Exposure and CVA for exotic derivatives

Alexandre Antonov, Serguei Issakov, Serguei Mechkov
Numerix*

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Abstract

We develop the algorithmic approach for Counterparty exposure calculation and automate its application to arbitrary complicated instruments. Assuming that the portfolio is priced by the backward (American) Monte-Carlo method, our approach allows calculating the credit exposure as a pricing by-product, essentially without modifications in the usual pricing procedure. In particular, for the exposure calculation of callable instruments we manage to avoid a cumbersome aggregation of exercise indicators, applying them sequentially in parallel with the main pricing.

We explain how the obtained exposure can be integrated into the Credit Valuation Adjustment (CVA), based on the extension of the pricing model with a Counterparty credit component. The presented approach to the exposure computation is formulated in an arbitrary probability measure. To perform the measure change we use the cross-currency model semantics and calibrate the model to the real-world measure using indexes projections.

1 Introduction

In this paper we propose a unified approach to computing Monte Carlo simulated measures for Market Risk and Counterparty Risk. For Market Risk, the corresponding measures are Monte Carlo VaR (Value at Risk) and Expected Shortfall. For Counterparty Risk, we consider two approaches, which are associated with Basel II and Basel III, respectively. The first approach (Basel II) consists of computing what is generally referred to as Counterparty Credit Exposure. Those risk measures include Potential Future Exposures (PFE), Expected Positive Exposures (EPE), and other related risk measures. Counterparty Credit Exposure is the maximum amount of money that can be lost

*e-mails: antonov@numerix.com, isakov@numerix.com, meshkov@numerix.com

if default occurs for Counterparty and/or Self. A more advanced measure of Counterparty Risk, required by Basel III, is Credit Valuation Adjustment (CVA). CVA is an adjustment to the price of any financial instrument due to the possible default of Counterparty and/or Self, which depends on the default probabilities of Counterparty and/or Self.

There are several components of this approach that need to be discussed. In Quantitative Finance, there is consensus about how to compute price. In contrast to that, for Risk computation, there are various approaches used by middle office practitioners, also provided by Risk system vendors, which do not necessarily agree with each other. Some of these approaches do not allow estimating the error they introduce since they are based on non-controllable approximations. There is also the fundamental challenge of growing dimensionality when computing advanced Risk measures for non-linear instruments and structured products that require Monte Carlo method for pricing. This is referred to as "Monte Carlo on Monte Carlo" in the Risk industry and "stochastic on stochastic" in the Insurance industry.

In this paper we provide a theoretical base that can be used to reconcile various approaches to computing Risk, establishing the Risk computation on the same level of accuracy as the price computation. It provides a common language for the front and middle offices. One of the fundamentals of this approach is the concept of the *Exposure*. Intuitively, exposure is the distribution of prices, rates, or indexes on future dates. We can speak about *Exposure-centric analytics*, which generalizes the existing *Price-centric analytics*, as a rigorous framework for computing advanced Market and Counterparty Risk. This also provides a natural way for computing scenarios for Economic Scenario Generators, by adding economic variables to the scenario generation framework.

The main subject of the present article is a calculation of exotic portfolio *exposure*. One can define it as possible values of the portfolio price at time t related to different scenarios of the market evolution. There are at least two significantly different approaches to the subject: one is the Scenario approach and the second is the Modeling based one.

A literature on the subject is quite rich. Many authors are contributed here especially during last years. We mention selected books, reviews and articles, where further references can be found: Canabarro and Duffie (2003), Pykhtin (2005), Cesari et al. (2010), Gregory (2010), Brigo and Capponi (2010), Pykhtin and Rosen (2010), Brigo et al. (2011).

The Scenario based approach can be presented by the following steps. First, we generate a set of markets (scenarios) $M_i(t)$ at time t including yield curves, implied volatilities etc. Then, for each market, we choose a model and calibrate it to the market. Finally we price our portfolio with the calibrated model for each market. If the portfolio contains plain vanilla instruments we do not need to pass by the model and can evaluate the price directly from the generated market. We will concentrate here on exotic deals portfolio which requires model based pricing.

The main drawback of the Scenario based approach is its big computation effort. Indeed, apart from the model construction one needs to price the exotic deals for each scenario. The pricing procedure often involves slow methods

like the Backward Monte Carlo[1] which is computationally intensive. Another difficulty is related to the scenarios generation. There is a large variety of theoretical and phenomenological approaches to the scenario generation which become rather ambiguous for large cross-asset systems. The last shortcoming is that a period from today to the observation time t is out of scope: events like exercises and payments are not taken into account.

The Modeling approach is free of these drawbacks and based on a strict arbitrage-free model properly calibrated to today implied market. We prefer the model generated scenarios due to the following reasons. First, the exotic instrument pricing can be done in a very efficient manner: we need basically only one pricing. Second, the model provides the full time coverage: scenarios are available for all the time-steps. Third, the exposure is automatically consistent with the pricing. Finally, the model measure change can deliver the real-world measure taking into account indexes projections.

The key idea of the Modeling approach was developed by Cesari et al (2010). It associates the portfolio exposure with its future price given by an arbitrage-free model. As mentioned above, this approach is very attractive for exotic portfolios while vanilla ones can, in principle, be treated using the Scenarios.

Technically, the Modeling approach reduces to calculation of future portfolio prices using Backward Monte Carlo method. This future price is identified with the exposure applying netting conditions. Contrary to the Present Values (PV's), the exposure distribution depends on the model measure, and practitioners often prefer the exposure calculation in a *real-world measure*. In the article we discuss possibilities of calibration to the real-world measure using a cross-currency model analogy.

For *non-callable* simple instruments the future price can be calculated analytically, or reusing Backward Monte Carlo pricing routines. On the other hand, a *callable* instrument future price requires a careful calculation of exercise conditions and leads to cumbersome modifications of the pricing routines: the Backward pricing calculating exercise condition should be followed by a Forward Monte Carlo aggregating the final result. However, we have managed to include the exposure calculus directly in the Backward pricing routine without external changes which leads to a high comfort of the software usage. This is one of our main results in the present article.

The backward pricing procedure including instructions on instruments execution such as cashflows addition or exercise application is usually written in terms of a pay-off pseudo-code. *Our achievement is that the exposure calculation can be done during the backward pricing without changing the pseudo-code structure.*

We note here that the *Forward* Monte-Carlo pricing routine suitable for *pricing* of vanillas and semi-exotic instruments is not appropriate for the *exposure* calculation unless analytical methods are available. Indeed, the term exposure refers to a future price, i.e. to conditional expectation of instrument cashflows. Thus an extra mechanism such as analytics or backward Monte Carlo is necessary for the exposure calculation. In the present article we do not

[1]The Backward Monte Carlo method is also referred to in the literature as the Least Squares Monte Carlo approach, the Longstaff-Schwartz method [9] or American Monte Carlo one.

consider the forward direction routine but concentrate exclusively on portfolio written and executed in the Backward terms.

A few comments about the concept of a model probability measure. The model calibrated to today market still has an extra degree of freedom – the model stochastic measure. The arbitrage-free theory guarantees that any instrument *present value* (PV) does not depend on the measure. However, the *distribution* of future prices generated by the model does depend on the measure. Various Risk measures, both for market risk and Counterparty risk (Value at Risk, Potential Future Exposures, etc.) are expressed in terms of these distributions and thus measure dependent. On the other hand, credit valuation adjustment (CVA) is linked with PV's of default dependent payments and is measure independent. In this work, we assume an arbitrary fixed measure for the model, addressing the *real world measure* at the end.

The article is organized as follows. Section 2 describes the Modeling approach to the exposure calculation, with several simple illustrations. Section 3 explains our algorithm of the algorithmic exposure calculation for exotic portfolios. Section 4 discusses the CVA computation. Section 5 provides formulas for the risk computation and propose a new method of the real-world measure calibration and usage in the risk estimation. In Section 6 we present the numerical results for simulation in the real world measure.

2 The Modeling approach

Consider a payment of a certain amount $\mathcal{A}$ at a date τ. The present value (PV) of the payment is its discounted expectation

$$A(0) = \mathbb{E}\left[\frac{\mathcal{A}}{N(\tau)}\right]$$

where $N(t)$ is the model numeraire and $\mathbb{E}[\cdots]$ is the pricing expectation in the model measure.

The payment *seen* from some observation date $t < \tau$ is a conditional expectation

$$A(t) = \mathbb{E}\left[N(t)\frac{\mathcal{A}}{N(\tau)}\middle|\,\mathcal{F}_t\right]$$

Before passing to instrument examples, it is suitable to explain the concept of *conditional expectation* (CE) $\mathbb{E}\left[\cdots|\,\mathcal{F}_t\right]$. Loosely speaking, it is an average over the stochastic evolution after time t. If we fix the Brownian increments $\mathrm{d}W(\tau)$ for $\tau \leq t$ and average over $\mathrm{d}W(\tau)$ for $\tau > t$, then our CE is a continuous function of $\mathrm{d}W(\tau)$ for $\tau \leq t$. On the other hand, for any Markov process, the dependence from the previous Brownian increments is completely absorbed in the Markov state variables. Thus, for any regular path-independent pay-off $\mathcal{P}(T)$ at time T, the discounted CE at time t is a continuous function of *model states* $x_i(t)$

$$\mathbb{E}\left[N(t)\frac{\mathcal{P}(T)}{N(T)}\middle|\,\mathcal{F}_t\right] = f(t; x_i(t))$$

For example, in a hybrid cross-currency setup, the model states can be short rates and FX-rate.

Below we assume the possibility of CE calculation embedded in the Monte Carlo (MC) simulation. A typical numerical method of CE calculation is a *regression* to state variables[2]

$$\mathbb{E}\left[N(t)\frac{\mathcal{P}(T)}{N(T)}\middle|\mathcal{F}_t\right] \to \sum_j \nu_j\ \beta_j(x_1(t), x_2(t), \cdots)$$

where ν_j are regression coefficients best fitted in the least-squares sense, and $\beta_j(x_1(t), x_2(t), \cdots)$ are the basis functions (their choice is the key art of the method).

2.1 General instrument backward pricing

A general instrument is defined as a payment stream of amounts $\mathcal{A}_j$ paid at τ_j for $j = 1, \cdots, M$. The amounts are expressed in domestic currency units. They are linked with financial indexes (e.g. Libor or CMS) and, eventually, obey to exercise/barrier conditions (i.e. being "path-dependent").

The instrument today's price is a PV (discounted expectation) of the payments

$$V = \mathbb{E}\left[\sum_{j=1}^{M} \frac{\mathcal{A}_j}{N(\tau_j)}\right]$$

Introduce now the instrument *exposure* at certain observation date t_{obs} as a conditional expectation of future payments w.r.t. the observation date

$$v = \mathbb{E}\left[N(t_{obs}) \sum_{j=1,\tau_j>t_{obs}}^{M} \frac{\mathcal{A}_j}{N(\tau_j)}\middle|\mathcal{F}_{t_{obs}}\right] \tag{1}$$

Here and below we use the same letter for the price and exposure but capitalize the price notation. Note also absence of time argument in the PV and exposure notation.

We suppose that the instrument can be calculated using the backward pricing procedure also known as pricing script. The first main object of the backward procedure we want to introduce is *Continuation Value* (CV). It is related with financial products such as legs, options etc. expressed in the domestic currency units. We denote CV's as $U_j(t)$ with index running over different products. A CV $U_j(t)$ is a certain function of the model state variables at t. Note also that a unit payment in some currency $X_c(t)$ is itself a CV, $U_{X_c}(t) = X_c(t)$.

The second main object is a dimensionless *index* which is obtained from the CV's by two ways

- Ratio (Libor, CMS, FX-rate)

[2]Note that for path-dependent pay-offs the model space is augmented, and the regression functions should cover the additional dependencies. For example, for asian options where a pay-off depends on a time average $a(t) = \int_0^t dt' x(t')$ of a state x the regression function will be $\beta_j(x(t), a(t))$.

- Indicator (barrier, exercise)

Obviously, an index is a certain function of the model state variables for corresponding time.

The backward pricing procedure consist of manipulations with the CV's on instruments dates (including payment, exercise and trigger dates) and their propagation (a discounted conditional expectation) between the dates. To distinguish these two stages, we use notation $U_j(t)$ for the CV right after the propagation to the date t and $\tilde{U}_j(T)$ as a result of manipulations of CV's on the date t.

Introduce also a CV $\tilde{U}_j(T)$ to distinguish it formally from the CV $U_j(T)$: on the same instrument date T the latter does not include a CV *update*. All such updates including payments and exercises on an instrument date T can be reduced to the following currency preserving linear combination

$$\tilde{U}_k(T) = \sum_m \gamma_{m,k} U_m(T) \tag{2}$$

where $\gamma_{m,k}$ are dimensionless *indexes* (e.g. numbers, Libors or barrier/exercise indicators).

The simplest example of such combinations is a swap CV as a difference of fixed and floating legs. Another example is an addition of a payment of index α_n in some currency c_n

$$\tilde{U}_n(T) = U_n(T) + \alpha_n X_{c_n}(T)$$

Note that a seeming "non-linearity" may come from the CV dependent indicators. For example, an optimal exercise rule

$$\tilde{U}_k(T) = \max(U_m(T), U_n(T)) \tag{3}$$

can be expressed though a linear CV combination with the exercise indicators

$$\tilde{U}_k(T) = U_m(T)\, 1_{U_m(T)-U_n(T)} + U_n(T)\, 1_{U_n(T)-U_m(T)} \tag{4}$$

The discounted conditional expectation between two instrument dates ($t < T$) reads

$$U_n(t) = \mathbb{E}\left[N(t)\, \frac{\tilde{U}_n(T)}{N(T)} \middle| \mathcal{F}_t \right] \tag{5}$$

As mentioned above, the common numerical method to compute the conditional expectation is regression. Note that an *updated* CV $\tilde{U}_n(T)$ is propagated into a *non-updated* one $U_n(t)$ on a previous instrument date. One can also interpolate the CV between the instrument dates t and T: for a date τ, $t < \tau < T$, we simply define the CV as

$$U_n(\tau) = \mathbb{E}\left[N(\tau)\, \frac{\tilde{U}_n(T)}{N(T)} \middle| \mathcal{F}_\tau \right]$$

Thus, the instrument pricing consists of iterative update procedure followed by the discounted conditional expectation. The instrument value is computed

as today's value of one of underlyings associated with the instrument in hand (for definiteness, we consider $U_0(t) = V(t)$).

Below we consider examples of application of backward pricing procedure to a barrier option, swap, Bermudan option and Autocap. Note that a natural way of pricing of a barrier option and a swap is a *Forward* procedure. However, we cannot use it for the exposure calculation without analytics or other methods giving conditional expectations.

2.2 Instrument examples

2.2.1 Barrier instrument example

Consider a toy barrier instrument paying one unit of currency c at maturity T_M provided that on barrier dates $T_i,\ i = 1, \cdots, M$, a rate L_i was above a barrier level B. The rate index L_i is supposed to be a function of model states at T_j.

The payment

$$X_c(T_M) \prod_{i=1}^{M} 1_{L_i > B}$$

at T_M gives rise to the PV

$$V = \mathbb{E}\left[X_c(T_M) \frac{\prod_{i=1}^{M} 1_{L_i > B}}{N(T_M)} \right] \tag{6}$$

In spite of its forward pricing structure, the instrument can be priced backward. Let us denote by $V(t)$ the instrument continuation value (CV) and additionally introduce a CV $\tilde{V}(t)$, an updated version of the CV $V(t)$.

The backward propagation is initialized on the payment date by

$$V(T_M) = X_c(T_M)$$

Then, between two neighboring instrument dates $T_{j-1}, T_j \in \{T_1, \cdots T_M\}$ we repeat the following iterations

- Update CV's on the barrier dates T_j by the indicator[3] multiplication

$$\tilde{V}(T_j) = V(T_j)\, 1_{L_j > B} \tag{7}$$

- Calculate discounted conditional expectation at T_{j-1}

$$V(T_{j-1}) = \mathbb{E}\left[N(T_{j-1})\, \frac{\tilde{V}(T_j)}{N(T_j)} \middle| \mathcal{F}_t \right] \tag{8}$$

Note that the update (7) is a special case of the general one (2).

The option PV can be obviously obtained as $V = V(0)$.

The instrument *exposure* for some observation date $t_{obs} < T_M$ is by definition (1) a discounted conditional expectation

$$v = \mathbb{E}\left[N(t_{obs})\, \frac{X_c(T_M) \prod_{i=1}^{M} 1_{L_i > B}}{N(T_M)} \middle| \mathcal{F}_{t_{obs}} \right] \tag{9}$$

[3]The indicator $1_{L_j > B}$ is a dimensionless *index* in our "slang".

Splitting the product into two parts before and after the observation date we obtain

$$v = \prod_{i=1, T_i \le t_{obs}}^{M} 1_{L_i > B} \, \mathbb{E}\left[N(t_{obs}) \, \frac{X_c(T_M) \prod_{i=1, T_i > t_{obs}}^{M} 1_{L_i > B}}{N(T_M)} \middle| \mathcal{F}_{t_{obs}} \right]$$

or

$$v = V(t_{obs}) \prod_{i=1, T_i \le t_{obs}}^{M} 1_{L_i > B} \tag{10}$$

where

$$V(t) = \mathbb{E}\left[N(t_{obs}) \, \frac{X_c(T_M) \prod_{i=1, T_i > t}^{M} 1_{L_i > B}}{N(T_M)} \middle| \mathcal{F}_t \right] \tag{11}$$

is the instrument CV calculated using the procedure above (7-8).

It is easy to see that the instrument exposure is "path-dependent".

As mentioned before, the barrier instrument price can be naturally calculated using the Forward Monte Carlo as far as it is presented in terms of a simple average (6). However, its exposure contains conditional averages and cannot be handled using pure Forward Monte Carlo but needs analytics or Backward Monte Carlo.

2.2.2 Swap example

Consider a swap paying a simple non path-dependent *index* α_j in some currency c_j at a date τ_j for $j = 1, \cdots, M$. The instrument today's price is a PV (discounted expectation) of the payments

$$S = \mathbb{E}\left[\sum_{j=1}^{M} \frac{\alpha_j \, X_{c_j}(\tau_j)}{N(\tau_j)} \right]$$

Let us refer to a swap CV including the payments after time t as $S(t)$. Introduce also CV $\tilde{S}(t)$, an updated version of the CV $S(t)$ on the payment dates

Initializing $S(T_M) = 0$ we perform the following backward iterative procedure procedure

- Update swap CV at the payment dates τ_k adding payment of index[4]

$$\tilde{S}(\tau_k) = S(\tau_k) + \alpha_k \, X_{c_k}(\tau_k)$$

- Calculate discounted conditional expectation between the payment dates

$$S(\tau_{k-1}) = \mathbb{E}\left[N(\tau_{k-1}) \frac{\tilde{S}(\tau_k)}{N(\tau_k)} \middle| \mathcal{F}_{\tau_{k-1}} \right]$$

[4] As mentioned above the FX-rate is naturally considered a CV expressed in domestic currency units.

The swap PV can be obviously obtained as $S = S(0)$. It is easy to see that the swap CV can be expressed via

$$S(t) = \sum_{j,\tau_j > t} \mathbb{E}\left[N(t)\, \frac{\alpha_j\, X_{c_j}(\tau_j)}{N(\tau_j)} \middle| \mathcal{F}_t \right] \tag{12}$$

The swap *exposure* s_{obs} for some observation date t is by definition (1) a conditional expectation

$$s = \mathbb{E}\left[N(t_{obs}) \sum_{j=1,\, \tau_j > t_{obs}}^{M} \frac{\alpha_j\, X_{c_j}(\tau_j)}{N(\tau_j)} \middle| \mathcal{F}_{t_{obs}} \right]$$

which obviously coincides with the swap CV $S(t)$ at time t_{obs}.

2.2.3 Swaption example

Consider a Bermudan swaption giving a right to enter into the swap defined in the previous subsection on exercise dates T_i. The swaption PV is a discounted non-conditional expectation of the payments subjected to the *exercise conditions*

$$V = \mathbb{E}\left[\sum_{j=1}^{M} \mathcal{I}(\tau_j) \frac{\alpha_j\, X_{c_j}(\tau_j)}{N(\tau_j)} \right]$$

where indicator $\mathcal{I}(\tau_j)$ equals to one if we have entered into the swap *before* the payment date τ_j and zero otherwise.

The backward pricing procedure permits calculating the PV and the exercise indicators. Let us denote by $V(t)$ the swaption CV. Introduce also the instrument days $\mathcal{T} = \{\tau_1, \tau_2, \cdots\} \bigcup \{T_1, T_2, \cdots\}$, a union of exercise dates T_j and payment dates τ_k. We also introduce the CV $\tilde{V}(t)$, an updated version of the swaption CV. The backward pricing is performed by repeating the steps

- Update CV's on the instrument dates

$$\tilde{V}(T_j) = \max(V(T_j), S(T_j)) \text{ on the exercise dates } T_j \tag{13}$$

$$\tilde{S}(\tau_k) = S(\tau_k) + \alpha_k\, X_{c_k}(\tau_k) \text{ on the payment dates } \tau_k \tag{14}$$

- Calculate discounted conditional expectation between the neighbor instrument dates

$$V(t) = \mathbb{E}\left[N(t)\, \frac{\tilde{V}(T)}{N(T)} \middle| \mathcal{F}_t \right], \quad S(t) = \mathbb{E}\left[N(t)\, \frac{\tilde{S}(T)}{N(T)} \middle| \mathcal{F}_t \right] \tag{15}$$

  for $t, T \in \mathcal{T}$.

We remind that the formulas (15) also define the CV's for arbitrary times t falling, for example, between the instrument dates.

The backward pricing procedure (13-15) delivers the swaption today's price $V(0)$ along with the exercise indicators. A *conditional exercise indicator* at time T_j

$$\mathcal{C}_j = 1_{S(T_j) > V(T_j)}$$

reflects a trader optimal choice between entering to the swap or continuing with the swaption provided that the swaption was not exercised before. The (unconditional) exercise indicator $\mathcal{I}(t)$ can be constructed from the conditional ones

$$\mathcal{I}(t) = \mathcal{I}_j \quad \text{for} \;\; T_j \leq t < T_{j+1}$$

where

$$\mathcal{I}_j = 1 - \prod_{i=1}^{j} (1 - \mathcal{C}_i) \tag{16}$$

It equals to one if we entered to the swap before or on T_j and zero otherwise. In other words, a fact that the indicator $\mathcal{I}_j = 1$ means that at least one of conditional exercise indicators $\mathcal{C}_i$ for $i = 1, \cdots j$ was one (obviously, we have a right to exercise only once).

The swaption *exposure* for some observation date t_{obs} is by definition (1) a discounted conditional expectation

$$v = \mathbb{E}\left[N(t_{obs}) \sum_{j=1,\, \tau_j > t_{obs}}^{M} \mathcal{I}(\tau_j) \frac{\alpha_j \, X_{c_j}(\tau_j)}{N(\tau_j)} \middle| \mathcal{F}_{t_{obs}} \right] \tag{17}$$

which can be split in two parts

$$v = V_{co}(t_{obs}) \, (1 - \mathcal{I}(t_{obs})) + V_{ex}(t_{obs}) \, \mathcal{I}(t_{obs})$$

including continuation part $V_{co}(t_{obs})$ (the option was not exercised till t_{obs}) and exercise part $V_{ex}(t_{obs})$ (the option was exercised before or on t_{obs}). We can intuitively identify these parts with the swaption CV and the swap one (12)

$$v = V(t_{obs}) \, (1 - \mathcal{I}(t_{obs})) + S(t_{obs}) \, \mathcal{I}(t_{obs}) \tag{18}$$

A formal proof of it is based on the indicators decomposition

$$\mathcal{I}_k = \left(1 - \prod_{i=n+1}^{k} (1 - \mathcal{C}_i) \right) (1 - \mathcal{I}_n) + \mathcal{I}_n \quad \text{for} \quad k > n$$

Indeed, substituting it into the exposure definition (17) for $t_{obs} = T_n$ we obtain

$$\begin{aligned} v \;\; &= \;\; \mathbb{E}\left[N(T_n) \sum_{j=1, \tau_j > T_n}^{M} \mathcal{I}(T_n, \tau_j) \frac{\alpha_j \, X_{c_j}(\tau_j)}{N(\tau_j)} \middle| \mathcal{F}_{T_n} \right] (1 - \mathcal{I}_n) \\ &+ \;\; \mathbb{E}\left[N(T_n) \sum_{j=1, \tau_j > T_n}^{M} \frac{\alpha_j \, X_{c_j}(\tau_j)}{N(\tau_j)} \middle| \mathcal{F}_{T_n} \right] \mathcal{I}_n \end{aligned}$$

The first term containing a global exercise indicator conditional to no-exercise before T_n

$$\mathcal{I}(T_n, \tau) = 1 - \prod_{i=n+1}^{k} (1 - \mathcal{C}_i) \;\; \text{for} \;\; T_k \leq \tau < T_{k+1}$$

obviously coincides with the swaption CV $V(T_n)$ or "forward started" Bermudan swaption future price at T_n. Comparing the second term with the swap CV (12) concludes the proof.

Note that the exposure is a path-dependent product (for callable deals). For example, the continuation part $V(t_{obs})\,(1-\mathcal{I}(t_{obs}))$ consists of two contributions. The first one is the continuation value $V(t_{obs})$, i.e. a continuous function of the model states at the observation date. The second one is the indicator $\mathcal{I}(t_{obs})$ which is path-dependent, a combination of indicators for exercise dates before the observation.

2.2.4 Autocap example

Autocap is a cap with a limited number of exercises. Namely, we have a right to enter N times into floating-fixed exchanges in domestic currency on a set of dates T_i. Each payment index at T_{i+1} is defined as

$$L_i - K \tag{19}$$

where L_i is a Libor fixed at time T_i. Thus, the payment as seen at time T_i equals to

$$H(T_i) = (L_i - K)\, P_d(T_i, T_{i+1}) \tag{20}$$

where $P_d(T_i, T_{i+1})$ is a zero bond in the domestic currency. For transparency we limit ourselves to two exercises[5] $N = 2$.

The Autocap pricing is done via a backward induction. Denote two continuation values (CV): the first one corresponds to the product with *one* exercise $V^{(1)}$, while the second one $V^{(2)}$ is a CV of the product with *two* exercises (the instrument to evaluate). As usual, we denote the corresponding CV's with tilde after updates.

The backward pricing is performed by repeating the steps

- Update CV's on the dates T_j

$$\tilde{V}^{(1)}(T_j) = \max(V^{(1)}(T_j), H(T_j)) \tag{21}$$

$$\tilde{V}^{(2)}(T_j) = \max(V^{(2)}(T_j), H(T_j) + V^{(1)}(T_j)) \tag{22}$$

- Calculate discounted conditional expectation between T_{j-1} and T_j

$$V^{(n)}(T_{j-1}) = \mathbb{E}\left[N(T_{j-1})\, \frac{\tilde{V}^{(n)}(T_j)}{N(T_j)} \middle| \mathcal{F}_{T_{j-1}} \right] \quad \text{for} \quad n = 1, 2 \tag{23}$$

Note that the floating-fixed payment (20) should be treated as a CV: we multiply the CV associated with the zero bond expressed in domestic currency by dimensionless index, Libor minus fixed rate.

The pricing logic is transparent: having two-exercise product we choose to stay with it or exercise into the floating-fixed payment and one-exercise

[5] See Antonov A. (2004) "Generic Instrument Pricing", Numerix working paper, for any number of exercises.

instrument (22). If we have already exercised once, we have $V^{(1)}$ and can exercise into the floating-fixed payment (21).

The iterative procedure which is initialized on T_M by $V^{(n)}(T_M) = 0$ for $n = 1, 2$ gives the Autocap PV at origin $V = V^{(2)}(0)$.

Introduce now conditional exercise indicators. The indicator $\mathcal{C}_i^{(2)}$ equals to one if we exercise at T_i provided that we have not exercised before T_i (and zero otherwise). Analogously, $\mathcal{C}_i^{(1)}$ equals to one if we exercise at T_i provided that we have already exercised once before T_i (and zero otherwise). Obviously, the indicators can be obtained from the CV's

$$\mathcal{C}_j^{(2)} = 1_{V^{(2)}(T_j) < H(T_j) + V^{(1)}(T_j)} \tag{24}$$

$$\mathcal{C}_j^{(1)} = 1_{V^{(1)}(T_j) < H(T_j)} \tag{25}$$

Using the above indicators, the update rules (21-22) can be rewritten as

$$\tilde{V}^{(1)}(T_j) = V^{(1)}(T_j)\,\bar{\mathcal{C}}_j^{(1)} + H(T_j)\,\mathcal{C}_j^{(1)} \tag{26}$$

$$\tilde{V}^{(2)}(T_j) = V^{(2)}(T_j)\,\bar{\mathcal{C}}_j^{(2)} + \Big(H(T_j) + V^{(1)}(T_j)\Big)\,\mathcal{C}_j^{(2)} \tag{27}$$

Before explaining the Autocap exposure we will address unconditional exercise indicators. Denote $\mathcal{I}_k^{(2)}$ an indicator that there was no exercise up to T_k, inclusively (i.e. it rests two exercises after T_k). Similarly, $\mathcal{I}_k^{(1)}$ is indicator of only one exercise up to T_k, inclusively (i.e. it rests one exercise after T_k). Finally, we call $\mathcal{E}_k$ indicator of an exercise at T_k (the first or the second).

These global (unconditional) exercise indicators can be obtained using the conditional ones (24-25). Indeed, it is easy to see that no-exercise indicator up to T_k equals to a product of conditional no-exercises

$$\mathcal{I}_k^{(2)} = \prod_{i=1}^{k} \bar{\mathcal{C}}_i^{(2)} \tag{28}$$

where $\bar{\mathcal{C}}_i^{(2)} = 1 - \mathcal{C}_i^{(2)}$. The indicator of one exercise $\mathcal{I}_k^{(1)}$ is naturally composed from one conditional exercise $\mathcal{C}^{(2)}$ (when we have right for two exercises) and no-exercise after that

$$\mathcal{I}_k^{(1)} = \sum_{j=1}^{k} \left(\prod_{i=1}^{j-1} \bar{\mathcal{C}}_i^{(2)}\right) \mathcal{C}_j^{(2)} \left(\prod_{i=j+1}^{k} \bar{\mathcal{C}}_i^{(1)}\right) \tag{29}$$

Finally, the exercise indicator which equals to one if we exercise at T_k is

$$\mathcal{E}_k = \mathcal{C}_k^{(2)} \prod_{p=1}^{k-1} \bar{\mathcal{C}}_p^{(2)} + \sum_{j=1}^{k-1} \left(\prod_{i=1}^{j-1} \bar{\mathcal{C}}_i^{(2)}\right) \mathcal{C}_j^{(2)} \left(\prod_{i=j+1}^{k-1} \bar{\mathcal{C}}_i^{(1)}\right) \mathcal{C}_k^{(1)} \tag{30}$$

Its first part corresponds to the first exercise at T_k while the second one indicates the second exercise.

Proceed now with the exposure[6] calculation at observation date $t_{obs} = T_k$. It is intuitively clear that the exposure is composed of the CV with two exercises $V^{(2)}(T_k)$ provided that we have right to do it ($\mathcal{I}_k^{(2)} = 1$), plus the CV with one exercise $V^{(1)}(T_k)$ provided that we have exercised once before ($\mathcal{I}_k^{(1)} = 1$), plus the floating-fixed payment at T_{k+1} (seen at T_k as $H(T_k)$) if we have exercised exactly on T_k, i.e.

$$v = \mathcal{I}_k^{(2)} V^{(2)}(T_k) + \mathcal{I}_k^{(1)} V^{(1)}(T_k) + \mathcal{E}_k H(T_k) \tag{31}$$

A formal proof of this result is based on a cumbersome but straightforward indicators logic, analogous to the Bermudan single-exercise case from the previous section.

3 The exposure calculation

In this section we proceed with exposure calculation for an arbitrary instrument which price can be written using the backward induction. As we have seen in the Section 2.2.2, an exposure of simple swaps coincides with its continuation value. However, the presence of exercise (callable or barrier instruments) requires more efforts for exposure calculations. Below we will analyze exotic instrument exposure treatment distinguishing two approaches

- Direct
  Calculate all components in the backward pricing procedure and assemble them afterwards, by a forward pass
- Algorithmic
  Change continuation values operations to obtain the exposure as by-product of the pricing procedure

Note that a good quality conditional expectation (regression) algorithm is essential.

The direct approach algorithm includes the following steps:

- Backward step
  Calculate by backward induction and store CV's (e.g. swap, the option) and conditional exercise indicators
- Forward step
  Calculate unconditional exercise indicators, for example, using (16) for the bermudan swaption and more complicated logic (28-30) for the Autocap
- Results aggregation
  Substitute the obtained components (CV's and indicators) into final formula, for example, (18) for the bermudan swaption and more complicated aggregation (31) for the Autocap

[6] Recall that according to our convention we do not include into the exposure an eventual floating-fixed exchange paid at T_k.

The direct approach for the single-exercise instruments was first proposed in Cesari et al. where authors have treated a callable deal as a pair of payment legs before and after the exercise. Encapsulating single-exercise deals permitted to come up with a universal computation of their exposures following the three steps above.

However, the direct exposure calculation approach proposed in Cesari et al. does not treat multi-exercise instruments, for example a call with a barrier trigger or multi-call features. Moreover, it relies on the hierarchical code structure encapsulating before- and after-exercise payment legs, leaving apart stand-alone pricing scripts also presented in financial institutions. For such scripts, the direct exposure calculation approach requires substantial modifications, including both backward and forward steps with a cumbersome logic of exercise indicators calculation and aggregation. It can be very complicated for exotic instruments with different types of exercises like callable instruments with automatic triggers.

Below we propose an alternative, an *algorithmic approach* to the exposure calculation performed in parallel with the pricing. We suppose that the backward pricing procedure includes both essential objects of the "scripting language": continuation values bearing currency units and dimensionless indexes. The idea is to add an extra entity (which we will call Exposure Value) to each Continuation Value. Then, each time we modify the Continuation Value, its Exposure counterpart is also modified according to simple algorithmic rules, giving at origin the product exposure.

Let us describe now the general procedure of the Algorithmic Exposure Calculation. We denote by $v(t)$ *Exposure Values* using small letters to distinguish them from the usual *Continuation Values* $V(t)$ denoted by capital letters. As usual, we write $\tilde{v}(t)$ for updates Exposure Values.

We assume a complex instrument containing multiple CV's $U_m(T)$ (legs, swaps, options etc. in different currencies) and possibly multiple exercise conditions (we have described various operations with the CV's in the Sect. 2.1). We recall that the CV's manipulations are currency unit preserving operations[7].

Let us fix an observation date t_{obs}. To calculate exposures u_m for all products, we define the corresponding Exposure Values (EV's) $u_m(T)$ and their updated versions after payment, exercise or another combination $\tilde{u}_m(T)$. The EV's calculated at zero $u_n(0)$ are closely related with the exposure u_n. Let us stress a difference between the product price (being a number) and the exposure (being a stochastic variable).

Along with the CV's (expressed in the domestic currency units and having a continuation sense) the pricing procedure also deals with *dimensionless indexes*. These objects are obtained as a ratio of two CV's U_n/U_m, their comparison $1_{U_n>U_m}$ or other dimensionless combinations. In any case the indexes do *not* have an exposure counterpart.

The EV's are initialized at the observation date to be equal to the corresponding CV's

$$u_m(t_{obs}) = U_m(t_{obs}) \tag{32}$$

[7]For example, the "script" cannot contain quadratic operations with CV's.

The EV's are kept "attached" to the CV's during the further backward propagation effectively forming a pair $\{U_m(t), u_m(t)\}$ for $t < t_{obs}$.

Now, the unit preserving backward propagation rules (2) and (5) should algorithmicly[8] induce the rules for the EV's for $t < t_{obs}$

- Linear update rules of the CV's lead to the same relation with the EV's

$$\begin{gathered}\tilde{U}_k(T) = \sum_m \gamma_{m,k} U_m(T) \\ \Downarrow \\ \tilde{u}_k(T) = \sum_m \gamma_{m,k} u_m(T)\end{gathered} \tag{33}$$

  where $\gamma_{m,k}$ are indexes (for example, numbers or barrier exercise indicators)

- No conditional expectation (regression) for the EV's

$$\begin{gathered}U_n(t) = \mathbb{E}\left[N(t)\, \frac{\tilde{U}_n(T)}{N(T)} \middle| \mathcal{F}_t \right] \\ \Downarrow \\ u_n(t) = N(t) \frac{\tilde{u}_n(T)}{N(T)}\end{gathered} \tag{34}$$

This way, the exposure is finalized once the backward procedure reached the origin

$$u_m = u_m(0)\, N(t_{obs})$$

The algorithmic exposure calculations can be adapted to any backward *pricing* script provided that it clearly distinguishes between the dimensionless indexes and the CV's bearing domestic currency units.

Consider a few practical examples. The first one is related with with optimal exercise update (3) which is equivalent to its "linearized" version (4). Thus, the maximum update rule of the CV's induces the following one for the EV's

$$\begin{gathered}\tilde{U}_k(T) = \max(U_m(T), U_n(T)) \\ \Downarrow \\ \tilde{u}_k(T) = u_m(T)\, 1_{U_m(T) > U_n(T)} + u_n(T)\, 1_{U_m(T) \leq U_n(T)}\end{gathered} \tag{35}$$

As another example, consider a payment of an index α in currency c at some date T applied to a leg U at time $t < t_{obs}$

$$\tilde{U}(t) = U(t) + \alpha\, P_c(t, T)$$

[8]The EV rules can be encapsulated inside the CV manipulation operations.

where $P_c(t,T)$ is a currency c zero bond converted to domestic currency. A creation of the zero bond CV $U_{zb}(t) = P_c(t,T)$ in the r.h.s. will automatically initialize its EV u_{zb} according to (32)

$$u_{zb}(t) = \begin{cases} P_c(t_{obs},T), & T > t_{obs} \\ 0, & T \leq t_{obs} \end{cases}$$

The further manipulations with the EV's in the r.h.s. will lead to

$$\begin{gathered} \tilde{U}(t) = U(t) + \alpha\, P_c(t,T) \\ \Downarrow \\ \tilde{u}(t) = \begin{cases} u(t) + \alpha P_c(t_{obs},T), & T > t_{obs} \\ u(t), & T \leq t_{obs} \end{cases} \end{gathered} \tag{36}$$

As mentioned above, the dimensionless indexes and the CV's (expressed in domestic currency units) should not be mixed but reflect the nature of the real payments. For illustration set $T > t_{obs}$. Then, in the purely pricing script we can formally replace the zero bond $P_c(t,T)$ as the *dimensionless index* paid at t in domestic currency. This replace will not change the leg price but will erroneously set the exposure to zero (the payment was formally treated as being done before the observation date).

As the last example, consider the payment date $T = t$. The general rule (36) gives

$$\begin{gathered} \tilde{U}(t) = U(t) + \alpha\, X_c(t) \\ \Downarrow \\ \tilde{u}(t) = u(t) \end{gathered} \tag{37}$$

where we replace the zero bond in its maturity by the corresponding exchange rate. This means that payments done before the observation do not change the EV and can be ignored.

3.1 Algorithmic exposure calculation: numerical performance

A few words about the method performance in numerical calculations. The computational effort is split between Monte Carlo simulations of model rates and conditional expectation calculation using the Least-Square Monte Carlo. Quite regularly, the latter is much more time consuming than the former. Indeed, it implies intensive calculus with a large number of basis functions, heavy linear algebra and possible nonlinear search procedures. Thus, the main CPU consumption is related with the Least-Square Monte Carlo. The numerical overhead of the algorithmic exposure procedure w.r.t. the pricing is limited to EV's arithmetic but not with the conditional averages computation which is the most CPU consuming part of the numerics. So we can state that our exposure calculation does not significantly affect the computational effort w.r.t. the usual backward pricing.

3.2 Algorithmic exposure calculation: examples

In this Section we propose several examples to illustrate the algorithmic exposure calculation method.

3.2.1 Barrier instrument example.

Following the general logic we introduce the barrier instrument Exposure Value $v(t)$ in addition to Continuation Value $V(t)$ which appeared during the pricing. In parallel with the main pricing manipulations (7-8) one should execute the following operations with the Exposure Value (EV).

We initialize the EV on the observation date $v(t_{obs}) = V(t_{obs})$. For dates *before* the observation date $t < t_{obs}$ the update and roll rules for the EV follow these of the CV according to the general recipe (33-34) with an indicator as dimensionless index. Putting all together, we

- The update rules on barrier dates T_j
  Pricing

$$\tilde{V}(T_j) = V(T_j)\, 1_{L_j > B} \tag{38}$$

  Exposure

$$\tilde{v}(T_j) = v(T_j)\, 1_{L_j > B} \tag{39}$$

- Roll procedure between the dates
  Pricing

$$V(t) = \mathbb{E}\left[N(t) \frac{\tilde{V}(T)}{N(T)} \middle| \mathcal{F}_t \right]$$

  Exposure

$$v(t) = N(t) \frac{\tilde{v}(T)}{N(T)} \tag{40}$$

  for $t, T \in \{T_1, \cdots, T_M\}$. In other words, we *skip* the conditional expectation for the exposure calculus.

Finally the exposure equals to

$$v = v(0)\, N(t_{obs}) \tag{41}$$

To see that this expression coincides with the previously obtained one (10) we notice that $v(t_{obs}) = V(t_{obs})$ on the observation date. Then, combining (39) with (40) on two barrier neighbor dates we obtain a recursion

$$\frac{v(T_{j-1})}{N(T_{j-1})} = \frac{v(T_j)}{N(T_j)}\, 1_{L_j > B} \tag{42}$$

which leads to

$$\frac{v(0)}{N(0)} = \frac{v(t_{obs})}{N(t_{obs})} \prod_{j=1, T_j \le t_{obs}}^{M} 1_{L_j > B} \tag{43}$$

and ends the proof.

3.2.2 Swaption example.

In this section we apply the algorithmic calculation method to the swaption exposure and prove that it coincides with the theoretical one (18).

Introduce Exposure Values (EV) for swaption $v(t)$ and for swap $s(t)$. We initialize the EV's on the observation date $v(t_{obs}) = V(t_{obs})$ and $s(t_{obs}) = S(t_{obs})$. For dates *before* the observation date $t < t_{obs}$ the update and roll rules for the EV follow these of the CV according to the general recipe (33-34), or, in particular, (35) and (37)

- The update rules on exercise dates T_j
  Pricing

$$\tilde{V}(T_j) = \max(V(T_j), S(T_j))$$

  Exposure

$$\tilde{v}(T_j) = v(T_j)\, 1_{S(T_j) \leq V(T_j)} + s(T_j)\, 1_{S(T_j) > V(T_j)}$$

  or

$$\tilde{v}(T_j) \quad = \quad v(T_j)\,(1 - \mathcal{C}_j) + s(T_j)\,\mathcal{C}_j \tag{44}$$

  for the exercise indicator $\mathcal{C}_j = 1_{S(T_j) > V(T_j)}$.

- Update rules on payment dates τ_k
  Pricing

$$\tilde{S}(\tau_k) = S(\tau_k) + \alpha_k\, X_{c_k}(\tau_k)$$

  Exposure

$$\tilde{s}(\tau_k) = s(\tau_k) \tag{45}$$

- Roll procedure between the dates
  Pricing

$$V(t) \quad = \quad \mathbb{E}\left[N(t) \frac{\tilde{V}(T)}{N(T)} \middle| \mathcal{F}_t \right] \tag{46}$$

$$S(t) \quad = \quad \mathbb{E}\left[N(t) \frac{\tilde{S}(T)}{N(T)} \middle| \mathcal{F}_t \right] \tag{47}$$

  Exposure

$$v(t) \quad = \quad N(t) \frac{\tilde{v}(T)}{N(T)} \tag{48}$$

$$s(t) \quad = \quad N(t) \frac{\tilde{s}(T)}{N(T)} \tag{49}$$

  for $t, T \in \mathcal{T}$. In other words we *skip* the conditional expectation for the exposure calculus.

Finally the exposure equals to

$$v = v(0)\, N(t_{obs}) \tag{50}$$

To understand the reason one can compare the option PV

$$V = \mathbb{E}\left[\sum_{j=1}^{M} \frac{\mathcal{I}(\tau_j)\alpha_j\, X_{c_j}(\tau_j)}{N(\tau_j)}\right]$$

with its exposure

$$v(0) = \mathbb{E}\left[N(t_{obs}) \sum_{j=1\ \tau_j > t_{obs}}^{M} \frac{\mathcal{I}(\tau_j)\,\alpha_j\, X_{c_j}(\tau_j)}{N(\tau_j)}\middle| \mathcal{F}_{t_{obs}}\right]$$

Apart from the numeraire the only differences are the *conditional* expectation of the exposure and absence of the past payments w.r.t. the observation date. That is why we have stopped the regression in the exposure calculation before the observation date along with the payment updates.

Let us prove explicitly that the algorithmic exposure (50) coincides with the previously obtained one (18). Given the swap EV procedures (45) and (49) we have

$$\frac{s(t)}{N(t)} = \frac{s(t_{obs})}{N(t_{obs})} \quad \text{for} \;\; t \le t_{obs}$$

The swaption EV update on the exercise dates (44) combined with its roll rule (48) leads to a recursion

$$\frac{v(T_{j-1})}{N(T_{j-1})} = \frac{v(T_j)}{N(T_j)}\,(1-\mathcal{C}_j) + \frac{s(t_{obs})}{N(t_{obs})}\,\mathcal{C}_j$$

for the terminal value $v(t_{obs}) = V(t_{obs})$. This linear recursion can be easily solved to obtain the desired expression for the exposure

$$v = v(0)\, N(t_{obs}) = V(t_{obs})\,(1-\mathcal{I}(t_{obs})) + S(t_{obs})\,\mathcal{I}(t_{obs})$$

Note that a similar recursion logic was applied in Egloff et al (2007) and Andersen-Piterbarg (2010) Sect 18.3.4. in a context of pricing of Callable Libor Exotics. Our contribution is that we explicitly target the exposure and also generalize it for arbitrarily complex instrument.

3.2.3 Autocap example

In this section we apply the algorithmic calculation method to the Autocap exposure and prove that it coincides with the theoretical one (31).

For observation date $t_{obs} = T_k$ introduce Exposure Values (EV) for the underlyings products $v^{(n)}(T_j)$ and $\tilde{v}^{(n)}(T_j)$ for $n = 1, 2$. and initialize them to the corresponding CV's

$$v^{(n)}(T_k) = V^{(n)}(T_k) \quad \text{and} \quad \tilde{v}^{(n)}(T_k) = \tilde{V}^{(n)}(T_k) \tag{51}$$

For dates *before* the observation date $t < t_{obs}$ the update and roll rules for the EV follow these of the CV according to the general recipe (33-34)

- Update on the dates T_j for $j = 1, \cdots, k$
  Pricing

$$\begin{aligned}\tilde{V}^{(1)}(T_j) &= V^{(1)}(T_j)\,\bar{\mathcal{C}}_j^{(1)} + H(T_j)\,\mathcal{C}_j^{(1)} \\ \tilde{V}^{(2)}(T_j) &= V^{(2)}(T_j)\,\bar{\mathcal{C}}_j^{(2)} + \Big(H(T_j) + V^{(1)}(T_j)\Big)\,\mathcal{C}_j^{(2)}\end{aligned}$$

  Exposure

$$\tilde{v}^{(1)}(T_j) = v^{(1)}(T_j)\,\bar{\mathcal{C}}_j^{(1)} + \delta_{jk}\,H(T_k)\,\mathcal{C}_j^{(1)} \tag{52}$$

$$\tilde{v}^{(2)}(T_j) = v^{(2)}(T_j)\,\bar{\mathcal{C}}_j^{(2)} + \Big(\delta_{jk}\,H(T_k) + v^{(1)}(T_j)\Big)\,\mathcal{C}_j^{(2)} \tag{53}$$

  where the Kronecker delta-symbol $\delta_{kj} = 1_{k=j}$ in front of the floating-fixed payment $H(T_k)$ reflects zero EV of the CV $H(T_j)$ for $j < k$.

- Evolution between the instrument dates
  Pricing

$$V^{(n)}(T_{j-1}) = \mathbb{E}\left[N(T_{j-1})\,\frac{\tilde{V}^{(n)}(T_j)}{N(T_j)}\,\middle|\,\mathcal{F}_{T_{j-1}}\right] \quad \text{for} \quad n = 1, 2$$

  Exposure

$$\frac{v^{(n)}(T_{j-1})}{N(T_{j-1})} = \frac{\tilde{v}^{(n)}(T_j)}{N(T_j)} \quad \text{for} \quad n = 1, 2 \tag{54}$$

Finally the Autocap exposure equals to

$$v = v^{(2)}(0)\,N(T_k) \tag{55}$$

Denoting the discounted values with subscribes, $v_j^{(n)} \equiv \frac{v^{(n)}(T_j)}{N(T_j)}$ and $h_k = \frac{H(T_k)}{N(T_k)}$, and combining (52-53) with (54) we obtain a simple recursion

$$v_{j-1}^{(1)} = v_j^{(1)}\,\bar{\mathcal{C}}_j^{(1)} + \delta_{jk}\,h_k\,\mathcal{C}_k^{(1)} \tag{56}$$

$$v_{j-1}^{(2)} = v_j^{(2)}\,\bar{\mathcal{C}}_j^{(2)} + \Big(\delta_{jk}\,h_k + v_j^{(1)}\Big)\,\mathcal{C}_j^{(2)} \tag{57}$$

This system of linear difference equations can be easily solved[9] given the terminal values $v_k^{(n)}$, $n = 1, 2$. Indeed, substituting into

$$v_0^{(2)} = v_k^{(2)} \prod_{m=1}^{k} \bar{\mathcal{C}}_m^{(2)} + \sum_{i=1}^{k} \Big(\delta_{ik}\,h_k + v_i^{(1)}\Big)\,\mathcal{C}_i^{(2)} \prod_{p=1}^{i-1} \bar{\mathcal{C}}_p^{(2)} \tag{58}$$

the solution of (56)

$$v_i^{(1)} = v_k^{(1)} \prod_{m=i+1}^{k} \bar{\mathcal{C}}_m^{(1)} + h_k\,\mathcal{C}_k^{(1)} \prod_{p=i+1}^{k-1} \bar{\mathcal{C}}_p^{(1)} \tag{59}$$

[9] We used a fact that $x_i = x_k \prod_{m=i+1}^{k} \alpha_m + \sum_{n=i+1}^{k} \beta_n \prod_{p=i+1}^{n-1} \alpha_p$ is the solution of a linear difference equation $x_{j-1} = \alpha_j\,x_j + \beta_j$.

we obtain

$$
\begin{aligned}
v_0^{(2)} &= v_k^{(2)} \prod_{m=1}^{k} \bar{\mathcal{C}}_m^{(2)} + v_k^{(1)} \sum_{i=1}^{k} \prod_{p=1}^{i-1} \bar{\mathcal{C}}_p^{(2)} \mathcal{C}_i^{(2)} \prod_{m=i+1}^{k} \bar{\mathcal{C}}_m^{(1)} \\
&+ h_k \left\{ \mathcal{C}_k^{(1)} \sum_{i=1}^{k-1} \prod_{p=1}^{i-1} \bar{\mathcal{C}}_p^{(2)} \mathcal{C}_i^{(2)} \prod_{p=i+1}^{k-1} \bar{\mathcal{C}}_p^{(1)} + \mathcal{C}_k^{(2)} \prod_{p=1}^{k-1} \bar{\mathcal{C}}_p^{(2)} \right\}
\end{aligned}
$$

Comparing it with (29-30) we conclude that

$$
\frac{v^{(2)}(0)}{N(0)} = \frac{v^{(2)}(T_k)}{N(T_k)} \mathcal{I}_k^{(2)} + \frac{v^{(1)}(T_k)}{N(T_k)} \mathcal{I}_k^{(1)} + \frac{H(T_k)}{N(T_k)} \mathcal{E}_k
$$

which permits to identify the algorithmic Autocap exposure (55) with its "manual" (theoretical) value (31).

4 CVA

Consider now the Credit Valuation Adjustment (CVA) calculation[10], see also Cesari et al (2010). We assume the Counterparty credit model be a single-name one with a stochastic intensity. Namely, denote the Counterparty default time by τ. Then the *survival* process $1_{t<\tau}$ is a Poisson process with some stochastic intensity (hazard rate) process $h(t)$, see Jeanblanc et al (2009). for more details. The initial pricing model containing arbitrage-free evolution of rates, equities and other market observables is augmented with the hazard rate. The resulting model filtration, i.e. information about the market observables and the hazard rate, is denoted as $\mathcal{F}_t$. Information about the jumps in the survival process is contained in filtration $\mathcal{D}_t$. Finally, we introduce the whole filtration $\mathcal{G}_t = \mathcal{D}_t \cup \mathcal{F}_t$ associated with all the information about the market and the Counterparty survival process. An average over it, $E\left[\cdots \,|\, \mathcal{G}_t\right]$, corresponds to averaging over both jumps and Brownian motions of the rates and the hazard rate.

The main property of the survival process is

$$
E\left[1_{T<\tau} \,|\, \mathcal{G}_t\right] = 1_{t<\tau}\, E\left[e^{-\int_t^T \mathrm{d}u\, h(u)} \,|\, \mathcal{F}_t\right] \tag{60}
$$

We adopt the "discount factor" notations for the exponential of the hazard rate integral

$$
H(t) = e^{-\int_0^t \mathrm{d}u\, h(u)} \tag{61}
$$

and refer to it as hazard discount factor.

Imagine we receive a payment of X at time T, provided that there were no defaults until T. The payment X is assumed to be dependent of the market observable and the hazard rate up to time T, i.e. being measurable w.r.t. the filtration $\mathcal{F}_t$. Then, its PV seen at t reads

$$
N(t)\, E\left[1_{T<\tau} \frac{X}{N(T)} \,|\, \mathcal{G}_t\right] = 1_{t<\tau}\, N(t)\; E\left[e^{-\int_t^T \mathrm{d}u\, h(u)} \frac{X}{N(T)} \,|\, \mathcal{F}_t\right]. \tag{62}
$$

[10] Our portfolio does not have an explicit default risk.

This is the main pricing tool we will use to evaluate default dependent payments.

Proceed now to the CVA calculus. As mentioned before, we should extend our initial pricing model with hazard rate process calibrated to the corresponding credit market. Note that the hazard rate evolution can be correlated with the other market observables. After simulation of the extended model we calculate the portfolio future values for a fine set of dates $\Pi(t)$ using the algorithm above. Note that we do not include the credit hazard rates into the regression variables.

Assume also that we have calculated the collateral, $C(t)$, which consists of a certain number $\Phi(t)$ of units of some asset $A(t)$ with known (simulated) evolution,

$$C(t) = \Phi(t)\, A(t).$$

The Self ("our") exposure at time t is

$$\mathcal{O}(t) = (\Pi(t) - C(t))^+$$

leading to the CVA expression

$$\mathrm{CVA} = (1 - \mathrm{RR}_C) \int_0^T \mathbb{E}\left[-\mathrm{d}1_{t<\tau} \frac{\mathcal{O}(t)}{N(t)} \right]$$

where RR_C is the Counterparty recovery rate which is assumed to be constant. Indeed, if the Counterparty defaults ($\mathrm{d}1_{t<\tau} = -1$) on the interval $[t, t+\mathrm{d}t]$, our loss will be equal to $(\Pi(t) - C(t))^+$, and this infinitesimal payment, as seen at the origin, is equal to the discounted expectation.

Averaging over jumps (independent from the hazard rate)

$$\begin{aligned} \mathrm{CVA} &= (1 - \mathrm{RR}_C) \int_0^T \mathbb{E}\left[-\mathrm{d}H(t)\, \frac{\mathcal{O}(t)}{N(t)} \right] && (63) \\ &= (1 - \mathrm{RR}_C) \int_0^T \mathrm{d}t\, \mathbb{E}\left[h(t)\, H(t)\, \frac{\mathcal{O}(t)}{N(t)} \right] && (64) \end{aligned}$$

reduces to a effective replacement of the survival processes by the hazard "discount factor"

$$H(t) = e^{-\int_0^t \mathrm{d}u\, h(u)}$$

Our extended model, containing the market observables and the hazard rate process equipped with the Least Squares MC, is able to compute the above averages. The future portfolio value is calculated as above (the credit hazard rates are not included into the regression variables). The collateral and the hazard discount factor are simulated and the integral element

$$\mathbb{E}\left[h(t)\, H(t)\, \frac{\mathcal{O}(t)}{N(t)} \right]$$

is estimated by averaging.

In a similar way, one can calculate the Debt Valuation Adjustment (DVA) and Bilateral CVA can be computed, see, for example [7] and [10].

5 Risk

The risk measure is a general term for statistical characteristics of the instrument exposure, i.e. *non-discounting* (simple) averages

$$\mathbb{E}[f(\mathcal{O}(t))]$$

A simple average is *measure-dependent*, contrary to the discounted one

$$\mathbb{E}\left[f(\mathcal{O}(t))/N(t)\right]$$

which is *measure-independent*. Here are several examples for the risk-measures

- Potential Future Exposure (PFE) for a confidence level α

$$q_\alpha(t) = \inf\{x : \mathbb{E}[1_{\mathcal{O}(t)<x}] \geq \alpha\}$$

- Expected Shortfall or Expected Tail Loss

$$\mathbb{E}[\mathcal{O}(t) \,|\, \mathcal{O}(t) > q_\alpha(t)]$$

- Expected Positive Exposure (EPE)

$$\mathbb{E}[\mathcal{O}(t)^+]$$

5.1 Right/Wrong way exposure

When the model evolution is highly correlated with the default process the exposure distribution *conditional* to the default is important, i.e. instead of the CDF of the exposure

$$CDF(t,x) = \mathbb{E}[1_{\mathcal{O}(t)<x}]$$

we need a *conditional* CDF to default happening at time t

$$CDF_D(t,x) = \mathbb{E}[1_{\mathcal{O}(t)<x} \mid \tau = t]$$

where τ is a stochastic default time.

Following our CVA considerations we have

$$CDF_D(t,x) = \frac{\mathbb{E}[1_{\mathcal{O}(t)<x}\, \mathrm{d}1_{t<\tau}]}{\mathbb{E}[\mathrm{d}1_{t<\tau}]} = \frac{\mathbb{E}[1_{\mathcal{O}(t)<x}\, h(t)\, H(t)]}{\mathbb{E}[h(t)\, H(t)]}$$

where $h(t)$ is the stochastic hazard rate and $H(t) = e^{-\int_0^t \mathrm{d}u\, h(u)}$ is the hazard "discount factor".

5.2 Real world measure

A model calibrated to today's market still has an extra degree of freedom – *the model probability measure*. Usually, we have to fix this measure to be the *real-world* one for the exposure distribution. The real-world (or physical) probability measure appeared early in Quantitative Finance has not been widely used due to its loose definition, contrary to the *risk-neutral measure*.

The risk-neutral approach assumes that tradable securities have drift coinciding with the short rate $r(t)$. For example, a zero-bond SDE reads

$$\mathrm{d}P(t,T) = P(t,T)\left(r(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t)\right)$$

In the "real world" this drift is supposed to be different but its time-series estimations are vague. We need to link our model to the real-world in a more rigorous way by requiring that some non-discounting (simple) averages hold.

Suppose that we have a set of indices I_j fixed at some times t_j and their *projections* in the future p_j

$$p_j = \mathbb{E}_{RW}[I_j]$$

as expectation in the real-world (RW) measure. Our initial arbitrage-free model in its risk-neutral measure cannot return a priori such averages

$$p_j \neq \mathbb{E}[I_j]$$

Thus we should modify the model measure in order to meet the projections. Note that the model is calibrated to the implied market (initial rates, implied volatilities, etc.) and the only freedom to reproduce the projection is in the measure choice.

To introduce the real-world measure we apply here the cross-currency analogy. Namely, set our initial model as *foreign* one w.r.t. some FX-rate process and a domestic currency model. Then, our initial model states will get a drift adjustment depending on FX vol and correlations. As a result the initial model in such cross-currency (CC) setup will have a different measure w.r.t. to its risk-neutral one.

To realize the idea we consider a domestic model with a factitious currency "RW" being a trivial model with zero rates and unit numeraire $N_{RW}(t) = 1$. Then we set an FX-rate model between the initial currency and the factitious one to be Black-Scholes (BS) one. Finally a foreign model (initial currency) will coincide with initial model.

The resulting CC model calibration is organized as follows. First, the foreign component (initial model) is calibrated to its implied market. Then, the free parameters, the FX-vol and correlations, are calibrated to meet the indexes projections. This way the calibrated resulting CC model can price the initial instrument (identical to the initial pricing up to numerical errors) and perform the exposure simulation in the real-world measure.

Note that the BS process for the FX-rate makes a deterministic drift change for the foreign (initial) model. More complicated FX-rates evolution, e.g. Heston model, gives a more rich family of the measure change and provides more degrees of freedom for the calibration to real-world projections.

6 Numerical experiments

This section is devoted to numerical analysis of the proposed framework. We consider a 10Y cancelable swap: we receive semi-annually a 6M Libor and pay annually a fixed rate (= 2.57%, a swap rate at origin) on 1 EUR notional and we have a right to cancel the swap annually from 4Y. The numerical output is the following.

- 6M Libors expectation in different measures
- distribution (CDF) of 6Y instrument exposure (including the future payments only) in different measures
- exposure profile in the risk-neutral measure
- PFE 97.5% in different measures

The CC Model includes the following components

- Domestic model (RW currency)
  Trivial model with zero interest rates
- Foreign model (EUR)
  Hull-White IR model with 3% rate, 5% mean-reversion and 1.5% volatility
- FX rate (RW currency/EUR)
  BS model with correlation with HW Brownian motion $\rho = -100\%$ and a set of FX-volatilities: $0\%, 25\%, 50\%$

Note that different FX-volatilities correspond to different measures and the zero FX-volatility gives the risk-neutral measure. For simplicity we consider the Counterparty default *uncorrelated* with the CC model factors.

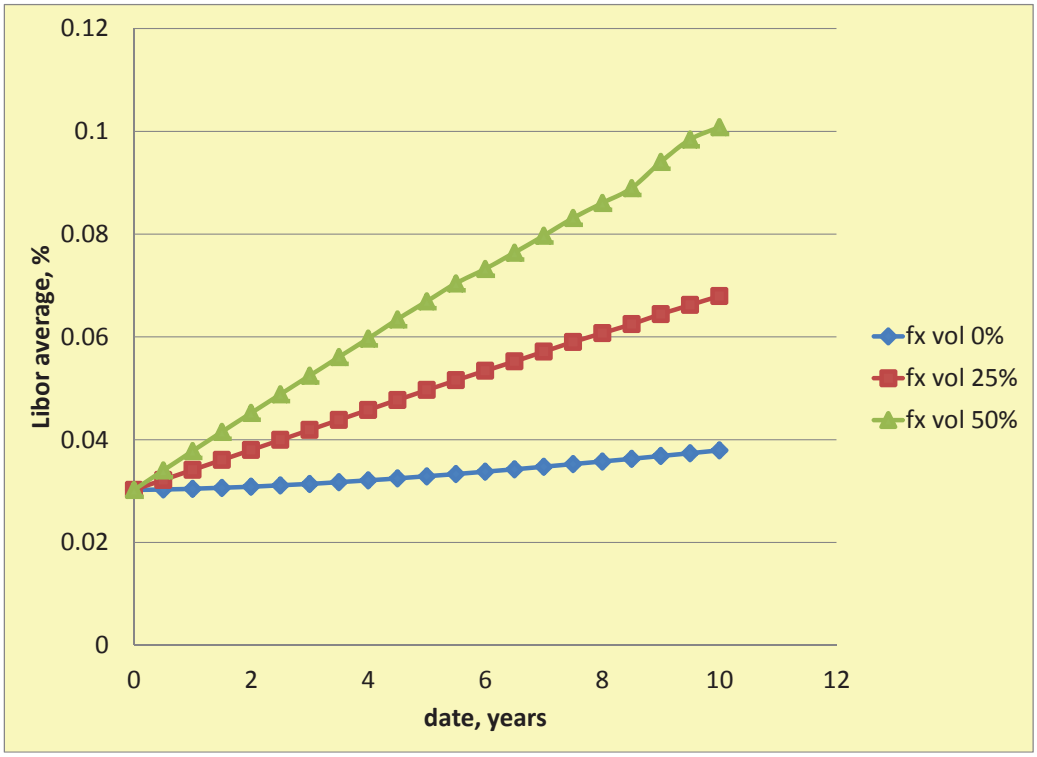

Figure 1: 6M Libor averages for different FX-vols (different measures).

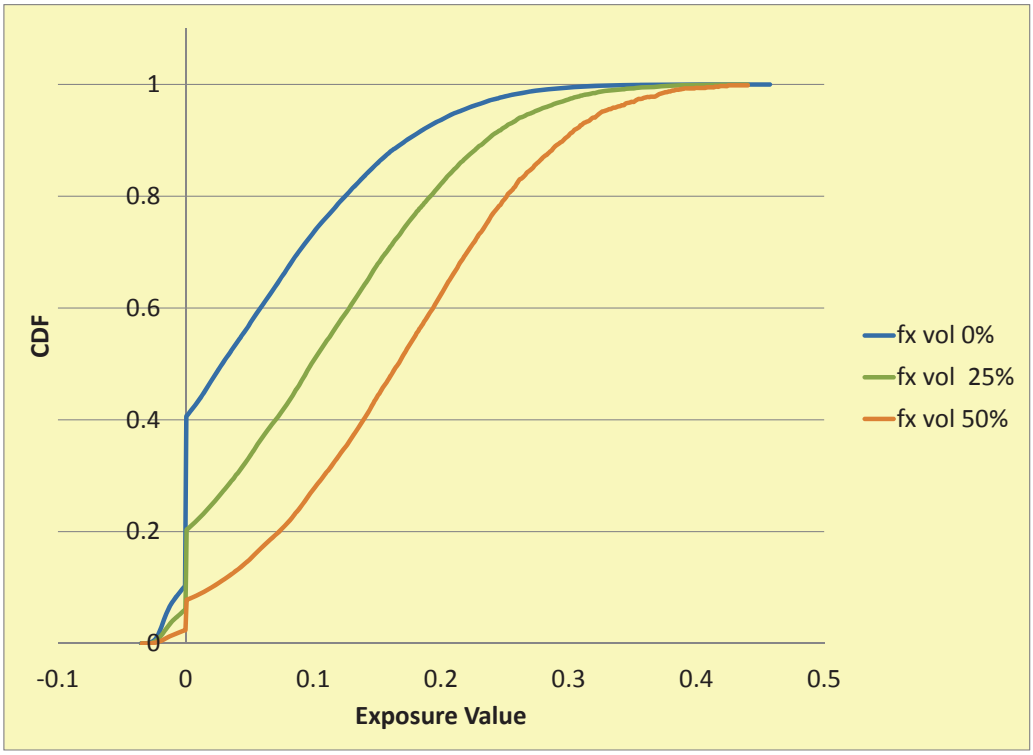

Figure 2: CDF of 6Y exposure for different FX-vols (different measures).

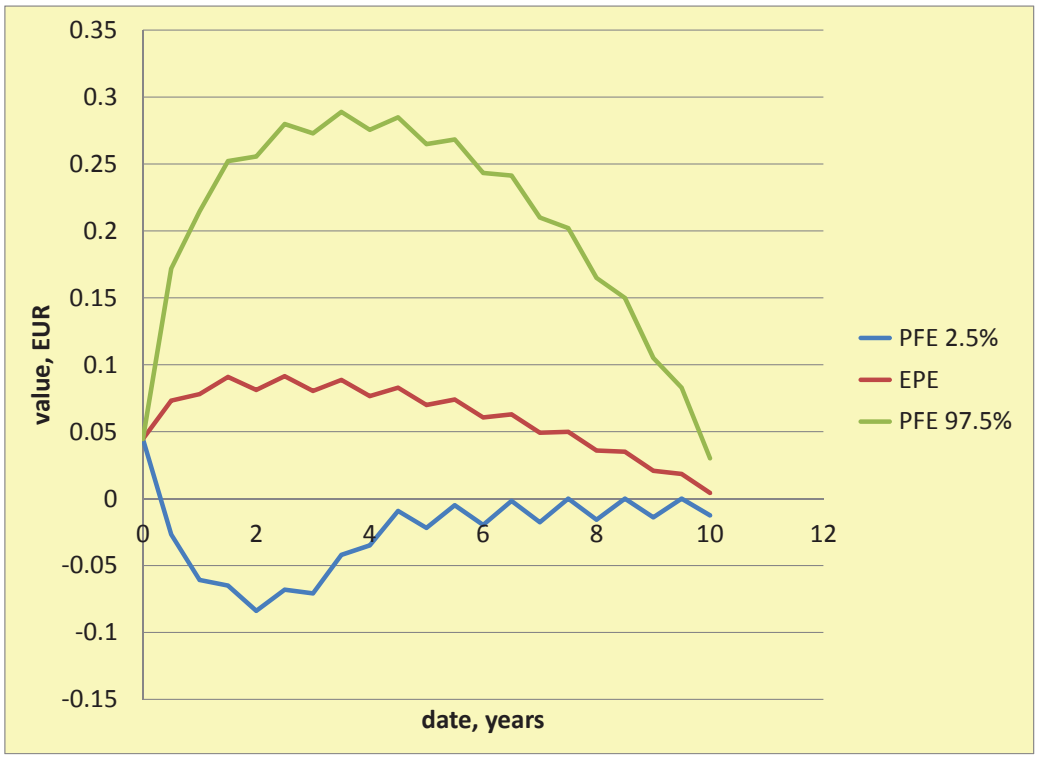

Figure 3: Risk profile in the risk-neutral measure.

Figure 4: PFE 97.5% for different FX-vols (different measures).

On the presented graphs we can observe typical shapes of the risk profile. We see that the measure change leads to a positive drift in rates: the bigger FX-vol, the bigger Libor average. We notice significant differences in the exposure distributions in different measures. Thus it is important to use the pricing model in the real-world measure (the factitious CC model should be calibrated to the rates projections) to get the correct exposure distribution.

7 Conclusion

In this article we presented efficient calculations of the portfolio exposure in a self-consistent way using arbitrage-free model calibrated to both implied market and real-world projections. We proposed a new algorithmic method of exposure calculations especially attractive for exotic portfolios avoiding cumbersome exercise aggregation. The new method permits efficient CVA calculation using the simulated information.

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