Model Validation for Interest Rate Models

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Introduction

What is a model?

Two types of financial assets:

- Liquid assets whose price is directly observable in the market.
  - Futures, e.g. commodities (oil, gas, etc.) and indices (SPX)
  - Interest rate swaps
- Instruments for which the price is not directly observable in the market
  - Financial derivatives - products related to the performance of simpler assets
  - Complex financial products, e.g.

Pricing the second type of products requires a model

Models express derivatives prices in terms of simpler liquid instruments, possibly also introducing unobservable parameters

Examples

1. *Local volatility model*. Calibrate to vanilla options on a given stock $S(t)$ with all strikes and maturities - liquid observable instruments. The resulting model can price any European payoff

   $$\text{Payoff} = f(S(t))$$

   The set of all vanilla calls and puts gives the terminal distribution of the asset (e.g. stock price) at any given maturity t

2. *Pricing CMS products by replication*. Calibrate the model to prices of European swaptions with all strikes at a given maturity T and tenor τ.

   The model can price any payoff depending on the swap rate $S(T, \tau)$, e.g. CMS swaps, CMS caps/floors.

Model risk

Model risk: assume that several models can be calibrated such that they price perfectly a set of liquid instruments, but produce different prices for the same exotic product. The spread of the prices among different models is a measure of the model risk.

Is there a unique "correct" model? If not, how do we measure model risk?

Model risk

In July 2009 the Basel Committee on Banking Supervision mandated [3] that financial institutions quantify model risk.

Two types of model risk should be taken into account:

- The model risk associated with using a possibly incorrect valuation
- The risk associated with using unobservable calibration parameters

How to quantify the model risk associated with a possibly incorrect valuation?

More on model risk

How is the model used?

- Just valuation, or are we also going to use the model for hedging? The latter case requires stable and accurate greeks.
- Does the model capture the correct dynamics? For example, European options require only the terminal distribution, which can be obtained from vanilla options.
- More complex dynamics may be needed. For example, a forward starting option (e.g. cliquet for equities) must describe the correct joint smile-forward dynamics.

Do we use the correct input data? I will assume this as given.

Regulatory mandates: OCC Bulletin 2011-12

Supervisory Guidance on Model Risk Management

1. Model risk should be managed like other types of risk: banks should identify the sources of that risk, assess its magnitude, and establish a framework for managing the risk.

2. Banks should objectively assess model risk using a sound model validation process, including evaluation of conceptual soundness, ongoing monitoring, and outcomes analysis.

3. A central principle for managing model risk is the need for "effective challenge" of models: critical analysis by objective, informed parties who can identify model limitations and assumptions and produce appropriate change.

Types of Model Validation

1. Examine the theoretical assumptions of the model and check soundness of model, e.g. absence of arbitrage.

2. Test run the front office implementation:

- Test pricing under different market conditions (stress-testing)
- Under what conditions does it calibrate?
- How stable are the prices and Greeks?

3. Check the performance of the model in the front office implementation:

- Backtesting of hedging performance under historical market conditions
- Compare performance of the model against alternative models used for same product

4. Replicate the model under a different simulation method (Monte Carlo) and check pricing and calibration

5. Compare against limiting cases where exact or approximative solutions are available.

Model Validation for Interest Rate Products

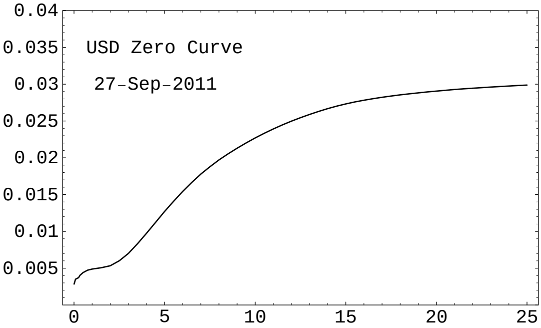
- Brief introduction to interest rate modeling
- Types of interest rate derivatives
- Model validation for interest rates derivatives
- New issues post-2008

Types of interest rates

- Treasury rates: rates implied by the US government bonds and T-bills
- Inter-bank rates: LIBOR (USD, GBP), Euribor (EUR)
  "The rate at which an individual Contributor Panel bank could borrow funds, were it to do so by asking for and then accepting inter-bank offers in reasonable market size, just prior to 11:00 London time."
- Swap rates. Effective rate used for a periodic payment over a longer time (2Y-30Y)
- Unsecured overnight lending rates:
  - Federal Funds rate (USD),
  - *EONIA* (Euro OverNight Index Average),
  - *SONIA* (Sterling OverNight Index Average),
  - *SARON* (Swiss Average Rate Overnight),
  - *Mutan* (same for JPY)

The yield curve

Naively, all these rates should be obtainable from a common yield curve. This could be defined for example as the zero rate curve $R(t)$.

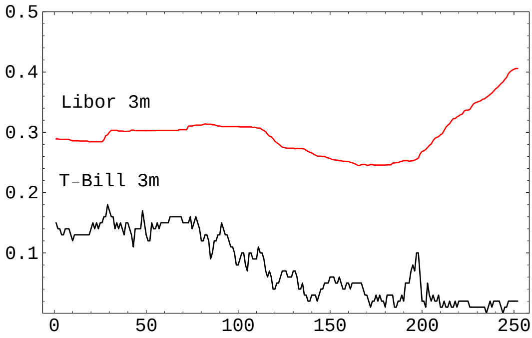


Example

Treasury zero curve $R(t)$ for USD as of 27-Sep-2011.

Different rates

3 month T-Bill and Libor rates.

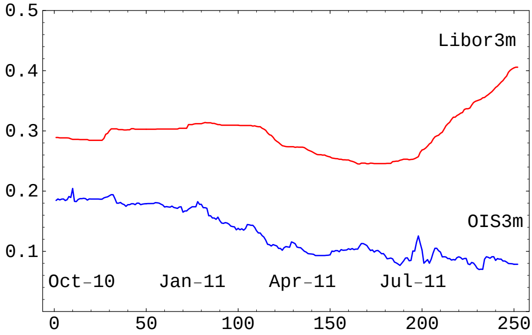


Example

Daily values between 18-Oct-2010 and 18-Oct-2011.

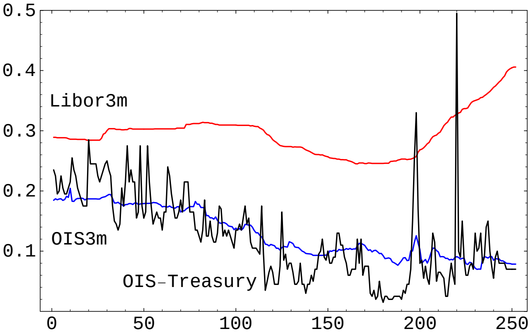
Libor-OIS spread

The Libor - OIS spread is a measure of the stress in the money markets. A high spread indicates decreased willingness to lend by major banks, while a lower spread indicates higher liquidity in the markets.



Daily values between 18-Oct-2010 and 18-Oct-2011.

Overnight rate secured by Treasuries



Example

Daily values between 18-Oct-2010 and 18-Oct-2011.

A multiplicity of curves

- All rates can be derived from a single yield curve only if all cash flows are free of default risk, and if we are guaranteed to be able to borrow at the respective rate (liquidity)
- Historically credit and liquidity issues have been considered insignificant for inter-bank lending.
- The credit crunch changed the market's perception, and banks are now more conservative about the possibility of other banks' default, and their own funding cost
- These risks are now being realized. They introduce spreads among different rates, which are not related anymore (e.g. Libor - OIS spread)
- Model validation of these curves: data sourcing, curve construction.

Interest rate modeling

- The need to hedge against changes in the interest rates contributed to the creation of interest rate derivatives.
- Corporations, banks, hedge funds can now enter into many types of contracts aiming to mitigate and/or exploit the effects of the interest rate movements
- Well-developed market. Daily turnover for[1]:
  - interest rate swaps $295bn
  - forward rate agreements $250bn
  - interest rate options (caps/floors, swaptions) $70bn
- This requires a very good understanding of the dynamics of the interest rates markets: interest rate models
- A good interest rate model should reproduce the observed dynamics of the interest rates, in a way which is compatible with the most liquid instruments

[1] The FX and IR Derivatives Markets: Turnover in the US, 2010, Federal Reserve Bank of New York

Setup and definitions

Consider a model for interest rates defined on a set of discrete dates

$$0 = t_0 < t_1 < t_2 \cdots t_{n-1} < t_n$$

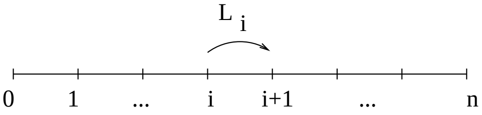
In the simplest setting, there is one yield curve at each time point, defined by the zero coupon bonds $P_{i,j}$

Definition

Zero coupon bond $P(t_i, t_j)$: price of bond paying \$1 at time t_j, as observed at time t_i

$L_i(t_i)$ = Libor rate set at time t_i for the period (t_i, t_{i+1})

$$L_i(t_i) = \frac{1}{\tau}\left(\frac{1}{P(t_i, t_{i+1})} - 1\right)$$



Interest rate products

- *Curve instruments*. Can be priced off the yield curve(s) alone
  - Forward rate agreements (FRA)
  - Vanilla swaps
- *Non-callable options*. Require models which capture the volatility
  - Caps and floors
  - CMS swaps
  - Spread options
  - Range accruals
- *Callable derivatives*: one party can exercise optionality to enter or cancel
  - Swaptions: options on swaps
  - European swaptions (one call date)
  - Bermudan (multiple call dates)

Model validation for curve products

The curve products are priced off the yield curve. Model validation should validate the curve construction. This should verify that the calibration instruments are correctly repriced.

There is a wide variety of curve construction methodologies, differing in

- Choice of instruments
- Interpolation and extrapolation techniques
- Bootstrapping methodology

The yield curve is constructed from the following instruments

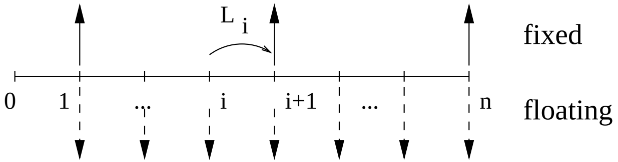
- Cash ($O/N, 1M, 3M, 6M, 12M$): floating rate reset information
- Money-market futures (e.g. 3M Eurodollar futures): Most liquid short rates
- Fixed-Floating Swaps: Most liquid long-term rates
- Basis Swaps: Long-term basis information
- FRAs and Short Swaps: Short-term basis information

Simplest interest rate product: vanilla swap

One of the most common interest rate products: *swap*

Floating leg: Pays/receives the amount $(L_i + \text{spread})\tau_i$ at times t_{i+1}, where L_i is set at times t_i. Accrual time $\tau_i = t_{i+1} - t_i$
Fixed leg: Receives/pays $K\tau_i$ where K is a fixed rate, agreed upon in advance

Pricing an interest rate swap - single curve pricing

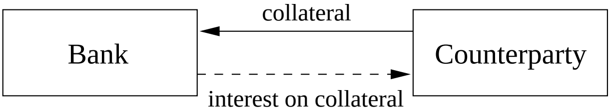


Swap Present Value $= \Sigma_{j=1}^{N} P(0, t_{j+1})(L_j^{\text{fwd}} + s)\tau_j - K\Sigma_{j=1}^{N} P(0, t_{j+1})\tau_j$

Both L_j^{fwd} and $P(0, t)$ are obtained from the same curve

Pricing under collateral agreements

After the 2008 credit crisis, banks started requesting collateral. This modifies the pricing of derivatives.



Collateral agreements are regulated by CSA (Credit Support Annex) agreements. Typically a CSA agreement will specify the following events:
1. If the net present value of the trade for Bank is positive, it posts a collateral call in the amount of the trade value *MtM*
2. Counterparty (CP) posts collateral in the amount *MtM*
3. Bank pays interest on the collateral back to the CP at the overnight cash rate.

> *The collateral rate is the greater of 0% and the Federal Funds Overnight Rate. For the purposes hereof, "Federal Funds Overnight Rate" means, for any day, an interest rate per annum equal to the rate published as the Federal Funds Effective Rate that appears on Reuters Page FEDM or on Bloomberg Page FEDL01 for such day.*

4. If counterparty default occurs, Bank takes ownership of collateral

Pricing under collateral agreements

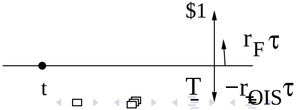
A general treatment was given by Piterbarg (2010). Here we illustrate the idea on a very simple example: fixed cash flow of \$1 paid at T.
What is the value P of this cash flow at time $t < T$?
Assume that the bank can invest cash at the funding rate r_F, and pays interest on the collateral at the OIS rate r_{OIS}

Without collateralization: the value of the cash flow must be such that we get back one dollar at time T by investing at the funding rate

$$P_{NC}(1 + r_F\tau) = 1$$
$$P_{NC} = \frac{1}{1 + r_F\tau}$$

With collateralization: the value of the cash flow is such that by investing it at the funding rate we get back the dollar, plus the funding rate accrued on the collateral, minus the interest paid back to the counterparty

$$1 + (r_F\tau)P_C - (r_{\mathrm{OIS}}\tau)P_C = (1 + r_F\tau)P_C$$
$$P_C = \frac{1}{1 + r_{\mathrm{OIS}}\tau}$$



Discounting at the OIS rate

Conclusion: In the presence of collateralization, cash flows must be discounted at the rate paid on the collateral (OIS rate), instead of the funding rate r_F

The construction of the yield curve must take this into account. A dual curve approach is used (Bianchetti (2009))

- Projection curve: determines the forward rates - Libor 1M, 3M, 6M, etc. Determined from basis swaps, e.g. a 10Y 6x3s swap exchanges Libor6M for Libor3M + spread for 10 years.
- Discounting curve: gives the discount factors. Can be determined from OIS swaps. In USD we have OIS swaps up to 10Y, and FedFunds vs Libor3M basis swaps up to 30Y. In GBP and EUR, we have OIS quoted to 30Y.

Additional complications when the collateral can be posted in multiple currencies. Multiple discount curves, must deal also with the optionality of collateral ccy.

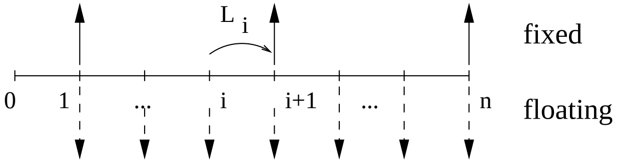
Vanilla swap pricing - post-2009

Floating leg: The Libor rate L_i is computed from the *projection* curve

$$L_i = \frac{1}{\tau_i}\left(\frac{1}{P^{\text{proj}}_{i,i+1}} - 1\right)$$

Both the floating and fixed coupons are discounted using a *common* discounting curve - constructed from the OIS swaps

Pricing an interest rate swap - dual curve pricing



Multiple curves. Bianchetti (2009)

- Projection curve: 1M, 3M, 6M, tenor-specific
- Discounting curve: single OIS curve

Interest rate options

Simplest interest rate derivatives which are sensitive to rates' volatility: *caplets and floorlets*

Caplet on the Libor L_i with strike K pays at time t_{i+1} the amount

$$\text{Pay} = \max(L_i(t_i) - K, 0)$$

Similar to a call option on the Libor L_i

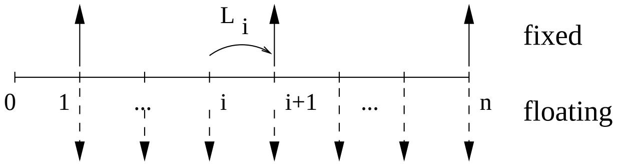
Caplet prices are parameterized in terms of caplet volatilities σ_i via the Black caplet formula

$$Caplet(K) = P_{0,i+1} C_{\mathrm{BS}}(L_i^{\mathrm{fwd}}, K, \sigma_i, t_i)$$

Analogous to the Black-Scholes formula.

Swaptions

European swaptions: can decide at time t_1 whether to enter into swap (t_1, t_n).



Example: payer swaption

$$\text{Swaption Present Value} = A^{t_1,t_n}(0)\mathbb{E}_A[(S^{t_1,t_n}(t_1) - K)_+]$$

where $A^{t_1,t_n}(0)$ is the annuity associated with the swap, and $S^{t_1,t_n}(t)$ is the swap rate, defined as

$$A^{t_1,t_n}(t) \equiv \Sigma_{j=1}^{n} P(t, t_j)\tau_j$$

$$S^{t_1,t_n}(t) = \frac{P(t, t_1) - P(t, t_n)}{A^{t_1,t_n}(t)}$$

The expectation value is taken in the annuity (swap) measure, with numeraire $A^{t_1,t_n}(t)$

Types of interest rate models

Construct an interest rate model compatible with a given yield curve $P_{0,i}$ and caplet/swaption volatilities $\sigma_i(K)$

1. *Short rate models.* Model the distribution of the short rates $L_i(t_i)$ at the setting time t_i.
   - Hull, White model - equivalent with the Linear Gaussian Model (LGM)
   - Markov Functional Models - can reproduce a given swaption/cap smile
2. *Forward rate models*: Heath-Jarrow-Morton (HJM) models
   - Generic HJM model with 1-, 2-, 3-factors.
   - Markovian HJM models: Sankarasubramanian-Ritchken, or Cheyette models
3. *Market models.* Describe the evolution of individual forward Libors $L_i(t)$
   - Libor Market Model, or the BGM model.

Short rate models

Short rate models are defined by the process for the short rate $r(t)$.

Hull-White model. Equivalent to the Linear Gaussian Model (LGM) of Hagan and Woodward (2001).
Normally distributed short rate

$$dr(t) = \sigma(t)dW(t) + (a(t) - b(t)r(t))dt$$

Used for simple products, such as Bermudan swaptions. Can be calibrated efficiently to today's curve, and caplet or swaption volatilities.

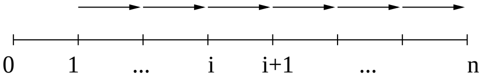
Shortcomings

- No correlation among rates of different tenors. Can be remedied by adding more factors.
- No skew control. The model has normal smile.
- Can not reproduce more than one line/column of the swaption volatility matrix

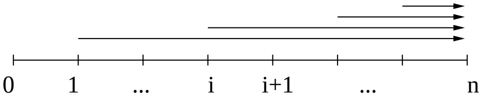
Issues with short rate models

Can only calibrate to a limited number of volatility instruments, because of a limited number of free adjustable parameters. Possible choices:

- Caplets on consecutive non-overlapping periods. Appropriate for caps/floors, instruments sensitive only to the terminal distributions of Libors on their setting dates, but not their correlations.



- Co-terminal swaptions. Appropriate for Bermudan swaptions.



- Swaption volatilities with same term, different expiries. Appropriate for CMS flows, CMS caps/floors.

Difficult to use for pricing a big portfolio, containing products of different types. Model validation should determine region of validity.

HJM type models

Fundamental variable: forward short rate $f(t, T)$

General dynamics of the forward short rate in the HJM model

$$df(t, T) = \sigma_f(t, T, f)dW(t) + \mu(t, T, f)dt$$

Used for path dependent products, such as multi-callable exotics. Can be calibrated efficiently to today's curve, and caplet or swaption volatilities.

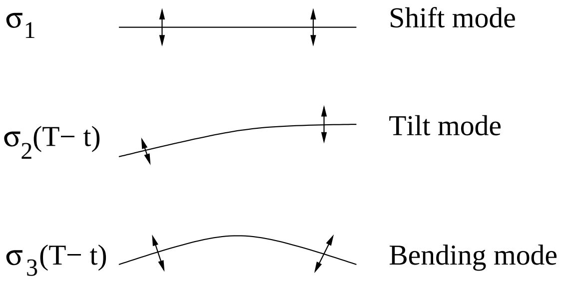
- A general HJM model is not Markovian: the future evolution of the rates depends on the past - "memory effect". The short rate $r(t)$ depends on the Markov driver $W(t)$ at all times prior to t
- Skew can be controlled through the choice of $\sigma_f(t, T, f)$
- An HJM model reduces to a short rate model (Markov) provided that the volatility factorizes $\sigma_f(t, T) = \sigma_1(t)\sigma_2(T)$

Multi-factor HJM models

More realistic models: include correlation structure among forward rates with different maturities T_i

The shape of the forward curve is driven by several Brownian drivers, each controlling a different feature of the curve

$$d_t f(t, T) = \Sigma_i \sigma_i(T - t) dW_i(t) + (\text{ drift })dt$$

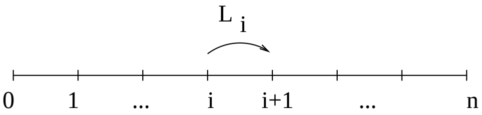


Libor market models

The forward instantaneous rates $f(t,T)$ are not directly observable in the market

Formulate the model in terms of the future Libor rates $L_i(t_i)$ = forward Libor rate for the period (t_i, t_{i+1})

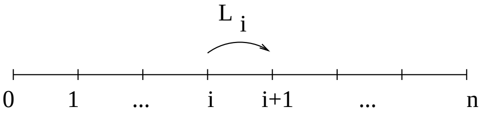
$$L_i(t) = \frac{1}{\tau}\Big(\frac{P_{t,t_i}}{P(t,t_{i+1})} - 1\Big)$$



The natural (forward) measure

Each Libor L_i has a different "natural" measure $\mathbb{P}_{i+1}$

Numeraire $= P_{t,i+1}$, the zero coupon bond maturing at time t_{i+1}



The forward Libor $L_i(t)$

$$L_i(t) = \frac{1}{\tau}\left(\frac{P_{t,i}}{P_{t,i+1}} - 1\right)$$

is a martingale in the $\mathbb{P}_{i+1}$ measure

$$L_i(0) = L_i^{\mathrm{fwd}} = \mathbb{E}[L_i(t_i)]$$

This is the analog for interest rates of the risk-neutral measure for equities

Simple Libor market model

Simplest model for the forward Libor $L_i(t)$ which is compatible with a given yield curve $P_{0,i}$ and given caplet volatilities σ_i

Log-normal diffusion for the forward Libor $L_i(t)$: each $L_i(t)$ driven by its own separate Brownian motion $W_i(t)$

$$dL_i(t) = L_i(t)\sigma_i dW_i(t)$$

with initial condition $L_i(0) = L_i^{\mathrm{fwd}}$, and $W_i(t)$ is a Brownian motion in the measure $\mathbb{P}_{i+1}$

Problem: each Libor $L_i(t)$ is described in a different measure.

We would like to describe the joint dynamics of all rates in a common measure.

Libor market model

Choosing the terminal measure with numeraire $P(t, t_n)$

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t)dW_i(t) - \Sigma_{k>i}\sigma_i(t)\sigma_k(t)\rho_{ik}\frac{L_k(t)\tau_k}{1+L_k(t)\tau_k}dt$$

The correlation matrix ρ_{ij} is usually provided as an exogenous (external) parameter.
Example: Rebonato parameterization

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty)\exp\Big(-\beta e^{-\gamma \min(t_i,t_j)}|t_i - t_j|\Big)$$

The number of factors is chosen such that main components of the correlation structure among rates are reproduced
Calibration: finding the Libor volatilities $\sigma_i(t)$, e.g. using the cascade algorithm of Brigo, Mercurio.
The LMM can calibrate to the entire swaption volatility matrix, since there are more free parameters for calibration. This is better suited as a pricing engine at portfolio level.

Practical implementation of the models

Model implementation

- How are the models used in practice?
- In a few cases we have analytical solutions
  - Linear Gaussian Model (HW): explicit formulas for zero coupon bonds
  - Small volatility approximation in the LMM: Rebonato approximation for swaption volatilities
- Generally numerical solutions are necessary
- Three main approaches:
  - Monte Carlo implementation
  - Tree implementation
  - Finite difference methods

Monte Carlo methods

Statistical approach

- Simulate paths for the Brownian motions $W_i(t)$ driving the quantities of the model, e.g. short rate $r(t)$ or the forward Libors $L_i(t)$ for the LMM
- Solve the evolution equations (SDEs) of the model for zero coupon bonds, Libors, swap rates, etc. along each path. They can be discretized using Euler scheme, predictor-corrector scheme, etc.

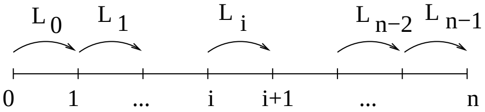
$$L_i(t+\tau) = L_i(t) + \sigma_i(t)\Delta W_i(t) + \mu_i(t)\Delta t$$

- Compute the discounted cash flows along each MC path
- Compute the present value and its standard deviation $PV = V \pm V_{\mathrm{err}}$ of the product by averaging over paths

$$V = \langle V \rangle \,, \qquad V_{\mathrm{err}} = \frac{1}{\sqrt{N-1}}\sqrt{\langle V^2 \rangle - \langle V \rangle^2} \simeq \frac{1}{\sqrt{N}}$$

Tree implementations

Example: Black-Derman-Toy model. This is a short rate model.
Describe the joint distribution of the Libors $L_i(t_i)$ at their setting times t_i



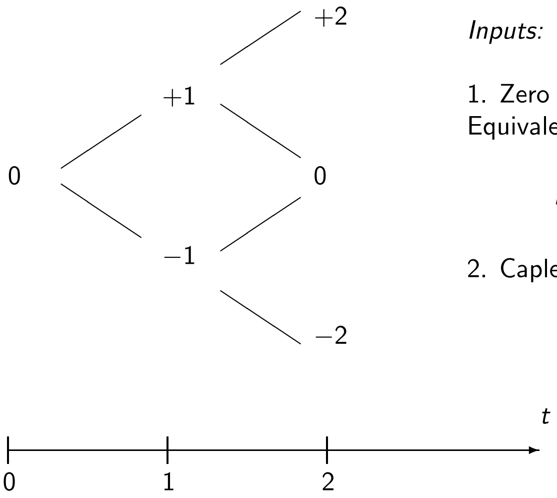
Libors (short rate) $L_i(t_i)$ are log-normally distributed

$$L_i(t_i) = \tilde{L}_i e^{\sigma_i x(t_i) - \frac{1}{2}\sigma_i^2 t_i}$$

where $\tilde{L}_i$ are constants to be determined such that the initial yield curve is correctly reproduced (calibration)

$x(t)$ is a Brownian motion. A given path for $x(t)$ describes a particular realization of the Libors $L_i(t_i)$

BDT tree - Markov driver $x(t)$



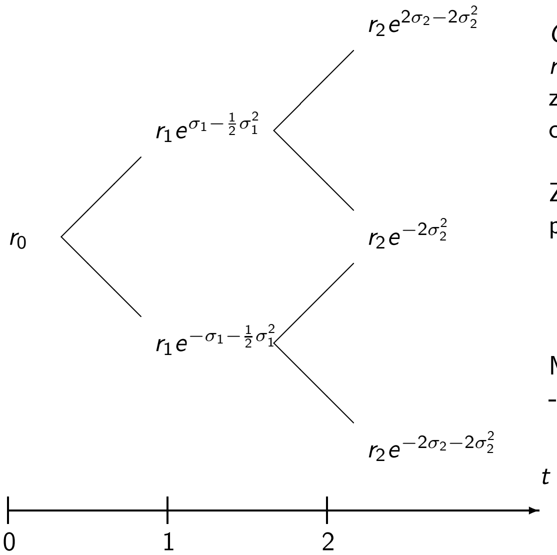
Inputs:

1. Zero coupon bonds $P_{0,i}$
Equivalent with zero rates R_i

$$P_{0,i} = \frac{1}{(1+R_i)^i}$$

2. Caplet volatilities σ_i

BDT tree - the short rate $r(t)$



Calibration: Determine $r_0, r_1, r_2, \cdots$ such that the zero coupon prices are correctly reproduced

Zero coupon bonds $P_{0,i}$ prices

$$P_{0,i} = \mathbb{E}\left[\frac{1}{B_i}\right]$$

Money market account $B(t)$ - node dependent

$$B_0 = 1$$
$$B_1 = 1 + r(1)$$
$$B_2 = (1 + r(1))(1 + r(2))\,.$$

Finite difference methods

Any stochastic differential equation is equivalent with a PDE, similar to the diffusion equation

$$dX(t) = \sigma(X(t), t)dt + \mu(X(t), t)dt$$

Knowing the distribution of X at time $t = 0$, its distribution $f(X, t)$ at a later time t is given by the solution of the Fokker-Planck (Kolmogoroff forward equation)

$$\partial_t f(X, t) = -\partial_X[\mu(X, t)f(X, t)] + \frac{1}{2}\partial_X^2[\sigma^2(X, t)f(X, t)]$$

Can be solved numerically on a grid using a variety of methods: explicit, implicit, Crank-Nicholson.

Numerical implementations of the models

- Finite difference methods
  - Fast and numerically efficient
  - Can be applied only to Markov models (all model quantities are state-dependent, but not path-dependent), with a limited number of factors. E.g. Vasicek-Hull-White models, Cheyette model, but not general HJM or LMM models
  - Numerical instabilities must be carefully avoided
- Tree methods
  - Fast and numerically efficient
  - Process specific: model changes can be difficult to accomodate. E.g. adding mean reversion to the BDT model requires re-designing the tree.
  - Can become very complex - e.g. willow trees.
- Monte Carlo methods
  - Versatile, generally applicable.
  - Computationally intensive
  - Callable features require special attention: Longstaff-Schwartz method

Model implementation for model validation

In my own experience, Monte Carlo is the most convenient implementation method for the purposes of model validation of interest rates models and products

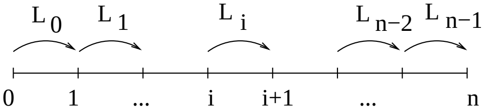
- Often encountered statement: MC can not deal with rates models where we do not have closed form results for discount factors, zero coupon bonds. Not true.
- Monte Carlo implementations are flexible and easy to modify, so as to explore possible model modifications.
- The calibration is more difficult. Work with the front office desk to expose calibration results, and validate them. Use them as inputs into the validation model, and reproduce the calibration instruments.
- Calculation speed may be slower, but this is not an issue for model validation.

Monte Carlo implementation of the BDT model

The Libors $L_i(t_i)$ are log-normally distributed in the spot measure

$$L_i(t_i) = L_i^{\text{fwd}} e^{\sigma_i(x(t_i)+\delta_i) - \frac{1}{2}\sigma_i^2 t_i}$$

where δ_i are constants to be determined such that the initial discount factors $P(0,t)$ are correctly reproduced (calibration)



Calibration conditions

$$P(0,t_i) = \mathbb{E}\left[\frac{1}{(1+L_0\tau_0)(1+L_1\tau_1)\cdots(1+L_{i-1}\tau_i)}\right], \qquad i = 1,2,\cdots,n$$

Monte Carlo implementation

1. Initialize $\delta_i = 0$ for $i = 1, \cdots, n-1$.
2. Generate N paths for the Brownian motion $x(t)$ and compute the $n \times N$ matrix of Libors $L_{ij} \equiv [L_i(t_i)]_{\mathrm{path\,j}} = L_i^{\mathrm{fwd}} e^{\sigma_i(x(t_i)+\delta_i) - \frac{1}{2}\sigma_i^2 t_i}$.
3. Compute the matrix of path-wise discount factors for maturity t_i

$$[D(t_i)]_{\mathrm{path\,j}} = \frac{1}{(1 + L_{0j}\tau_0)(1 + L_{1j}\tau_1) \cdots (1 + L_{i-1,j}\tau_{i-1})}$$

4. Compute the realized zero coupon bond $P(0, t_j)$ by averaging over the MC paths

$$P(0, t_j) = \mathbb{E}[D(t_j)]$$

starting with $j = 1$. Iterate over values of δ_j (using e.g. the bisection method) until the two sides agree. Repeat for $j = 2, 3, \cdots, n-1$. The model is now calibrated.
5. Compute all the zero coupon bonds $P(t_i, t_j)$ by regressing $D(t_j)/D(t_i)$ against $x(t_i)$, using e.g. linear regression with a polynomial basis.
6. Store the N paths for $D(t_i)$ and $P(t_i, t_j)$ and use for pricing derivatives.

Case study: Analytical calibration for a short rate model with log-normal rates

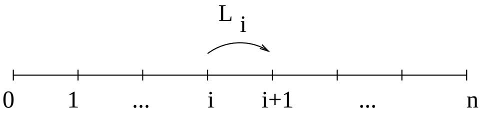
BDT model in the terminal measure

Keep the same log-normal distribution of the short rate L_i as in the BDT model, but work in the terminal measure

$$L_i(t_i) = \tilde{L}_i e^{\psi_i x(t_i) - \frac{1}{2}\psi_i^2 t_i}$$

$L_i(t_i) =$ Libor rate set at time t_i for the period (t_i, t_{i+1})

Numeraire in the terminal measure: $P_{t,n}$, the zero coupon bond maturing at the last time t_n



Why the terminal measure?

Why formulate the Libor distribution in the terminal measure?

- *Numerical convenience.* The calibration of the model is simpler than in the spot measure: no need to solve a nonlinear equation at each time step
- The model is a particular parametric realization of the so-called Market functional model (MFM), which is a short rate model aiming to reproduce exactly the caplet smile. MFM usually formulated in the terminal measure.
- More general functional distributions can be considered in the Markov functional model $L_i(t_i) = \tilde{L}_i f(x_i)$, parameterized by an arbitrary function $f(x)$. This allows more general Libor distributions.

Main results

1. The BDT model in the terminal measure can be solved analytically for the case of uniform Libor volatilities $\psi_i = \psi$ (Pirjol (2010)). Solution possible (in principle) also for arbitrary ψ_i, but messy results.

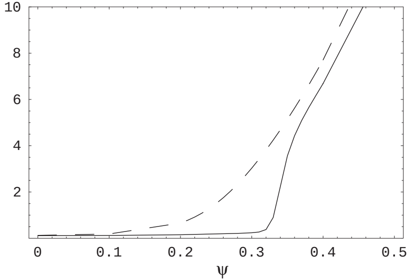
2. The analytical solution has a surprising behaviour at large volatility:
   - The convexity adjustment explodes at a critical volatility, such that the average Libors in the terminal measure (convexity-adjusted Libors) $\tilde{L}_i$ become tiny (below machine precision)
   - This is very unusual, as the convexity adjustments are "supposed" to be well-behaved (increasing) functions of volatility
   - The model has two regimes, of low and large volatility, separated by a sharp transition
   - Practical implication: the convexity-adjusted Libors $\tilde{L}_i$ become very small, below machine precision, and the simulation truncates them to zero

Explanation

The size of the convexity adjustment is given by an expectation value

$$N_i = \mathbb{E}[\hat{P}_{i,i+1} e^{\psi x - \frac{1}{2}\psi^2 t_i}]$$

Recall that the convexity-adjusted Libors are $\tilde{L}_i = L_i^{\rm fwd}/N_i$



Plot of $\log N_i$ vs the volatility ψ

Simulation with $n = 40$ quarterly time steps

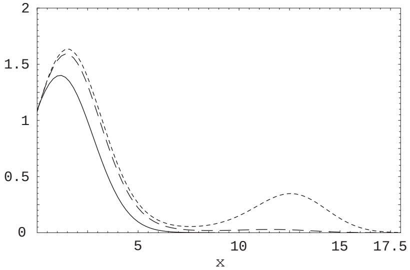
$i = 30, t = 7.5, r_0 = 5\%$

Note the sharp increase after a critical volatility $\psi_{\rm cr} \sim 0.33$

Explanation

The expectation value as integral

$$N_i = \mathbb{E}[\hat{P}_{i,i+1} e^{\psi x - \frac{1}{2}\psi^2 t_i}] = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t_i}} e^{-\frac{1}{2t_i}x^2} \hat{P}_{i,i+1}(x) e^{\psi x - \frac{1}{2}\psi^2 t_i}$$



The integrand

Simulation with $n = 20$ quarterly time steps $i = 10$, $t_i = 2.5$

$$\psi = \begin{cases} 0.4 & (solid) \\ 0.5 & (dashed) \\ 0.52 & (dotted) \end{cases}$$

Note the secondary maximum which appears for super-critical volatility at $x \sim 10\sqrt{t_i}$. This will be missed in usual simulations of the model.

Analytical study of the model was crucial for uncovering this shortcoming.

Concluding comments

- Model validation is an on-going effort. Market conditions change, model performance may be different under changed market environments.
- A good relationship with the front office group can help a lot.
- Need to be aware of the main assumptions going into the model. This may be apparent from the model description, but discussions with the modeler/FO quants will help.
- It is often useful to be aware of the front office testing of the models.

In general, model validation is a mix of several ingredients:

1. Process. The model validation must follow the regulatory guidance and mandates.
2. Science. We use mathematics, financial mathematics, statistics.
3. Art. Experience plays an important role.

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