Lecture 8. Weak Derivatives.

From now on we work on an open set $U \subset \mathbb{R}^n$.
Notation. If U and V are open in $\mathbb{R}^n$ we write

$$V \subset\subset U$$

to mean $\bar{V} \subset U$ and $\bar{V}$ is compact.

Definition. For $1 \leq p < \infty$, there are two ways to define the space $L^p(U)$. If you know measure theory, we consider the space of measurable functions $u : U \to \mathbb{R}$, and we consider elements u and v to be equivalent if $u = v$ almost everywhere, and write $[u]$ for those functions equivalent to u. Then $[u] \in L^p(U)$ if and only if

$$\int_U |u|^p \, dx \; < \; \infty.$$

We define the norm

$$(*) \qquad \|u\|_p \; = \; \left(\int_U |u|^p \, dx \right)^{1/p}.$$

For $[u] \in L^p(U)$, we say that $[u] \in C^k(U)$ if there exists $v \in [u]$ with $v \in C^k(U)$. We usually omit mention of the equivalence class.

Exercise. $C_c(U)$ is dense in $L^p(U)$.

Hint: Approximate L^p functions by step functions and approximate step functions by continuous functions.

If you don't know measure theory, then $L^p(U)$ is the completion of $C_c(U)$ in the norm (*).

Definition. U is an open subset of $\mathbb{R}^n$. Then $L^1_{\text{loc}}(U)$ is the space of functions u such that every open subset V with $V \subset\subset U$, we have $u \in L^1(V)$.

Remark. $L^p(U) \subset L^1_{\text{loc}}(U)$ for every $p \geq 1$.

Definition. The set of **test functions** on U is the space $C_c^\infty(U)$ of smooth functions $\phi : U \to \mathbb{R}$ with compact support.

Excercise. The function

$$\psi(x) \; = \; \begin{cases} e^{-1/t}, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth. Hence the function

$$\psi(r^2 - |x|^2)$$

is a smooth function supported on $\{|x| \leq r\}$, and strictly positive on $\{|x| < r\}$.

Lemma. *$C_c^\infty(U)$ is dense in $L^p(U)$.*

Proof. Given $u \in L^p(U)$ and $\varepsilon > 0$, there exists $v \in C_c(U)$ with $\|u - v\|_p < \varepsilon$. Choose a function $\phi \in C_c^\infty(\mathbb{R}^n)$ supported on $\{|x| \le 1\}$ and positive on $\{|x| < 1\}$ with

$$\int \phi \, dx = 1.$$

Set

$$\phi_\delta(x) \;=\; \frac{1}{\delta^n}\phi(x/\delta).$$

Then ϕ_δ is supported on $\{|x| \le \delta\}$ and

$$\int \phi_\delta \, dx \;=\; 1.$$

Then consider the convolution

$$v * \phi_\delta(x) \;=\; \int_U v(y)\phi_\delta(x-y)\, dy.$$

This function is in $C^\infty(\mathbb{R}^n)$, and it is supported in U when

$$2\delta \;<\; \text{dist}(\text{supp}(v)\;,\;\mathbb{R}^n \setminus U).$$

Moreover,

$$\begin{aligned} |v(x) - v * \phi_\delta(x)| \;&=\; \left| \int_U (v(x) - v(y))\phi_\delta(x-y)\, dy \right| \\ &\le\; \int_U |v(x) - v(y)|\, \phi_\delta(x-y)\, dy \;\le\; \sup_{|x-y|<\delta} |v(x) - v(y)| \;=:\; \omega(\delta). \end{aligned}$$

Since a continuous function on a compact set is uniformly continuous, $\omega(\delta) \to 0$ as $\delta \to 0$. If $\text{supp}\, v \subset B(0, R)$, then

$$\|v - v * \phi_\delta\|_p \;\le\; \omega(\delta)|B(0, R + 2\delta)|^{1/p} \;\to\; 0 \text{ as } \delta \to 0.$$

Hence we can find $v' \in C_c^\infty(U)$ with $\|u - v'\|_p < 2\varepsilon$.

Excercise. Suppose that $v \in L^1_{\text{loc}}(U)$ and

$$\int_U v\phi\, dx \;=\; 0 \qquad\qquad \text{for every } \phi \in C_c^\infty(U).$$

Then $v = 0$.

Definition. If $u, v \in L^1_{\text{loc}}(U)$, then v is the α^{th} weak partial derivative of u, written

$$D^\alpha u \;=\; v$$

if

$$\int_U u D^\alpha \phi \, dx \;=\; (-1)^{|\alpha|} \int_U v\phi \, dx$$

for every function $\phi \in C_c^\infty(U)$.

Lemma. There is at most one function $v \in L^1_{\text{loc}}(U)$ (modulo the values on a set of measure zero) such that v is the weak derivative of u.

Examples 1 and 2 in 5.2.1.

Dimension 1. Suppose $U = \mathbb{R}^1$. Suppose that $u, v \in L^1_{\text{loc}}(\mathbb{R}^1)$ and v is the weak derivative of u, that is $Du = v$. Then (up to a set of measure zero)

$$u(x) \;=\; C \;+\; \int_0^x v(y)\, dy.$$

In particular, u is continuous and differentiable almost everywhere.

Proof. Certainly defining $w(x)$ to be the integral in the right hand side, then $w(x)$ satisfies $Dw = v$. Indeed, if $\phi \in C_c^\infty(\mathbb{R})$ is supported on $[-N, N]$, then

$$\begin{aligned} -\int_{-\infty}^{\infty} w(x) D\phi(x)\, dx \;&=\; -\left(\int_0^N + \int_{-N}^0\right) \int_0^x D\phi(x) v(y)\, dy dx \\ &=\; -\int_0^N v(y) \int_y^N D\phi(x)\, dx dy \;+\; \int_{-N}^0 v(y) \int_{-N}^y D\phi(x)\, dx dy \\ &=\; \int_0^N \phi(y) v(y)\, dx \;+\; \int_{-N}^0 \phi(y) v(y)\, dy \;=\; \int_{-N}^N \phi(y) v(y)\, dy. \end{aligned}$$

Now if $Du = v$ then setting $f = u - w$, we have $Df = 0$. We wish to imply from this that f is constant.

Choose $\psi \in C_c^\infty(\mathbb{R})$ with $\int \psi = 1$. Set

$$g \;:=\; f \;-\; \int f\psi \, dx$$

Note that

$$Dg = 0, \qquad \int g\psi \, dx = 0.$$

We will show that $g = 0$.

Exercise. Suppose $\chi \in C_c^\infty(\mathbb{R})$ with $\int \chi = 0$. Then there exists $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ with $\chi = D\tilde{\chi}$.

Now suppose $\phi \in C_c^\infty(M)$. Then

$$\chi \;:=\; \phi \;-\; \left(\int \phi\psi\right) \psi$$

is in $C_c^\infty(M)$ with $\int \chi = 0$. Hence

$$\int g\chi \, dx \;=\; \int gD\tilde{\chi} \, dx \;=\; 0.$$

Thus

$$\int g\phi \, dx \;=\; \int g\chi \, dx \;+\; \int g\psi \, dx \left(\int \phi\psi \right) \;=\; 0.$$

Thus $g \equiv 0$.

In particular we see that the function $\mathrm{sgn}(x)$ on $\mathbb{R}$ has no weak derivative in $L^1_{\mathrm{loc}}(\mathbb{R})$, because it is not everywhere continuous (or more precisely, it is not equal almost everywhere to a function which is everywhere continuous).