Defintion of Weak derivatives notes

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Today we want to generalize the notion of derivative to the derivative of a function in $L^1_{\text{loc}})(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. (Recall that $u \in L^1_{\text{loc}}(\Omega)$ if $\|u\|_{L^1(\Omega')} < \infty$ for all $\Omega' \subset\subset \Omega$.) First let's consider the derivative of a function $u \in C^k(\Omega)$. Let α with $|\alpha| \leq k$. By integration by parts

$$\int_\Omega D^\alpha u \zeta = (-1)^{|\alpha|} \int_\Omega u D^\alpha \zeta. \tag{1}$$

for all $\zeta \in C^\infty_c(\Omega)$. The left hand side of (1) is not defined if $u \in L^1_{\text{loc}}(\Omega)$, but the right hand side of (1) *is* defined if $u \in L^1_{\text{loc}}(\Omega)$. This leads to the following definition:

Definition 1. *Let Ω be an open set in $\mathbb{R}^n$, $u \in L^1_{loc}(\Omega)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index. The α-th distributional derivative or weak derivative of u is a linear functional $T : C^\infty_c(\Omega) \to \mathbb{R}$ defined by*

$$T(\zeta) = (-1)^{|\alpha|} \int_\Omega u D^\alpha \zeta$$

for all $\zeta \in C^\infty_c(\Omega)$. We say $v \in L^1_{loc}(\Omega)$ is the α-th weak derivative of u if

$$T(\zeta) = \int_\Omega v \zeta$$

for all $\zeta \in C^\infty_c(\Omega)$; that is,

$$\int_\Omega v \zeta = (-1)^{|\alpha|} \int_\Omega u D^\alpha \zeta.$$

for all $\zeta \in C^\infty_c(\Omega)$. Note that such a v is unique. We will also denote v by $D^\alpha u$.

Example 1: If $u \in C^k(\Omega)$ and α with $|\alpha| \leq k$, as discussed above by integration by parts the α-th weak derivative of u is just the standard derivative $D^\alpha u$ of u defined using difference quotients. Thus weak derivatives generalize the classical notion of derivative.

Example 2: Suppose $\Omega = \mathbb{R}$ and

$$u(t) = \begin{cases} -t & \text{for } t \leq 0, \\ +t & \text{for } t \geq 0. \end{cases}$$

Then u has no classical derivative at $t = 0$. But

$$\int_{-\infty}^\infty u(t)\zeta'(t)dt = \int_{-\infty}^0 -t\zeta'(t)dt + \int_0^\infty t\zeta'(t)dt = -\int_{-\infty}^0 -1 \cdot \zeta(t)dt - \int_0^\infty 1 \cdot \zeta(t)dt$$

for all $\zeta \in C_c^\infty(\Omega)$, so the weak derivative v of u is

$$v(t) = \begin{cases} -1 & \text{for } t < 0, \\ +1 & \text{for } t > 0, \end{cases}$$

and $v \in L^1_{\text{loc}}(\mathbb{R})$.

Example 3: Not every distributional derivative is in L^1_{loc}. Suppose $\Omega = \mathbb{R}$ and

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t > 0. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} u(t)\zeta'(t)dt = \int_0^{\infty} \zeta'(t)dt = -\zeta(0)$$

for all $\zeta \in C_c^\infty(\Omega)$. We can regard the weak derivative of u to be the Borel measure μ such that $\delta(A) = 1$ if $0 \in A$ and $\delta(A) = 0$ if $0 \notin A$ for every Borel set $A \subseteq \mathbb{R}$ since

$$\int_{\mathbb{R}} u(t)\zeta'(t)dt = -\int_{\mathbb{R}} \zeta d\delta.$$

δ is not in $L^1_{\text{loc}}(\mathbb{R})$! Recall that the space $\mathcal{M}(\mathbb{R})$ of Borel measures on $\mathbb{R}$ is a Frechet space with semi-norm

$$\|\mu\|_{\mathcal{M},I} = \mu(I)$$

for all finite Borel measures μ and for all closed bounded intervals $I \subset \mathbb{R}$. There is an embedding

$$E : L^1_{\text{loc}}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}).$$

given by

$$E(f) = f d\mathcal{L}^1$$

for all $f \in L^1_{\text{loc}}(\mathbb{R})$, where $\mathcal{L}^1$ is the Lebesgue measure, such that

$$\|f\|_{L^1(I)} = \|f d\mathcal{L}^1\|_{\mathcal{M},I}$$

for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all closed bounded intervals $I \subset \mathbb{R}$. E is not surjective, i.e. not every Borel measure is in $L^1_{\text{loc}}(\mathbb{R})$, since $L^1_{\text{loc}}(\mathbb{R})$ consists of precisely those Borel measures that are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$. δ is not absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ and thus δ is not in $L^1_{\text{loc}}(\mathbb{R})$.