FUNCTIONS DEFINED BY IMPROPER INTEGRALS

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This is a supplement to the author's

Introduction to Real Analysis

1 Foreword

This is a revised version of Section 7.5 of my *Advanced Calculus* (Harper & Row, 1978). It is a supplement to my textbook *Introduction to Real Analysis*, which is referenced several times here. You should review Section 3.4 (Improper Integrals) of that book before reading this document.

2 Introduction

In Section 7.2 (pp. 462–484) we considered functions of the form

$$F(y) = \int_a^b f(x,y)\,dx, \quad c \le y \le d.$$

We saw that if f is continuous on $[a,b]\times[c,d]$, then F is continuous on $[c,d]$ (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

$$\int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$$

to evaluate it as

$$\int_c^d F(y)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

(Corollary 7.2.3, p. 466).

Here is another important property of F.

Theorem 1 *If f and f_y are continuous on $[a,b]\times[c,d]$, then*

$$F(y) = \int_a^b f(x,y)\,dx, \quad c \le y \le d, \tag{1}$$

is continuously differentiable on $[c,d]$ and $F'(y)$ can be obtained by differentiating (1) under the integral sign with respect to y; that is,

$$F'(y) = \int_a^b f_y(x,y)\,dx, \quad c \le y \le d. \tag{2}$$

Here $F'(a)$ and $f_y(x,a)$ are derivatives from the right and $F'(b)$ and $f_y(x,b)$ are derivatives from the left.

Proof If y and $y+\Delta y$ are in $[c,d]$ and $\Delta y \ne 0$, then

$$\frac{F(y+\Delta y)-F(y)}{\Delta y} = \int_a^b \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\,dx. \tag{3}$$

From the mean value theorem (Theorem 2.3.11, p. 83), if $x \in [a,b]$ and $y,\ y+\Delta y \in [c,d]$, there is a $y(x)$ between y and $y+\Delta y$ such that

$$f(x,y+\Delta y)-f(x,y) = f_y(x,y)\Delta y = f_y(x,y(x))\Delta y + (f_y(x,y(x)-f_y(x,y))\Delta y.$$

From this and (3),

$$\left| \frac{F(y+\Delta y)-F(y)}{\Delta y} - \int_a^b f_y(x,y)\,dx \right| \le \int_a^b |f_y(x,y(x)) - f_y(x,y)|\,dx. \quad (4)$$

Now suppose $\epsilon > 0$. Since f_y is uniformly continuous on the compact set $[a,b]\times[c,d]$ (Corollary 5.2.14, p. 314) and $y(x)$ is between y and $y+\Delta y$, there is a $\delta > 0$ such that if $|\Delta| < \delta$ then

$$|f_y(x,y) - f_y(x,y(x))| < \epsilon, \quad (x,y) \in [a,b]\times[c,d].$$

This and (4) imply that

$$\left| \frac{F(y+\Delta y - F(y))}{\Delta y} - \int_a^b f_y(x,y)\,dx \right| < \epsilon(b-a)$$

if y and $y+\Delta y$ are in $[c,d]$ and $0 < |\Delta y| < \delta$. This implies (2). Since the integral in (2) is continuous on $[c,d]$ (Exercise 7.2.3, p. 481, with f replaced by f_y), F' is continuous on $[c,d]$. □

Example 1 Since

$$f(x,y) = \cos xy \quad \text{and} \quad f_y(x,y) = -x\sin xy$$

are continuous for all (x,y), Theorem 1 implies that if

$$F(y) = \int_0^\pi \cos xy\,dx, \quad -\infty < y < \infty, \quad (5)$$

then

$$F'(y) = -\int_0^\pi x\sin xy\,dx, \quad -\infty < y < \infty. \quad (6)$$

(In applying Theorem 1 for a specific value of y, we take $R = [0,\pi]\times[-\rho,\rho]$, where $\rho > |y|$.) This provides a convenient way to evaluate the integral in (6): integrating the right side of (5) with respect to x yields

$$F(y) = \left.\frac{\sin xy}{y}\right|_{x=0}^{\pi} = \frac{\sin \pi y}{y}, \quad y \ne 0.$$

Differentiating this and using (6) yields

$$\int_0^\pi x\sin xy\,dx = \frac{\sin \pi y}{y^2} - \frac{\pi\cos \pi y}{y}, \quad y \ne 0.$$

To verify this, use integration by parts. ■

We will study the continuity, differentiability, and integrability of

$$F(y) = \int_a^b f(x,y)\,dx, \quad y \in S,$$

where S is an interval or a union of intervals, and F is a convergent improper integral for each $y \in S$. If the domain of f is $[a,b) \times S$ where $-\infty < a < b \leq \infty$, we say that F is *pointwise convergent on* S or simply *convergent on* S, and write

$$\int_a^b f(x,y)\,dx = \lim_{r \to b-} \int_a^r f(x,y)\,dx \tag{7}$$

if, for each $y \in S$ and every $\epsilon > 0$, there is an $r = r_0(y)$ (which also depends on ϵ) such that

$$\left| F(y) - \int_a^r f(x,y)\,dx \right| = \left| \int_r^b f(x,y)\,dx \right| < \epsilon, \quad r_0(y) \leq y < b. \tag{8}$$

If the domain of f is $(a,b] \times S$ where $-\infty \leq a < b < \infty$, we replace (7) by

$$\int_a^b f(x,y)\,dx = \lim_{r \to a+} \int_r^b f(x,y)\,dx$$

and (8) by

$$\left| F(y) - \int_r^b f(x,y)\,dx \right| = \left| \int_a^r f(x,y)\,dx \right| < \epsilon, \quad a < r \leq r_0(y).$$

In general, pointwise convergence of F for all $y \in S$ does not imply that F is continuous or integrable on $[c,d]$, and the additional assumptions that f_y is continuous and $\int_a^b f_y(x,y)\,dx$ converges do not imply (2).

Example 2 The function

$$f(x,y) = ye^{-|y|x}$$

is continuous on $[0,\infty) \times (-\infty,\infty)$ and

$$F(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty ye^{-|y|x}\,dx$$

converges for all y, with

$$F(y) = \begin{cases} -1 & y < 0, \\ 0 & y = 0, \\ 1 & y > 0; \end{cases}$$

therefore, F is discontinuous at $y = 0$.

Example 3 The function

$$f(x,y) = y^3 e^{-y^2 x}$$

is continuous on $[0,\infty) \times (-\infty,\infty)$. Let

$$F(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty y^3 e^{-y^2 x}\,dx = y, \quad -\infty < y < \infty.$$

Then

$$F'(y) = 1, \quad -\infty < y < \infty.$$

However,

$$\int_0^\infty \frac{\partial}{\partial y}(y^3 e^{-y^2 x})\, dx = \int_0^\infty (3y^2 - 2y^4 x)e^{-y^2 x}\, dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

so

$$F'(y) \neq \int_0^\infty \frac{\partial f(x, y)}{\partial y}\, dx \quad \text{if} \quad y = 0.$$

3 Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that f is locally integrable on an interval I if it is integrable on every finite closed subinterval of I.

Theorem 2 (Cauchy Criterion for Convergence of an Improper Integral I) *Suppose g is locally integrable on $[a, b)$ and denote*

$$G(r) = \int_a^r g(x)\, dx, \quad a \leq r < b.$$

Then the improper integral $\int_a^b g(x)\, dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$|G(r) - G(r_1)| < \epsilon, \quad r_0 \leq r, r_1 < b. \tag{9}$$

Proof For necessity, suppose $\int_a^b g(x)\, dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0 \in [a, b)$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad r_0 \leq r, r_1 < b.$$

Therefore

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - L) - (G(r_1) - L)| \\ &\leq |G(r) - L| + |G(r_1) - L| < \epsilon, \quad r_0 \leq r, r_1 < b. \end{aligned}$$

For sufficiency, (9) implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| < |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon,$$

$r_0 \leq r \leq r_1 < b$. Since G is also bounded on the compact set $[a, r_0]$ (Theorem 5.2.11, p. 313), G is bounded on $[a, b)$. Therefore the monotonic functions

$$\overline{G}(r) = \sup\left\{G(r_1) \,\middle|\, r \leq r_1 < b\right\} \quad \text{and} \quad \underline{G}(r) = \inf\left\{G(r_1) \,\middle|\, r \leq r_1 < b\right\}$$

are well defined on $[a,b)$, and

$$\lim_{r\to b-}\overline{G}(r)=\overline{L}\quad\text{and}\quad\lim_{r\to b-}\underline{G}(r)=\underline{L}$$

both exist and are finite (Theorem 2.1.11, p. 47). From (9),

$$\begin{aligned}|G(r)-G(r_1)| &= |(G(r)-G(r_0))-(G(r_1)-G(r_0))| \\ &\le |G(r)-G(r_0)|+|G(r_1)-G(r_0)|<2\epsilon,\end{aligned}$$

so

$$\overline{G}(r)-\underline{G}(r)\le 2\epsilon,\quad r_0\le r,r_1<b.$$

Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r\to b-}(\overline{G}(r)-\underline{G}(r))=0,$$

so $\overline{L}=\underline{L}$. Let $L=\overline{L}=\underline{L}$. Since

$$\underline{G}(r)\le G(r)\le\overline{G}(r),$$

it follows that $\lim_{r\to b-}G(r)=L$. ☐

We leave the proof of the following theorem to you (Exercise 2).

Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II) *Suppose g is locally integrable on $(a,b]$ and denote*

$$G(r)=\int_r^b g(x)\,dx,\quad a\le r<b.$$

Then the improper integral $\int_a^b g(x)\,dx$ converges if and only if, for each $\epsilon>0$, there is an $r_0\in(a,b]$ such that

$$|G(r)-G(r_1)|<\epsilon,\quad a<r,r_1\le r_0.$$

To see why we associate Theorems 2 and 3 with Cauchy, compare them with Theorem 4.3.5 (p. 204)

4 Uniform convergence of improper integrals

Henceforth we deal with functions $f=f(x,y)$ with domains $I\times S$, where S is an interval or a union of intervals and I is of one of the following forms:

- $[a,b)$ with $-\infty<a<b\le\infty$;
- $(a,b]$ with $-\infty\le a<b<\infty$;
- (a,b) with $-\infty\le a\le b\le\infty$.

In all cases it is to be understood that f is locally integrable with respect to x on I. When we say that the improper integral $\int_a^b f(x, y)\, dx$ has a stated property "on S" we mean that it has the property for every $y \in S$.

Definition 1 *If the improper integral*

$$\int_a^b f(x, y)\, dx = \lim_{r \to b-} \int_a^r f(x, y)\, dx \tag{10}$$

converges on S, *it is said to converge uniformly* (*or be uniformly convergent*) *on* S *if, for each* $\epsilon > 0$, *there is an* $r_0 \in [a, b)$ *such that*

$$\left| \int_a^b f(x, y)\, dx - \int_a^r f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r < b,$$

or, equivalently,

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r < b. \tag{11}$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in (8) may depend upon the particular value of y, while the r_0 in (11) does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

Theorem 4 (**Cauchy Criterion for Uniform Convergence I**) *The improper integral in* (10) *converges uniformly on* S *if and only if, for each* $\epsilon > 0$, *there is an* $r_0 \in [a, b)$ *such that*

$$\left| \int_r^{r_1} f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r, r_1 < b. \tag{12}$$

Proof Suppose $\int_a^b f(x, y)\, dx$ converges uniformly on S and $\epsilon > 0$. From Definition 1, there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \frac{\epsilon}{2} \text{ and } \left| \int_{r_1}^b f(x, y)\, dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \le r, r_1 < b. \tag{13}$$

Since

$$\int_r^{r_1} f(x, y)\, dx = \int_r^b f(x, y)\, dx - \int_{r_1}^b f(x, y)\, dx,$$

(13) and the triangle inequality imply (12).

For the converse, denote

$$F(y) = \int_a^r f(x, y)\, dx.$$

Since (12) implies that

$$|F(r,y)-F(r_1,y)|<\epsilon, \quad y\in S, \quad r_0\le r,r_1<b, \tag{14}$$

Theorem 2 with $G(r)=F(r,y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x,y)\,dx$ converges pointwise for $y\in S$. Therefore, if $\epsilon>0$ then, for each $y\in S$, there is an $r_0(y)\in[a,b)$ such that

$$\left|\int_r^b f(x,y)\,dx\right|<\epsilon, \quad y\in S, \quad r_0(y)\le r<b. \tag{15}$$

For each $y\in S$, choose $r_1(y)\ge\max[r_0(y),r_0]$. (Recall (14)). Then

$$\int_r^b f(x,y)\,dx=\int_r^{r_1(y)} f(x,y)\,dx+\int_{r_1(y)}^b f(x,y)\,dx,$$

so (12), (15), and the triangle inequality imply that

$$\left|\int_r^b f(x,y)\,dx\right|<2\epsilon, \quad y\in S, \quad r_0\le r<b.$$

□

In practice, we don't explicitly exhibit r_0 for each given ϵ. It suffices to obtain estimates that clearly imply its existence.

Example 4 For the improper integral of Example 2,

$$\left|\int_r^\infty f(x,y)\,dx\right|=\int_r^\infty |y|e^{-|y|x}=e^{-r|y|}, \quad y\ne 0.$$

If $|y|\ge\rho$, then

$$\left|\int_r^\infty f(x,y)\,dx\right|\le e^{-r\rho},$$

so $\int_0^\infty f(x,y)\,dx$ converges uniformly on $(-\infty,\rho]\cup[\rho,\infty)$ if $\rho>0$; however, it does not converge uniformly on any neighborhood of $y=0$, since, for any $r>0$, $e^{-r|y|}>\frac{1}{2}$ if $|y|$ is sufficiently small.

Definition 2 *If the improper integral*

$$\int_a^b f(x,y)\,dx=\lim_{r\to a+}\int_r^b f(x,y)\,dx$$

converges on S, it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon>0$, there is an $r_0\in(a,b]$ such that

$$\left|\int_a^b f(x,y)\,dx-\int_r^b f(x,y)\,dx\right|<\epsilon, \quad y\in S, \quad a<r\le r_0,$$

or, equivalently,

$$\left|\int_a^r f(x,y)\,dx\right|<\epsilon, \quad y\in S, \quad a<r\le r_0.$$

We leave proof of the following theorem to you (Exercise 3).

Theorem 5 **(Cauchy Criterion for Uniform Convergence II)** *The improper integral*

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+}\int_r^b f(x,y)\,dx$$

converges uniformly on S *if and only if, for each* $\epsilon > 0$, *there is an* $r_0 \in (a,b]$ *such that*

$$\left|\int_{r_1}^{r} f(x,y)\,dx\right| < \epsilon, \quad y \in S, \quad a < r, r_1 \le r_0.$$

We need one more definition, as follows.

Definition 3 *Let* $f = f(x,y)$ *be defined on* $(a,b)\times S$, *where* $-\infty \le a < b \le \infty$. *Suppose* f *is locally integrable on* (a,b) *for all* $y \in S$ *and let* c *be an arbitrary point in* (a,b). *Then* $\int_a^b f(x,y)\,dx$ *is said to converge uniformly on* S *if* $\int_a^c f(x,y)\,dx$ *and* $\int_c^b f(x,y)\,dx$ *both converge uniformly on* S.

We leave it to you (Exercise 4) to show that this definition is independent of c; that is, if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on S for some $c \in (a,b)$, then they both converge uniformly on S for every $c \in (a,b)$.

We also leave it you (Exercise 5) to show that if f is bounded on $[a,b]\times[c,d]$ and $\int_a^b f(x,y)\,dx$ exists as a proper integral for each $y \in [c,d]$, then it converges uniformly on $[c,d]$ according to all three Definitions 1–3.

Example 5 Consider the improper integral

$$F(y) = \int_0^\infty x^{-1/2}e^{-xy}\,dx,$$

which diverges if $y \le 0$ (verify). Definition 3 applies if $y > 0$, so we consider the improper integrals

$$F_1(y) = \int_0^1 x^{-1/2}e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_1^\infty x^{-1/2}e^{-xy}\,dx$$

separately. Moreover, we could just as well define

$$F_1(y) = \int_0^c x^{-1/2}e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_c^\infty x^{-1/2}e^{-xy}\,dx, \tag{16}$$

where c is any positive number.

Definition 2 applies to F_1. If $0 < r_1 < r$ and $y \ge 0$, then

$$\left|\int_r^{r_1} x^{-1/2}e^{-xy}\,dx\right| < \int_{r_1}^{r} x^{-1/2}\,dx < 2r^{1/2},$$

so $F_1(y)$ converges for uniformly on $[0,\infty)$.

Definition 1 applies to F_2. Since

$$\left|\int_r^{r_1} x^{-1/2}e^{-xy}\,dx\right| < r^{-1/2}\int_r^\infty e^{-xy}\,dx = \frac{e^{-ry}}{yr^{1/2}},$$

$F_2(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$. It does not converge uniformly on $(0,\rho)$, since the change of variable $u = xy$ yields

$$\int_r^{r_1} x^{-1/2}e^{-xy}\,dx = y^{-1/2}\int_{ry}^{r_1y} u^{-1/2}e^{-u}\,du,$$

which, for any fixed $r > 0$, can be made arbitrarily large by taking y sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho,\infty)$ if $\rho > 0$.

Note that that the constant c in (16) plays no role in this argument.

Example 6 Suppose we take

$$\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2} \tag{17}$$

as given (Exercise 31**(b)**). Substituting $u = xy$ with $y > 0$ yields

$$\int_0^\infty \frac{\sin xy}{x}\,dx = \frac{\pi}{2}, \quad y > 0. \tag{18}$$

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition 1 and Theorem 4 apply here. If $0 < r < r_1$ and $y > 0$, then

$$\int_r^{r_1} \frac{\sin xy}{x}\,dx = -\frac{1}{y}\left(\left.\frac{\cos xy}{x}\right|_r^{r_1} + \int_r^{r_1}\frac{\cos xy}{x^2}\,dx\right), \text{ so } \left|\int_r^{r_1}\frac{\sin xy}{x}\,dx\right| < \frac{3}{ry}.$$

Therefore (18) converges uniformly on $[\rho,\infty)$ if $\rho > 0$. On the other hand, from (17), there is a $\delta > 0$ such that

$$\int_{u_0}^\infty \frac{\sin u}{u}\,du > \frac{\pi}{4}, \quad 0 \le u_0 < \delta.$$

This and (18) imply that

$$\int_r^\infty \frac{\sin xy}{x}\,dx = \int_{yr}^\infty \frac{\sin u}{u}\,du > \frac{\pi}{4}$$

for any $r > 0$ if $0 < y < \delta/r$. Hence, (18) does not converge uniformly on any interval $(0,\rho]$ with $\rho > 0$.

5 Absolutely Uniformly Convergent Improper Integrals

Definition 4 (**Absolute Uniform Convergence I**) *The improper integral*

$$\int_a^b f(x,y)\,dx = \lim_{r\to b-}\int_a^r f(x,y)\,dx$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x,y)|\,dx = \lim_{r\to b-}\int_a^r |f(x,y)|\,dx$$

converges uniformly on S; that is, if, for each $\epsilon > 0$, there is an $r_0 \in [a,b)$ such that

$$\left|\int_a^b |f(x,y)|\,dx - \int_a^r |f(x,y)|\,dx\right| < \epsilon, \quad y \in S, \quad r_0 < r < b.$$

To see that this definition makes sense, recall that if f is locally integrable on $[a,b)$ for all y in S, then so is $|f|$ (Theorem 3.4.9, p. 161). Theorem 4 with f replaced by $|f|$ implies that $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a,b)$ such that

$$\int_r^{r_1} |f(x,y)|\,dx < \epsilon, \quad y \in S, \quad r_0 \le r < r_1 < b.$$

Since

$$\left|\int_r^{r_1} f(x,y)\,dx\right| \le \int_r^{r_1} |f(x,y)|\,dx,$$

Theorem 4 implies that if $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S then it converges uniformly on S.

Theorem 6 (**Weierstrass's Test for Absolute Uniform Convergence I**) *Suppose $M = M(x)$ is nonnegative on $[a,b)$, $\int_a^b M(x)\,dx < \infty$, and*

$$|f(x,y)| \le M(x), \quad y \in S, \quad a \le x < b. \tag{19}$$

Then $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S.

Proof Denote $\int_a^b M(x)\,dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0 \in [a,b)$ such that

$$L - \epsilon < \int_a^r M(x)\,dx \le L, \quad r_0 < r < b.$$

Therefore, if $r_0 < r \le r_1$, then

$$0 \le \int_r^{r_1} M(x)\,dx = \left(\int_a^{r_1} M(x)\,dx - L\right) - \left(\int_a^r M(x)\,dx - L\right) < \epsilon$$

This and (19) imply that

$$\int_r^{r_1} |f(x,y)|\,dx \le \int_r^{r_1} M(x)\,dx < \epsilon, \quad y \in S, \quad a \le r_0 < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. ☐

Example 7 Suppose $g = g(x,y)$ is locally integrable on $[0,\infty)$ for all $y \in S$ and, for some $a_0 \ge 0$, there are constants K and p_0 such that

$$|g(x,y)| \le Ke^{p_0 x}, \quad y \in S, \quad x \ge a_0.$$

If $p > p_0$ and $r \ge a_0$, then

$$\begin{aligned}\int_r^\infty e^{-px}|g(x,y)|\,dx &= \int_r^\infty e^{-(p-p_0)x}e^{-p_0x}|g(x,y)|\,dx \\ &\le K\int_r^\infty e^{-(p-p_0)x}\,dx = \frac{Ke^{-(p-p_0)r}}{p-p_0},\end{aligned}$$

so $\int_0^\infty e^{-px}g(x,y)\,dx$ converges absolutely on S. For example, since

$$|x^\alpha \sin xy| < e^{p_0x} \quad \text{and} \quad |x^\alpha \cos xy| < e^{p_0x}$$

for x sufficiently large if $p_0 > 0$, Theorem 4 implies that $\int_0^\infty e^{-px}x^\alpha \sin xy\,dx$ and $\int_0^\infty e^{-px}x^\alpha \cos xy\,dx$ converge absolutely uniformly on $(-\infty,\infty)$ if $p > 0$ and $\alpha \ge 0$. As a matter of fact, $\int_0^\infty e^{-px}x^\alpha \sin xy\,dx$ converges absolutely on $(-\infty,\infty)$ if $p > 0$ and $\alpha > -1$. (Why?)

Definition 5 (**Absolute Uniform Convergence II**) *The improper integral*

$$\int_a^b f(x,y)\,dx = \lim_{r\to a+}\int_r^b f(x,y)\,dx$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x,y)|\,dx = \lim_{r\to a+}\int_r^b |f(x,y)|\,dx$$

converges uniformly on S; that is, if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that

$$\left|\int_a^b |f(x,y)|\,dx - \int_r^b |f(x,y)|\,dx\right| < \epsilon, \quad y \in S, \quad a < r < r_0 \le b.$$

We leave it to you (Exercise 7) to prove the following theorem.

Theorem 7 (**Weierstrass's Test for Absolute Uniform Convergence II**) *Suppose $M = M(x)$ is nonnegative on $(a,b]$, $\int_a^b M(x)\,dx < \infty$, and*

$$|f(x,y)| \le M(x), \quad y \in S, \quad x \in (a,b].$$

Then $\int_a^b f(x,y)\,dx$ converges absolutely uniformly on S.

Example 8 If $g = g(x, y)$ is locally integrable on $(0, 1]$ for all $y \in S$ and

$$|g(x, y)| \le Ax^{-\beta}, \quad 0 < x \le x_0,$$

for each $y \in S$, then

$$\int_0^1 x^\alpha g(x, y)\, dx$$

converges absolutely uniformly on S if $\alpha > \beta - 1$. To see this, note that if $0 < r < r_1 \le x_0$, then

$$\int_{r_1}^r x^\alpha |g(x, y)|\, dx \le A \int_{r_1}^r x^{\alpha-\beta}\, dx = \left. \frac{Ax^{\alpha-\beta+1}}{\alpha - \beta + 1} \right|_{r_1}^r < \frac{Ar^{\alpha-\beta+1}}{\alpha - \beta + 1}.$$

Applying this with $\beta = 0$ shows that

$$F(y) = \int_0^1 x^\alpha \cos xy\, dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -1$ and

$$G(y) = \int_0^1 x^\alpha \sin xy\, dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -2$.

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems 6 and 7 with Weierstrass.

6 Dirichlet's Tests

Weierstrass's test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where $\int_a^b f(x, y)\, dx$ converges uniformly on S, but $\int_a^b |f(x, y)|\, dx$ does not.

Theorem 8 **(Dirichlet's Test for Uniform Convergence I)** *If g, g_x, and h are continuous on $[a, b) \times S$, then*

$$\int_a^b g(x, y) h(x, y)\, dx$$

converges uniformly on S if the following conditions are satisfied:

(a) $\displaystyle \lim_{x \to b-} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0;$

(b) *There is a constant M such that*

$$\sup_{y \in S} \left| \int_a^x h(u, y)\, du \right| < M, \quad a \le x < b;$$

(c) $\int_a^b |g_x(x,y)|\,dx$ *converges uniformly on* S.

Proof If

$$H(x,y) = \int_a^x h(u,y)\,du, \tag{20}$$

then integration by parts yields

$$\begin{aligned} \int_r^{r_1} g(x,y)h(x,y)\,dx &= \int_r^{r_1} g(x,y)H_x(x,y)\,dx \\ &= g(r_1,y)H(r_1,y) - g(r,y)H(r,y) \\ &\quad - \int_r^{r_1} g_x(x,y)H(x,y)\,dx. \end{aligned} \tag{21}$$

Since assumption **(b)** and (20) imply that $|H(x,y)| \le M$, $(x,y) \in (a,b] \times S$, Eqn. (21) implies that

$$\left| \int_r^{r_1} g(x,y)h(x,y)\,dx \right| < M \left(2 \sup_{x \ge r} |g(x,y)| + \int_r^{r_1} |g_x(x,y)|\,dx \right) \tag{22}$$

on $[r, r_1] \times S$.

Now suppose $\epsilon > 0$. From assumption **(a)**, there is an $r_0 \in [a,b)$ such that $|g(x,y)| < \epsilon$ on S if $r_0 \le x < b$. From assumption **(c)** and Theorem 6, there is an $s_0 \in [a,b)$ such that

$$\int_r^{r_1} |g_x(x,y)|\,dx < \epsilon, \quad y \in S, \quad s_0 < r < r_1 < b.$$

Therefore (22) implies that

$$\left| \int_r^{r_1} g(x,y)h(x,y) \right| < 3M\epsilon, \quad y \in S, \quad \max(r_0, s_0) < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. ☐

The statement of this theorem is complicated, but applying it isn't; just look for a factorization $f = gh$, where h has a bounded antderivative on $[a,b)$ and g is "small" near b. Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

Example 9 Let

$$I(y) = \int_0^\infty \frac{\cos xy}{x+y}\,dx, \quad y > 0.$$

The obvious inequality

$$\left| \frac{\cos xy}{x+y} \right| \le \frac{1}{x+y}$$

is useless here, since

$$\int_0^\infty \frac{dx}{x+y} = \infty.$$

However, integration by parts yields

$$\begin{aligned}\int_r^{r_1} \frac{\cos xy}{x+y}\,dx &= \left.\frac{\sin xy}{y(x+y)}\right|_r^{r_1} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2}\,dx \\ &= \frac{\sin r_1 y}{y(r_1+y)} - \frac{\sin ry}{y(r+y)} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2}\,dx.\end{aligned}$$

Therefore, if $0 < r < r_1$, then

$$\left|\int_r^{r_1} \frac{\cos xy}{x+y}\,dx\right| < \frac{1}{y}\left(\frac{2}{r+y} + \int_r^{\infty} \frac{1}{(x+y)^2}\right) \le \frac{3}{y(r+y)^2} \le \frac{3}{\rho(r+\rho)}$$

if $y \ge \rho > 0$. Now Theorem 4 implies that $I(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

We leave the proof of the following theorem to you (Exercise 10).

Theorem 9 **(Dirichlet's Test for Uniform Convergence II)** *If g, g_x, and h are continuous on $(a,b] \times S$, then*

$$\int_a^b g(x,y)h(x,y)\,dx$$

converges uniformly on S if the following conditions are satisfied:

(a) $\lim_{x\to a+} \left\{\sup_{y\in S} |g(x,y)|\right\} = 0;$

(b) *There is a constant M such that*

$$\sup_{y\in S}\left|\int_x^b h(u,y)\,du\right| \le M, \quad a < x \le b;$$

(c) $\int_a^b |g_x(x,y)|\,dx$ *converges uniformly on S.*

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems 8 and 9 with Dirichlet.

7 Consequences of uniform convergence

Theorem 10 *If $f = f(x,y)$ is continuous on either $[a,b) \times [c,d]$ or $(a,b] \times [c,d]$ and*

$$F(y) = \int_a^b f(x,y)\,dx \tag{23}$$

converges uniformly on $[c,d]$, then F is continuous on $[c,d]$. Moreover,

$$\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx. \tag{24}$$

Proof We will assume that f is continuous on $(a,b]\times[c,d]$. You can consider the other case (Exercise 14).

We will first show that F in (23) is continuous on $[c,d]$. Since F converges uniformly on $[c,d]$, Definition 1 (specifically, (11)) implies that if $\epsilon>0$, there is an $r\in[a,b)$ such that

$$\left|\int_r^b f(x,y)\,dx\right|<\epsilon,\quad c\le y\le d.$$

Therefore, if $c\le y, y_0\le d]$, then

$$\begin{aligned}|F(y)-F(y_0)| &= \left|\int_a^b f(x,y)\,dx-\int_a^b f(x,y_0)\,dx\right|\\ &\le \left|\int_a^r [f(x,y)-f(x,y_0)]\,dx\right|+\left|\int_r^b f(x,y)\,dx\right|\\ &\quad+\left|\int_r^b f(x,y_0)\,dx\right|,\end{aligned}$$

so

$$|F(y)-F(y_0)|\le\int_a^r|f(x,y)-f(x,y_0)|\,dx+2\epsilon. \tag{25}$$

Since f is uniformly continuous on the compact set $[a,r]\times[c,d]$ (Corollary 5.2.14, p. 314), there is a $\delta>0$ such that

$$|f(x,y)-f(x,y_0)|<\epsilon$$

if (x,y) and (x,y_0) are in $[a,r]\times[c,d]$ and $|y-y_0|<\delta$. This and (25) imply that

$$|F(y)-F(y_0)|<(r-a)\epsilon+2\epsilon<(b-a+2)\epsilon$$

if y and y_0 are in $[c,d]$ and $|y-y_0|<\delta$. Therefore F is continuous on $[c,d]$, so the integral on left side of (24) exists. Denote

$$I=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy. \tag{26}$$

We will show that the improper integral on the right side of (24) converges to I. To this end, denote

$$I(r)=\int_a^r\left(\int_c^d f(x,y)\,dy\right)dx.$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[a,r]\times[c,d]$ (Corollary 7.2.2, p. 466),

$$I(r)=\int_c^d\left(\int_a^r f(x,y)\,dx\right)dy.$$

From this and (26),

$$I - I(r) = \int_c^d \left(\int_r^b f(x, y)\, dx \right) dy.$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x, y)\, dx$ converges uniformly on $[c, d]$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad r_0 < r < b,$$

so $|I - I(r)| < (d - c)\epsilon$ if $r_0 < r < b$. Hence,

$$\lim_{r \to b-} \int_a^r \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy,$$

which completes the proof of (24). □

Example 10 It is straightforward to verify that

$$\int_0^\infty e^{-xy}\, dx = \frac{1}{y}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\begin{aligned} \int_{y_1}^{y_2} \frac{dy}{y} &= \int_{y_1}^{y_2} \left(\int_0^\infty e^{-xy}\, dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} e^{-xy}\, dy \right) dy \\ &= \int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x}\, dx. \end{aligned}$$

Since

$$\int_{y_1}^{y_2} \frac{dy}{y} = \log \frac{y_2}{y_1}, \quad y_2 \ge y_1 > 0,$$

it follows that

$$\int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x}\, dx = \log \frac{y_2}{y_1}, \quad y_2 \ge y_1 > 0.$$

Example 11 From Example 6,

$$\int_0^\infty \frac{\sin xy}{x}\, dx = \frac{\pi}{2}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore, Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\begin{aligned} \frac{\pi}{2}(y_2 - y_1) &= \int_{y_1}^{y_2} \left(\int_0^\infty \frac{\sin xy}{x}\, dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} \frac{\sin xy}{x}\, dy \right) dx \\ &= \int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2}\, dx. \qquad (27) \end{aligned}$$

The last integral converges uniformly on $(-\infty, \infty)$ (Exercise 10**(h)**), and is therefore continuous with respect to y_1 on $(-\infty, \infty)$, by Theorem 10; in particular, we can let $y_1 \to 0+$ in (27) and replace y_2 by y to obtain

$$\int_0^\infty \frac{1 - \cos xy}{x^2}\, dx = \frac{\pi y}{2}, \quad y \ge 0.$$

The next theorem is analogous to Theorem 4.4.20 (p. 252).

Theorem 11 *Let f and f_y be continuous on either $[a, b) \times [c, d]$ or $(a, b] \times [c, d]$. Suppose that the improper integral*

$$F(y) = \int_a^b f(x, y)\, dx$$

converges for some $y_0 \in [c, d]$ and

$$G(y) = \int_a^b f_y(x, y)\, dx$$

converges uniformly on $[c, d]$. Then F converges uniformly on $[c, d]$ and is given explicitly by

$$F(y) = F(y_0) + \int_{y_0}^y G(t)\, dt, \quad c \le y \le d.$$

Moreover, F is continuously differentiable on $[c, d]$; specifically,

$$F'(y) = G(y), \quad c \le y \le d, \tag{28}$$

where $F'(c)$ and $f_y(x, c)$ are derivatives from the right, and $F'(d)$ and $f_y(x, d)$ are derivatives from the left.

Proof We will assume that f and f_y are continuous on $[a, b) \times [c, d]$. You can consider the other case (Exercise 15).

Let

$$F_r(y) = \int_a^r f(x, y)\, dx, \quad a \le r < b, \quad c \le y \le d.$$

Since f and f_y are continuous on $[a, r] \times [c, d]$, Theorem 1 implies that

$$F_r'(y) = \int_a^r f_y(x, y)\, dx, \quad c \le y \le d.$$

Then

$$\begin{aligned} F_r(y) &= F_r(y_0) + \int_{y_0}^y \left(\int_a^r f_y(x, t)\, dx \right) dt \\ &= F(y_0) + \int_{y_0}^y G(t)\, dt \\ &\quad + (F_r(y_0) - F(y_0)) - \int_{y_0}^y \left(\int_r^b f_y(x, t)\, dx \right) dt, \quad c \le y \le d. \end{aligned}$$

Therefore,

$$\left| F_r(y) - F(y_0) - \int_{y_0}^{y} G(t)\,dt \right| \le |F_r(y_0) - F(y_0)| + \left| \int_{y_0}^{y} \int_r^b f_y(x,t)\,dx \right| dt. \qquad (29)$$

Now suppose $\epsilon > 0$. Since we have assumed that $\lim_{r\to b-} F_r(y_0) = F(y_0)$ exists, there is an r_0 in (a,b) such that

$$|F_r(y_0) - F(y_0)| < \epsilon, \quad r_0 < r < b.$$

Since we have assumed that $G(y)$ converges for $y \in [c,d]$, there is an $r_1 \in [a,b)$ such that

$$\left| \int_r^b f_y(x,t)\,dx \right| < \epsilon, \quad t \in [c,d], \quad r_1 \le r < b.$$

Therefore, (29) yields

$$\left| F_r(y) - F(y_0) - \int_{y_0}^{y} G(t)\,dt \right| < \epsilon(1 + |y - y_0|) \le \epsilon(1 + d - c)$$

if $\max(r_0, r_1) \le r < b$ and $t \in [c,d]$. Therefore $F(y)$ converges uniformly on $[c,d]$ and

$$F(y) = F(y_0) + \int_{y_0}^{y} G(t)\,dt, \quad c \le y \le d.$$

Since G is continuous on $[c,d]$ by Theorem 10, (28) follows from differentiating this (Theorem 3.3.11, p. 141). ☐

Example 12 Let

$$I(y) = \int_0^\infty e^{-yx^2}\,dx, \quad y > 0.$$

Since

$$\int_0^r e^{-yx^2}\,dx = \frac{1}{\sqrt{y}} \int_0^{r\sqrt{y}} e^{-t^2}\,dt,$$

it follows that

$$I(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-t^2}\,dt,$$

and the convergence is uniform on $[\rho,\infty)$ if $\rho > 0$ (Exercise 8**(i)**). To evaluate the last integral, denote $J(\rho) = \int_0^\rho e^{-t^2}\,dt$; then

$$J^2(\rho) = \left(\int_0^\rho e^{-u^2}\,du \right) \left(\int_0^\rho e^{-v^2}\,dv \right) = \int_0^\rho \int_0^\rho e^{-(u^2+v^2)}\,du\,dv.$$

Transforming to polar coordinates $r = r\cos\theta$, $v = r\sin\theta$ yields

$$J^2(\rho) = \int_0^{\pi/2} \int_0^\rho r e^{-r^2}\,dr\,d\theta = \frac{\pi(1 - e^{-\rho^2})}{4}, \quad \text{so} \quad J(\rho) = \frac{\sqrt{\pi(1 - e^{-\rho^2})}}{2}.$$

Therefore

$$\int_0^\infty e^{-t^2}\,dt = \lim_{\rho\to\infty} J(\rho) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \int_0^\infty e^{-yx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{y}}, \quad y > 0.$$

Differentiating this n times with respect to y yields

$$\int_0^\infty x^{2n} e^{-yx^2}\,dx = \frac{1\cdot 3\cdots(2n-1)\sqrt{\pi}}{2^n y^{n+1/2}} \quad y > 0, \quad n = 1, 2, 3, \dots,$$

where Theorem 11 justifies the differentiation for every n, since all these integrals converge uniformly on $[\rho, \infty)$ if $\rho > 0$ (Exercise 8(i)).

Some advice for applying this theorem: Be sure to check first that $F(y_0) = \int_a^b f(x, y_0)\,dx$ converges for at least one value of y. If so, differentiate $\int_a^b f(x, y)\,dx$ formally to obtain $\int_a^b f_y(x, y)\,dx$. Then $F'(y) = \int_a^b f_y(x, y)\,dx$ if y is in some interval on which this improper integral converges uniformly.

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8 Applications to Laplace transforms

The *Laplace transform* of a function f locally integrable on $[0, \infty)$ is

$$F(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

for all s such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise 26).

Theorem 12 *Suppose f is locally integrable on $[0, \infty)$ and $|f(x)| \le Me^{s_0x}$ for sufficiently large x. Then the Laplace transform of F converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.*

Theorem 13 *If f is continuous on $[0, \infty)$ and $H(x) = \int_0^\infty e^{-s_0u} f(u)\,du$ is bounded on $[0, \infty)$, then the Laplace transform of f converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.*

Proof If $0 \le r \le r_1$,

$$\int_r^{r_1} e^{-sx} f(x)\,dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0x} f(x)\,dt = \int_r^{r_1} e^{-(s-s_0)t} H'(x)\,dt.$$

Integration by parts yields

$$\int_r^{r_1} e^{-sx} f(x)\,dt = e^{-(s-s_0)x} H(x)\Big|_r^{r_1} + (s - s_0)\int_r^{r_1} e^{-(s-s_0)x} H(x)\,dx.$$

Therefore, if $|H(x)| \leq M$, then

$$\begin{aligned}\left|\int_r^{r_1} e^{-sx} f(x)\,dx\right| &\leq M\left|e^{-(s-s_0)r_1} + e^{-(s-s_0)r} + (s-s_0)\int_r^{r_1} e^{-(s-s_0)x}\,dx\right| \\ &\leq 3Me^{-(s-s_0)r} \leq 3Me^{-(s_1-s_0)r}, \quad s \geq s_1.\end{aligned}$$

Now Theorem 4 implies that $F(s)$ converges uniformly on $[s_1, \infty)$.

The following theorem draws a considerably stonger conclusion from the same assumptions.

Theorem 14 *If f is continuous on $[0, \infty)$ and*

$$H(x) = \int_0^x e^{-s_0 u} f(u)\,du$$

is bounded on $[0, \infty)$, then the Laplace transform of f is infinitely differentiable on (s_0, ∞), with

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n f(x)\,dx; \tag{30}$$

that is, the n-th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^n x^n f(x)$.

Proof First we will show that the integrals

$$I_n(s) = \int_0^\infty e^{-sx} x^n f(x)\,dx, \quad n = 0, 1, 2, \ldots$$

all converge uniformly on $[s_1, \infty)$ if $s_1 > s_0$. If $0 < r < r_1$, then

$$\int_r^{r_1} e^{-sx} x^n f(x)\,dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} x^n f(x)\,dx = \int_r^{r_1} e^{-(s-s_0)x} x^n H'(x)\,dx.$$

Integrating by parts yields

$$\begin{aligned}\int_r^{r_1} e^{-sx} x^n f(x)\,dx &= r_1^n e^{-(s-s_0)r_1} H(r) - r^n e^{-(s-s_0)r} H(r) \\ &\quad - \int_r^{r_1} H(x)\left(e^{-(s-s_0)x} x^n\right)'\,dx,\end{aligned}$$

where $'$ indicates differentiation with respect to x. Therefore, if $|H(x)| \leq M \leq \infty$ on $[0, \infty)$, then

$$\left|\int_r^{r_1} e^{-sx} x^n f(x)\,dx\right| \leq M\left(e^{-(s-s_0)r} r^n + e^{-(s-s_0)r} r^n + \int_r^\infty |(e^{-(s-s_0)x}) x^n)'|\,dx\right).$$

Therefore, since $e^{-(s-s_0)r} r^n$ decreases monotonically on (n, ∞) if $s > s_0$ (check!),

$$\left|\int_r^{r_1} e^{-sx} x^n f(x)\,dx\right| < 3Me^{-(s-s_0)r} r^n, \quad n < r < r_1,$$

so Theorem 4 implies that $I_n(s)$ converges uniformly $[s_1, \infty)$ if $s_1 > s_0$. Now Theorem 11 implies that $F_{n+1} = -F'_n$, and an easy induction proof yields (30) (Exercise 25). ∎

Example 13 Here we apply Theorem 12 with $f(x) = \cos ax$ $(a \neq 0)$ and $s_0 = 0$. Since

$$\int_0^x \cos au\, du = \frac{\sin ax}{a}$$

is bounded on $(0, \infty)$, Theorem 12 implies that

$$F(s) = \int_0^\infty e^{-sx} \cos ax\, dx$$

converges and

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n \cos ax\, dx, \quad s > 0. \tag{31}$$

(Note that this is also true if $a = 0$.) Elementary integration yields

$$F(s) = \frac{s}{s^2 + a^2}.$$

Hence, from (31),

$$\int_0^\infty e^{-sx} x^n \cos ax = (-1)^n \frac{d^n}{ds^n} \frac{s}{s^2 + a^2}, \quad n = 0, 1, \ldots.$$

9 Exercises

1. Suppose g and h are differentiable on $[a, b]$, with

$$a \le g(y) \le b \quad \text{and} \quad a \le h(y) \le b, \quad c \le y \le d.$$

Let f and f_y be continuous on $[a, b] \times [c, d]$. Derive *Liebniz's rule*:

$$\begin{aligned}\frac{d}{dy}\int_{g(y)}^{h(y)} f(x, y)\, dx &= f(h(y), y)h'(y) - f(g(y), y)g'(y) \\ &\quad + \int_{g(y)}^{h(y)} f_y(x, y)\, dx.\end{aligned}$$

(Hint: Define $H(y, u, v) = \int_u^v f(x, y)\, dx$ and use the chain rule.)

2. Adapt the proof of Theorem 2 to prove Theorem 3.

3. Adapt the proof of Theorem 4 to prove Theorem 5.

4. Show that Definition 3 is independent of c; that is, if $\int_a^c f(x, y)\, dx$ and $\int_c^b f(x, y)\, dx$ both converge uniformly on S for some $c \in (a, b)$, then they both converge uniformly on S and every $c \in (a, b)$.

5. **(a)** Show that if f is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y)\, dx$ exists as a proper integral for each $y \in [c, d]$, then it converges uniformly on $[c, d]$ according to all of Definition 1–3.

(b) Give an example to show that the boundedness of f is essential in **(a)**.

6. Working directly from Definition 1, discuss uniform convergence of the following integrals:

(a) $\displaystyle\int_0^\infty \frac{dx}{1 + y^2x^2}\, dx$ **(b)** $\displaystyle\int_0^\infty e^{-xy}x^2\, dx$

(c) $\displaystyle\int_0^\infty x^{2n}e^{-yx^2}\, dx$ **(d)** $\displaystyle\int_0^\infty \sin xy^2\, dx$

(e) $\displaystyle\int_0^\infty (3y^2 - 2xy)e^{-y^2x}\, dx$ **(f)** $\displaystyle\int_0^\infty (2xy - y^2x^2)e^{-xy}\, dx$

7. Adapt the proof of Theorem 6 to prove Theorem 7.

8. Use Weierstrass's test to show that the integral converges uniformly on S:

(a) $\displaystyle\int_0^\infty e^{-xy}\sin x\, dx, \quad S = [\rho, \infty), \quad \rho > 0$

(b) $\displaystyle\int_0^\infty \frac{\sin x}{x^y}\, dx, \quad S = [c, d], \quad 1 < c < d < 2$

(c) $\displaystyle\int_1^\infty e^{-px}\frac{\sin xy}{x}\,dx, \quad p>0, \quad S=(-\infty,\infty)$

(d) $\displaystyle\int_0^1 \frac{e^{xy}}{(1-x)^y}\,dx, \quad S=(-\infty,b), \quad b<1$

(e) $\displaystyle\int_{-\infty}^\infty \frac{\cos xy}{1+x^2y^2}\,dx, \quad S=(-\infty,-\rho]\cup[\rho,\infty), \quad \rho>0.$

(f) $\displaystyle\int_1^\infty e^{-x/y}\,dx, \quad S=[\rho,\infty), \quad \rho>0$

(g) $\displaystyle\int_{-\infty}^\infty e^{xy}e^{-x^2}\,dx, \quad S=[-\rho,\rho], \quad \rho>0$

(h) $\displaystyle\int_0^\infty \frac{\cos xy-\cos ax}{x^2}\,dx, \quad S=(-\infty,\infty)$

(i) $\displaystyle\int_0^\infty x^{2n}e^{-yx^2}\,dx, \quad S=[\rho,\infty), \quad \rho>0, \quad n=0,1,2,\ldots$

9. **(a)** Show that

$$\Gamma(y)=\int_0^\infty x^{y-1}e^{-x}\,dx$$

converges if $y>0$, and uniformly on $[c,d]$ if $0<c<d<\infty$.

(b) Use integration by parts to show that

$$\Gamma(y)=\frac{\Gamma(y+1)}{y}, \quad y\geq 0,$$

and then show by induction that

$$\Gamma(y)=\frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)}, \quad y>0, \quad n=1,2,3,\ldots.$$

How can this be used to define $\Gamma(y)$ in a natural way for all $y\neq 0,-1,-2,\ldots$? (This function is called the *gamma function*.)

(c) Show that $\Gamma(n+1)=n!$ if n is a positive integer.

(d) Show that

$$\int_0^\infty e^{-st}t^\alpha\,dt=s^{-\alpha-1}\Gamma(\alpha+1), \quad \alpha>-1, \quad s>0.$$

10. Show that Theorem 8 remains valid with assumption **(c)** replaced by the assumption that $|g_x(x,y)|$ is monotonic with respect to x for all $y\in S$.

11. Adapt the proof of Theorem 8 to prove Theorem 9.

12. Use Dirichlet's test to show that the following integrals converge uniformly on $S=[\rho,\infty)$ if $\rho>0$:

(a) $\displaystyle\int_1^\infty \frac{\sin xy}{x^y}\,dx$ **(b)** $\displaystyle\int_2^\infty \frac{\sin xy}{\log x}\,dx$

(c) $\displaystyle\int_0^\infty \frac{\cos xy}{x+y^2}\,dx$ **(d)** $\displaystyle\int_1^\infty \frac{\sin xy}{1+xy}\,dx$

13. Suppose g, g_x and h are continuous on $[a,b)\times S$, and denote $H(x,y) = \int_a^x h(u,y)\,du$, $a \le x < b$. Suppose also that

$$\lim_{x\to b-}\left\{\sup_{y\in S}|g(x,y)H(x,y)|\right\} = 0 \quad \text{and} \quad \int_a^b g_x(x,y)H(x,y)\,dx$$

converges uniformly on S. Show that $\int_a^b g(x,y)h(x,y)\,dx$ converges uniformly on S.

14. Prove Theorem 10 for the case where $f = f(x,y)$ is continuous on $(a,b]\times[c,d]$.

15. Prove Theorem 11 for the case where $f = f(x,y)$ is continuous on $(a,b]\times[c,d]$.

16. Show that

$$C(y) = \int_{-\infty}^\infty f(x)\cos xy\,dx \quad \text{and} \quad S(y) = \int_{-\infty}^\infty f(x)\sin xy\,dx$$

are continuous on $(-\infty,\infty)$ if

$$\int_{-\infty}^\infty |f(x)|\,dx < \infty.$$

17. Suppose f is continuously differentiable on $[a,\infty)$, $\lim_{x\to\infty} f(x) = 0$, and

$$\int_a^\infty |f'(x)|\,dx < \infty.$$

Show that the functions

$$C(y) = \int_a^\infty f(x)\cos xy\,dx \quad \text{and} \quad S(y) = \int_a^\infty f(x)\sin xy\,dx$$

are continuous for all $y \ne 0$. Give an example showing that they need not be continuous at $y = 0$.

18. Evaluate $F(y)$ and use Theorem 11 to evaluate I:

(a) $\displaystyle F(y) = \int_0^\infty \frac{dx}{1+y^2x^2},\ y \ne 0;\quad I = \int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,dx,$ $a,b > 0$

(b) $F(y) = \int_0^\infty x^y \, dx, \; y > -1; \quad I = \int_0^\infty \frac{x^a - x^b}{\log x} \, dx, \quad a, b > -1$

(c) $F(y) = \int_0^\infty e^{-xy} \cos x \, dx, \quad y > 0$

$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos x \, dx, \quad a, b > 0$

(d) $F(y) = \int_0^\infty e^{-xy} \sin x \, dx, \quad y > 0$

$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin x \, dx, \quad a, b > 0$

(e) $F(y) = \int_0^\infty e^{-x} \sin xy \, dx; \; I = \int_0^\infty e^{-x} \frac{1 - \cos ax}{x} \, dx$

(f) $F(y) = \int_0^\infty e^{-x} \cos xy \, dx; \; I = \int_0^\infty e^{-x} \frac{\sin ax}{x} \, dx$

19. Use Theorem 11 to evaluate:

(a) $\int_0^1 (\log x)^n x^y \, dx, \quad y > -1, \quad n = 0, 1, 2, \ldots$

(b) $\int_0^\infty \frac{dx}{(x^2 + y)^{n+1}} \, dx, \quad y > 0, \quad n = 0, 1, 2, \ldots.$

(c) $\int_0^\infty x^{2n+1} e^{-yx^2} \, dx, \quad y > 0, \quad n = 0, 1, 2, \ldots$

(d) $\int_0^\infty x y^x \, dx, \quad 0 < y < 1.$

20. **(a)** Use Theorem 11 and integration by parts to show that

$$F(y) = \int_0^\infty e^{-x^2} \cos 2xy \, dx$$

satisfies

$$F' + 2yF = 0.$$

(b) Use part **(a)** to show that

$$F(y) = \frac{\sqrt{\pi}}{2} e^{-y^2}.$$

21. Show that

$$\int_0^\infty e^{-x^2} \sin 2xy \, dx = e^{-y^2} \int_0^y e^{u^2} \, du.$$

(Hint: See Exercise 20.)

22. State a condition implying that

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx$$

are n times differentiable on for all $y \neq 0$. (Your condition should imply the hypotheses of Exercise 16.)

23. Suppose f is continuously differentiable on $[a,\infty)$,

$$\int_a^\infty |(x^k f(x))'|\,dx < \infty, \quad 0 \le k \le n,$$

and $\lim_{x\to\infty} x^n f(x) = 0$. Show that if

$$C(y) = \int_a^\infty f(x)\cos xy\,dx \quad \text{and} \quad S(y) = \int_a^\infty f(x)\sin xy\,dx,$$

then

$$C^{(k)}(y) = \int_a^\infty x^k f(x)\cos xy\,dx \quad \text{and} \quad S^{(k)}(y) = \int_a^\infty x^k f(x)\sin xy\,dx,$$

$0 \le k \le n$.

24. Differentiating

$$F(y) = \int_1^\infty \cos\frac{y}{x}\,dx$$

under the integral sign yields

$$-\int_1^\infty \frac{1}{x}\sin\frac{y}{x}\,dx,$$

which converges uniformly on any finite interval. (Why?) Does this imply that F is differentiable for all y?

25. Show that Theorem 11 and induction imply Eq. (30).

26. Prove Theorem 12.

27. Show that if $F(s) = \int_0^\infty e^{-sx} f(x)\,dx$ converges for $s = s_0$, then it converges uniformly on $[s_0,\infty)$. (What's the difference between this and Theorem 13?)

28. Prove: If f is continuous on $[0,\infty)$ and $\int_0^\infty e^{-s_0x} f(x)\,dx$ converges, then

$$\lim_{s\to s_0+} \int_0^\infty e^{-sx} f(x)\,dx = \int_0^\infty e^{-s_0x} f(x)\,dx.$$

(Hint: See the proof of Theorem 4.5.12, p. 273.)

29. Under the assumptions of Exercise 28, show that

$$\lim_{s\to s_0+} \int_r^\infty e^{-sx} f(x)\,dx = \int_r^\infty e^{-s_0x} f(x)\,dx, \quad r > 0.$$

30. Suppose f is continuous on $[0, \infty)$ and

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

converges for $s = s_0$. Show that $\lim_{s\to\infty} F(s) = 0$. (Hint: Integrate by parts.)

31. **(a)** Starting from the result of Exercise 18**(d)**, let $b \to \infty$ and invoke Exercise 30 to evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, \quad a > 0.$$

(b) Use **(a)** and Exercise 28 to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

32. **(a)** Suppose f is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \le Me^{s_0 x}, \quad 0 \le x \le \infty.$$

Show that

$$G(s) = \int_0^\infty e^{-sx} f'(x)\, dx$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

(b) Show from part **(a)** that

$$G(s) = \int_0^\infty e^{-sx} xe^{x^2} \sin e^{x^2}\, dx$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem 6 or 8.)

33. Suppose f is continuous on $[0, \infty)$,

$$\lim_{x\to 0+} \frac{f(x)}{x}$$

exists, and

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

converges for $s = s_0$. Show that

$$\int_{s_0}^\infty F(u)\, du = \int_0^\infty e^{-s_0 x} \frac{f(x)}{x}\, dx.$$

10 Answers to selected exercises

5. **(b)** If $f(x, y) = 1/y$ for $y \neq 0$ and $f(x, 0) = 1$, then $\int_a^b f(x, y)\, dx$ does not converge uniformly on $[0, d]$ for any $d > 0$.

6. **(a)**, **(d)**, and **(e)** converge uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; **(b)**, **(c)**, and **(f)** converge uniformly on $[\rho, \infty)$ if $\rho > 0$.

17. Let $C(y) = \displaystyle\int_1^\infty \frac{\cos xy}{x}\, dx$ and $S(y) = \displaystyle\int_1^\infty \frac{\sin xy}{x}\, dx$. Then $C(0) = \infty$ and $S(0) = 0$, while $S(y) = \pi/2$ if $y \neq 0$.

18. **(a)** $F(y) = \dfrac{\pi}{2|y|}$; $\quad I = \dfrac{\pi}{2} \log \dfrac{a}{b}$ $\quad$ **(b)** $F(y) = \dfrac{1}{y+1}$; $\quad I = \log \dfrac{a+1}{b+1}$

(c) $F(y) = \dfrac{y}{y^2+1}$; $\quad I = \dfrac{1}{2} \dfrac{b^2+1}{a^2+1}$

(d) $F(y) = \dfrac{1}{y^2+1}$; $\quad I = \tan^{-1} b - \tan^{-1} a$

(e) $F(y) = \dfrac{y}{y^2+1}$; $\quad I = \dfrac{1}{2} \log(1 + a^2)$

(f) $F(y) = \dfrac{1}{y^2+1}$; $\quad I = \tan^{-1} a$

19. **(a)** $(-1)^n n!(y+1)^{-n-1}$ $\quad$ **(b)** $\pi 2^{-2n-1} \dbinom{2n}{n} y^{-n-1/2}$

(c) $\dfrac{n!}{2y^{n+1}} (\log y)^{-2}$ **(d)** $\dfrac{1}{(\log x)^2}$

22. $\displaystyle\int_{-\infty}^{\infty} |x^n f(x)|\, dx < \infty$

24. No; the integral defining F diverges for all y.

31. **(a)** $\dfrac{\pi}{2} - \tan^{-1} a$