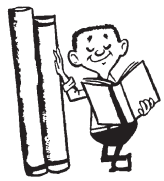
Chebyshev Polynomials



Reading **Problems**

Differential Equation and Its Solution

The Chebyshev differential equation is written as

$$(1 - x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2\,y = 0 \qquad\qquad n = 0, 1, 2, 3, \ldots$$

If we let $x = \cos t$ we obtain

$$\frac{d^2y}{dt^2} + n^2 y = 0$$

whose general solution is

$$y = A\cos nt + B\sin nt$$

or as

$$y = A\cos(n\cos^{-1} x) + B\sin(n\cos^{-1} x) \qquad\qquad |x| < 1$$

or equivalently

$$y = A\mathrm{T}_n(x) + B\mathrm{U}_n(x) \qquad\qquad |x| < 1$$

where $\mathrm{T}_n(x)$ and $\mathrm{U}_n(x)$ are defined as Chebyshev polynomials of the first and second kind of degree n, respectively.

If we let $x = \cosh t$ we obtain

$$\frac{d^2y}{dt^2} - n^2 y = 0$$

whose general solution is

$$y = A \cosh nt + B \sinh nt$$

or as

$$y = A \cosh(n \cosh^{-1} x) + B \sinh(n \cosh^{-1} x) \qquad\qquad |x| > 1$$

or equivalently

$$y = A\mathrm{T}_n(x) + B\mathrm{U}_n(x) \qquad\qquad |x| > 1$$

The function $\mathrm{T}_n(x)$ is a polynomial. For $|x| < 1$ we have

$$\begin{aligned}
\mathrm{T}_n(x) + i\mathrm{U}_n(x) &= (\cos t + i \sin t)^n = \left(x + i\sqrt{1 - x^2}\right)^n \\
\mathrm{T}_n(x) - i\mathrm{U}_n(x) &= (\cos t - i \sin t)^n = \left(x - i\sqrt{1 - x^2}\right)^n
\end{aligned}$$

from which we obtain

$$\mathrm{T}_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1 - x^2}\right)^n + \left(x - i\sqrt{1 - x^2}\right)^n\right]$$

For $|x| > 1$ we have

$$\begin{aligned}
\mathrm{T}_n(x) + \mathrm{U}_n(x) &= e^{nt} = \left(x \pm \sqrt{x^2 - 1}\right)^n \\
\mathrm{T}_n(x) - \mathrm{U}_n(x) &= e^{-nt} = \left(x \mp \sqrt{x^2 - 1}\right)^n
\end{aligned}$$

The sum of the last two relationships give the same result for $\mathrm{T}_n(x)$.

Chebyshev Polynomials of the First Kind of Degree n

The Chebyshev polynomials $\mathbf{T}_n(x)$ can be obtained by means of Rodrigue's formula

$$\mathbf{T}_n(x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\,\frac{d^n}{dx^n}\,(1-x^2)^{n-1/2} \qquad n = 0, 1, 2, 3, \ldots$$

The first twelve Chebyshev polynomials are listed in Table 1 and then as powers of x in terms of $\mathbf{T}_n(x)$ in Table 2.

Table 1: Chebyshev Polynomials of the First Kind

$$
\begin{aligned}
\mathbf{T}_0(x) &= 1 \\
\mathbf{T}_1(x) &= x \\
\mathbf{T}_2(x) &= 2x^2 - 1 \\
\mathbf{T}_3(x) &= 4x^3 - 3x \\
\mathbf{T}_4(x) &= 8x^4 - 8x^2 + 1 \\
\mathbf{T}_5(x) &= 16x^5 - 20x^3 + 5x \\
\mathbf{T}_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
\mathbf{T}_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
\mathbf{T}_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
\mathbf{T}_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
\mathbf{T}_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \\
\mathbf{T}_{11}(x) &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}
$$

Table 2: Powers of x as functions of $T_n(x)$

$$
\begin{aligned}
1 &= T_0 \\
x &= T_1 \\
x^2 &= \frac{1}{2}(T_0 + T_2) \\
x^3 &= \frac{1}{4}(3T_1 + T_3) \\
x^4 &= \frac{1}{8}(3T_0 + 4T_2 + T_4) \\
x^5 &= \frac{1}{16}(10T_1 + 5T_3 + T_5) \\
x^6 &= \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6) \\
x^7 &= \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7) \\
x^8 &= \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8) \\
x^9 &= \frac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9) \\
x^{10} &= \frac{1}{512}(126T_0 + 210T_2 + 120T_4 + 45T_6 + 10T_8 + T_{10}) \\
x^{11} &= \frac{1}{1024}(462T_1 + 330T_3 + 165T_5 + 55T_7 + 11T_9 + T_{11})
\end{aligned}
$$

Generating Function for $T_n(x)$

The Chebyshev polynomials of the first kind can be developed by means of the generating function

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n$$

Recurrence Formulas for $T_n(x)$

When the first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials $T_n(x),\ n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

The derivative of $T_n(x)$ with respect to x can be obtained from

$$(1 - x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x)$$

Special Values of $T_n(x)$

The following special values and properties of $T_n(x)$ are often useful:

$$T_n(-x) = (-1)^n T_n(x) \qquad T_{2n}(0) = (-1)^n$$

$$T_n(1) = 1 \qquad T_{2n+1}(0) = 0$$

$$T_n(-1) = (-1)^n$$

Orthogonality Property of $T_n(x)$

We can determine the orthogonality properties for the Chebyshev polynomials of the first kind from our knowledge of the orthogonality of the cosine functions, namely,

$$\int_0^\pi \cos(m\theta)\ \cos(n\ \theta)\ d\theta = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

Then substituting

$$\begin{aligned} T_n(x) &= \cos(n\theta) \\ \cos\theta &= x \end{aligned}$$

to obtain the orthogonality properties of the Chebyshev polynomials:

$$\int_{-1}^{1} \frac{T_m(x)\ T_n(x)\ dx}{\sqrt{1-x^2}} = \begin{cases} 0 & (m \neq n) \\ \pi/2 & (m = n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

We observe that the Chebyshev polynomials form an orthogonal set on the interval $-1 \leq x \leq 1$ with the weighting function $(1-x^2)^{-1/2}$

Orthogonal Series of Chebyshev Polynomials

An arbitrary function $f(x)$ which is continuous and single-valued, defined over the interval $-1 \leq x \leq 1$, can be expanded as a series of Chebyshev polynomials:

$$\begin{aligned} f(x) &= A_0T_0(x) + A_1T_1(x) + A_2T_2(x) + \ldots \\ &= \sum_{n=0}^{\infty} A_nT_n(x) \end{aligned}$$

where the coefficients A_n are given by

$$A_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)\ dx}{\sqrt{1-x^2}} \qquad\qquad n = 0$$

and

$$A_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\ T_n(x)\ dx}{\sqrt{1-x^2}} \qquad\qquad n = 1, 2, 3, \ldots$$

The following definite integrals are often useful in the series expansion of $f(x)$:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi \qquad\qquad \int_{-1}^{1} \frac{x^3\ dx}{\sqrt{1-x^2}} = 0$$

$$\int_{-1}^{1} \frac{x\ dx}{\sqrt{1-x^2}} = 0 \qquad\qquad \int_{-1}^{1} \frac{x^4\ dx}{\sqrt{1-x^2}} = \frac{3\pi}{8}$$

$$\int_{-1}^{1} \frac{x^2\ dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \qquad\qquad \int_{-1}^{1} \frac{x^5\ dx}{\sqrt{1-x^2}} = 0$$

Chebyshev Polynomials Over a Discrete Set of Points

A continuous function over a continuous interval is often replaced by a set of discrete values of the function at discrete points. It can be shown that the Chebyshev polynomials $T_n(x)$ are orthogonal over the following discrete set of $N+1$ points x_i, equally spaced on θ,

$$\theta_i = 0,\ \frac{\pi}{N},\ \frac{2\pi}{N}, \ldots (N-1)\ \frac{\pi}{N},\ \pi$$

where

$$x_i = \arccos\ \theta_i$$

We have

$$\frac{1}{2}T_m(-1)T_n(-1)+\sum_{i=2}^{N-1}T_m(x_i)T_n(x_i)+\frac{1}{2}T_m(1)T_n(1) = \begin{cases} 0 & (m \neq n) \\ N/2 & (m = n \neq 0) \\ N & (m = n = 0) \end{cases}$$

The $T_m(x)$ are also orthogonal over the following N points t_i equally spaced,

$$\theta_i = \frac{\pi}{2N}, \frac{3\pi}{2N}, \frac{5\pi}{2N}, \ldots, \frac{(2N-1)\pi}{2N}$$

and

$$t_i = \arccos \theta_i$$

$$\sum_{i=1}^{N} T_m(t_i)T_n(t_i) = \begin{cases} 0 & (m \neq n) \\ N/2 & (m = n \neq 0) \\ N & (m = n = 0) \end{cases}$$

The set of points t_i are clearly the midpoints in θ of the first case. The unequal spacing of the points in $x_i(Nt_i)$ compensates for the weight factor

$$W(x) = (1 - x^2)^{-1/2}$$

in the continuous case.

Additional Identities of Chebyshev Polynomials

The Chebyshev polynomials are both orthogonal polynomials and the trigonometric $\cos nx$ functions in disguise, therefore they satisfy a large number of useful relationships.

The differentiation and integration properties are very important in analytical and numerical work. We begin with

$$T_{n+1}(x) = \cos[(n+1)\ \cos^{-1} x]$$

and

$$T_{n-1}(x) = \cos[(n-1)\ \cos^{-1} x]$$

Differentiating both expressions gives

$$\frac{1}{(n+1)}\frac{d[T_{n+1}(x)]}{dx} = \frac{-\sin[(n+1)\ \cos^{-1} x}{-\sqrt{1-x^2}}$$

and

$$\frac{1}{(n-1)}\frac{d[T_{n-1}(x)]}{dx} = \frac{-\sin[(n-1)\ \cos^{-1} x}{-\sqrt{1-x^2}}$$

Subtracting the last two expressions yields

$$\frac{1}{(n+1)}\frac{d[T_{n+1}(x)]}{dx} - \frac{1}{(n-1)}\frac{d[T_{n-1}(x)]}{dx} = \frac{\sin(n+1)\theta - \sin(n-1)\theta}{\sin\theta}$$

or

$$\frac{T'_{n+1}(x)}{(n+1)} - \frac{T'_{n-1}(x)}{(n-1)} = \frac{2\cos n\theta\ \sin\theta}{\sin\theta} = 2T_n(x) \qquad n \geq 2$$

Therefore

$$
\begin{aligned}
T_2'(x) &= 4T_1 \\
T_1'(x) &= T_0 \\
T_0'(x) &= 0
\end{aligned}
$$

We have the formulas for the differentiation of Chebyshev polynomials, therefore these formulas can be used to develop integration for the Chebyshev polynomials:

$$
\begin{aligned}
\int T_n(x)dx &= \frac{1}{2}\left[\frac{T_{n+1}(x)}{(n+1)} - \frac{T_{n-1}(x)}{(n-1)}\right] + C \qquad n \geq 2 \\
\int T_1(x)dx &= \frac{1}{4}T_2(x) + C \\
\int T_0(x)dx &= T_1(x) + C
\end{aligned}
$$

The Shifted Chebyshev Polynomials

For analytical and numerical work it is often convenient to use the half interval $0 \leq x \leq 1$ instead of the full interval $-1 \leq x \leq 1$. For this purpose the shifted Chebyshev polynomials are defined:

$$T_n^*(x) = T_n * (2x - 1)$$

Thus we have for the first few polynomials

$$
\begin{aligned}
T_0^* &= 1 \\
T_1^* &= 2x - 1 \\
T_2^* &= 8x^2 - 8x + 1 \\
T_3^* &= 32x^3 - 48x^2 + 18x - 1 \\
T_4^* &= 128x^4 - 256x^3 + 160x^2 - 32x + 1
\end{aligned}
$$

and the following powers of x as functions of $T_n^*(x)$;

$$\begin{aligned}
1 &= T_0^* \\
x &= \frac{1}{2}(T_0^* + T_1^*) \\
x^2 &= \frac{1}{8}(3T_0^* + 4T_1^* + T_2^*) \\
x^3 &= \frac{1}{32}(10T_0^* + 15T_1^* + 6T_2^* + T_3^*) \\
x^4 &= \frac{1}{128}(35T_0^* + 56T_1^* + 28T_2^* + 8T_3^* + T_4^*)
\end{aligned}$$

The recurrence relationship for the shifted polynomials is:

$$T_{n+1}^*(x) = (4x - 2)T_n^*(x) - T_{n-1}^*(x) \qquad T_0^*(x) = 1$$

or

$$xT_n^*(x) = \frac{1}{4}T_{n+1}^*(x) + \frac{1}{2}T_n^*(x) + \frac{1}{4}T_{n-1}^*(x)$$

where

$$T_n^*(x) = \cos\left[n\cos^{-1}(2x-1)\right] = T_n(2x-1)$$

Expansion of x^n in a Series of $T_n(x)$

A method of expanding x^n in a series of Chebyshev polynomials employes the recurrence relation written as

$$\begin{aligned}
xT_n(x) &= \frac{1}{2}\left[T_{n+1}(x) + T_{n-1}(x)\right] \qquad n = 1, 2, 3 \ldots \\
xT_0(x) &= T_1(x)
\end{aligned}$$

To illustrate the method, consider x^4

$$
\begin{aligned}
x^4 &= x^2(xT_1) = \frac{x^2}{2}[T_2 + T_0] = \frac{x}{4}[T_1 + T_3 + 2T_1] \\
&= \frac{1}{4}[3xT_1 + xT_3] = \frac{1}{8}[3T_0 + 3T_2 + T_4 + T_2] \\
&= \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0
\end{aligned}
$$

This result is consistent with the expansion of x^4 given in Table 2.

Approximation of Functions by Chebyshev Polynomials

Sometimes when a function $f(x)$ is to be approximated by a polynomial of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + E_N(x) \qquad |x| \leq 1$$

where $|E_n(x)|$ does not exceed an allowed limit, it is possible to reduce the degree of the polynomial by a process called economization of power series. The procedure is to convert the polynomial to a linear combination of Chebyshev polynomials:

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} b_n T_n(x) \qquad n = 0, 1, 2, \ldots$$

It may be possible to drop some of the last terms without permitting the error to exceed the prescribed limit. Since $|T_n(x)| \leq 1$, the number of terms which can be omitted is determined by the magnitude of the coefficient b.

The Chebyshev polynomials are useful in numerical work for the interval $-1 \leq x \leq 1$ because

1. $|T_n(x)] \leq 1$ within $-1 \leq x \leq 1$
2. The maxima and minima are of comparable value.

3. The maxima and minima are spread reasonably uniformly over the interval $\boldsymbol{-1 \leq x \leq 1}$

4. All Chebyshev polynomials satisfy a three term recurrence relation.

5. They are easy to compute and to convert to and from a power series form.

These properties together produce an approximating polynomial which minimizes error in its application. This is different from the least squares approximation where the sum of the squares of the errors is minimized; the maximum error itself can be quite large. In the Chebyshev approximation, the average error can be large but the maximum error is minimized. Chebyshev approximations of a function are sometimes said to be mini-max approximations of the function.

The following table gives the Chebyshev polynomial approximation of several power series.

Table 3: Power Series and its Chebyshev Approximation

1. $f(x) = a_0$

   $f(x) = a_0 T_0$

2. $f(x) = a_0 + a_1 x$

   $f(x) = a_0 T_0 + a_1 T_1$

3. $f(x) = a_0 + a_1 x + a_2 x^2$

   $f(x) = \left(a_0 + \frac{a_2}{2}\right) T_0 + a_1 T_1 + \left(\frac{a_2}{2}\right) T_2$

4. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

   $f(x) = \left(a_0 + \frac{a_2}{2}\right) T_0 + \left(a_1 + \frac{3a_3}{4}\right) T_1 + \left(\frac{a_2}{2}\right) T_2 + \left(\frac{a_3}{4}\right) T_3$

5. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

   $$f(x) = \left(a_0 + \frac{a_2}{2} + \frac{a_3}{8}\right) T_0 + \left(a_1 + \frac{3a_3}{4}\right) T_1 + \left(\frac{a_2}{2} + \frac{a_4}{2}\right) T_2 + \left(\frac{a_3}{8}\right) T_3 + \left(\frac{a_4}{8}\right) T_4$$

6. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

   $$f(x) = \left(a_0 + \frac{a_2}{2} + \frac{3a_4}{8}\right) T_0 + \left(a_1 + \frac{3a_3}{4} + \frac{5a_5}{8}\right) T_1 + \left(\frac{a_2}{2} + \frac{a_4}{2}\right) T_2 + \left(\frac{a_3}{4} + \frac{5a_5}{16}\right) T_3 + \left(\frac{a_4}{8}\right) T_4 + \left(\frac{a_5}{16}\right) T_5$$

Table 4: Formulas for Economization of Power Series

$$
\begin{aligned}
x &= T_1 \\
x^2 &= \frac{1}{2}(1 + T_2) \\
x^3 &= \frac{1}{4}(3x + T_3) \\
x^4 &= \frac{1}{8}(8x^2 - 1 + T_4) \\
x^5 &= \frac{1}{16}(20x^3 - 5x + T_5) \\
x^6 &= \frac{1}{32}(48x^4 - 18x^2 + 1 + T_6) \\
x^7 &= \frac{1}{64}(112x^5 - 56x^3 + 7x + T_7) \\
x^8 &= \frac{1}{128}(256x^6 - 160x^4 + 32x^2 - 1 + T_8) \\
x^9 &= \frac{1}{256}(576x^7 - 432x^5 + 120x^3 - 9x + T_9) \\
x^{10} &= \frac{1}{512}(1280x^8 - 1120x^6 + 400x^4 - 50x^2 + 1 + T_{10}) \\
x^{11} &= \frac{1}{1024}(2816x^9 - 2816x^7 + 1232x^5 - 220x^3 + 11x + T_{11})
\end{aligned}
$$

For easy reference the formulas for economization of power series in terms of Chebyshev are given in Table 4.

Assigned Problems

Problem Set for Chebyshev Polynomials

1. Obtain the first three Chebyshev polynomials $T_0(x),\ T_1(x)$ and $T_2(x)$ by means of the Rodrigue's formula.

2. Show that the Chebyshev polynomial $T_3(x)$ is a solution of Chebyshev's equation of order 3.

3. By means of the recurrence formula obtain Chebyshev polynomials $T_2(x)$ and $T_3(x)$ given $T_0(x)$ and $T_1(x)$.

4. Show that $T_n(1) = 1$ and $T_n(-1) = (-1)^n$

5. Show that $T_n(0) = 0$ if n is odd and $(-1)^{n/2}$ if n is even.

6. Setting $x = \cos\theta$ show that

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right]$$

   where $i = \sqrt{-1}$.

7. Find the general solution of Chebyshev's equation for $n = 0$.

8. Obtain a series expansion for $f(x) = x^2$ in terms of Chebyshev polynomials $T_n(x)$,

$$x^2 = \sum_{n=0}^{3} A_n T_n(x)$$

9. Express x^4 as a sum of Chebyshev polynomials of the first kind.