Finance Stochast. 3, 391–412 (1999)

Finance and Stochastics

Applications of Malliavin calculus to Monte Carlo methods in finance

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Abstract. This paper presents an original probabilistic method for the numerical computations of Greeks (i.e. price sensitivities) in finance. Our approach is based on the *integration-by-parts* formula, which lies at the core of the theory of variational stochastic calculus, as developed in the Malliavin calculus. The Greeks formulae, both with respect to initial conditions and for smooth perturbations of the local volatility, are provided for general discontinuous path-dependent payoff functionals of multidimensional diffusion processes. We illustrate the results by applying the formula to exotic European options in the framework of the Black and Scholes model. Our method is compared to the Monte Carlo finite difference approach and turns out to be very efficient in the case of discontinuous payoff functionals.

Key words: Monte Carlo methods, Malliavin calculus, hedge ratios and Greeks

JEL classification : G13

Mathematics Subject Classification (1991):60H07, 60J60, 65C05

1 Introduction

In frictionless markets, the arbitrage price of most financial derivatives (European, Asian, etc. ...) can be expressed as expected values of the associated payoff which is usually defined as a functional of the underlying asset process.

Manuscript received: July 1997; final version received: September 1998

In this paper, we will show how one can use Malliavin calculus to devise efficient Monte Carlo methods for these expected values and their differentials. Other applications of Malliavin calculus for numerical figure and risk management will appear in companion papers.

In order to precise our goal, we need to introduce some mathematical notations. The underlying assets are assumed to be given by $\{X(t);\ 0 \le t \le T\}$ which is a markov process with values in $\mathbb{R}^n$ and whose dynamics are described by the stochastic differential equation

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t)\,, \tag{1}$$

where $\{W(t),\ 0 \le t \le T\}$ is a Brownian motion with values in $\mathbb{R}^n$. The coefficients b and σ are assumed to satisfy usual conditions in order to ensure the existence and uniqueness of a continuous adapted solution of equation (1).

Given $0 < t_1 \le \ldots \le t_m = T$, we consider the function

$$u(x) = \mathbf{E}\left[\,\phi(X(t_1),\ldots,X(t_m)) \mid X(0) = x\,\right]\,, \tag{2}$$

where ϕ satisfies some technical conditions to be described later on. In financial applications, $u(x)$ describes the price of a contingent claim defined by the payoff function ϕ involving the times $(t_1,\ \ldots\ ,t_m)$. Examples of such contingent claims include both usual and path dependent options and sophisticated objects such as MBS or CMO's. The function $u(x)$ can then be computed by Monte Carlo methods. However, financial applications require not only to compute the function $u(x)$ but also to compute its differentials with respect to the initial condition x, the drift coefficient b and the volatility coefficient σ.

A natural approach to this numerical problem is to compute by Monte Carlo simulation the finite difference approximation of the differentials. To simplify the discussion, let us specialize it to the case of the Delta, i.e. the derivative with respect to the initial condition x. Then, one has to compute a Monte Carlo estimator of $u(x)$ and a Monte Carlo estimator for $u(x+\varepsilon)$ for some small ε; the Delta is then estimated by $[u(x+\varepsilon) - u(x)]/\varepsilon$. If the simulations of the two estimators are drawn independently, then it is proved in Glynn (1989) that the best possible convergence rate is typically $n^{-1/4}$. Replacing the forward finite difference estimator by the central difference $[u(x+\varepsilon) - u(x-\varepsilon)]/(2\varepsilon)$ improves the optimal convergence rate to $n^{-1/3}$. However, by using common random numbers for both Monte Carlo estimators, one can achieve the convergence rate $n^{-1/2}$ which is the best that can be expected from (ordinary) Monte Carlo methods, see Glasserman and Yao (1992), Glynn (1989) and L'Ecuyer and Perron (1994). An important drawback of the common random numbers finite difference method is that it may perform very poorly when ϕ is not smooth enough, as for instance if one computes the delta of a digital or the gamma of European call options.

An alternative method which allows to achieve the $n^{-1/2}$ convergence rate is suggested by Broadie and Glasserman (1996) : for simple payoff functionals ϕ, an expectation representation of the Greek of interest can be obtained by simple differentiation inside the expectation operator; the resulting expectation is estimated by usual Monte Carlo methods. An important limitation of this method is that it can only be applied to simple payoff functionals.

In this paper, using Malliavin calculus we will show that all the differentials of interest can be expressed as

$$\mathbf{E}\left[\, \pi\, \phi(X(t_1), \ldots, X(t_m)) \;|\; X(0) = x \,\right], \tag{3}$$

where π is a random variable to be determined later on. Therefore, the required differential can be computed numerically by Monte Carlo simulation and the estimator achieves the $n^{-1/2}$ usual convergence rate. An important advantage of our differential formula is that the weight π does not depend on the payoff function ϕ.

While the aim of this paper is to design efficient numerical scheme, let us point out a theorical aspect of the formulations (3) that our use of Malliavin calculus leads us to set up. As it is well known, risk neutral probability is the technical tool by which one introduces observed market prices in a given model : this is done in practice through a calibration process, i.e. the computation of the Arrow-Debreu prices over future various states of the world. Hence asset prices can be written

$$price = \mathbf{E}_{Q_0}\left[\text{pay-offs}\right],$$

where *price* is today's value of the contingent claim, $\mathbf{E}_{Q_0}$ is the expected value under the risk neutral probability Q_0, and the discounted pay-offs are the future contingent cash amounts. Hedging is trying to protect the portfolio against at least some of the possible changes. But changes in market will come through the calibration process as changes of the risk neutral probability Q. So marginal changes of Q will lead to new prices according to

$$\begin{aligned} variation\ of\ prices &= new\ price - old\ price, \\ &= \mathbf{E}_Q\left[\text{pay-offs}\right] - \mathbf{E}_{Q_0}\left[\text{pay-offs}\right], \\ &= \mathbf{E}_{Q_0}\left[\text{pay-offs} \times \pi\right], \end{aligned}$$

where π is

$$\pi = \frac{dQ - dQ_0}{dQ_0}.$$

Now suppose that the probability Q lies within a parametrized family (Q_λ), $\lambda = (\lambda_1, \ldots, \lambda_n)$. In the typical diffusion case studied in this paper Q is parameterized by the drift and the volatility functions which may be specified in some parameterized family. Then the marginal moves of the market can be assessed through the derivatives

$$\frac{\partial}{\partial \lambda_i}(price) = \mathbf{E}_{Q_0}\left[\text{pay-offs} \times \pi_i\right]. \tag{4}$$

where $G = \frac{dQ}{dQ_0}$ and $\pi_i = \frac{\partial G}{\partial \lambda_i}$, i.e π_i is the logarithmic derivative of Q at Q_0 in the λ_i direction. Our use of Malliavin calculus helps to set up the formula (4) and other various formulas based on the various derivatives or primitives of the pay offs. But even if in many cases, it might be analytically easier to start with the formula of derivative before (4), our opinion is that (4) is likely to be a more fundamental hedging formula than other ones.

Finally let us observe that the case of stochastic interest rates is easily accomodated in the framework of this paper by working under the so-called forward measure or by extending the state space to include the additional state variable $\exp \int_0^t r(u)du$.

The paper is organized as follows. We first present in Sect. 2 a few basics of Malliavin calculus. Then, in Sect. 3, we derive the formulae for various differentials which correspond to the quantities called *greeks* in Finance. These cases have to be seen as an illustration of a general method which can be adapted and applied to all other pratical differentials. Finally, Sect. 4 is devoted to some numerical examples and further comments on the operational use of our method.

2 A primer of Malliavin calculus for finance

This section briefly reviews the Malliavin calculus and presents the efficient rules to use it in financial examples (see Nualart [9] for other expositions).

The Malliavin calculus defines the derivative of functions on Wiener space and can be seen as a theory of integration by parts on this space. Thanks to the Malliavin calculus, we can compute the derivatives of a large set of random variables and processes (adapted or not to the filtration) defined on the Wiener space. We present the following notations which shall be used in the rest of the paper.

Let $\{W(t),\, 0 \leq t \leq T\}$ be a n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and we shall denote by $\{\mathscr{F}_t\}$ the augmentation with respect to P of the filtration generated by W. Let $\mathscr{C}$ be the set of random variables F of the form :

$$F = f\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right), \qquad f \in \mathscr{S}(\mathbb{R}^n)$$

where $\mathscr{S}(\mathbb{R}^n)$ denotes the set of infinitly differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and $h_1, \ldots, h_n \in L^2(\Omega \times \mathbb{R}_+)$. For $F \in \mathscr{C}$, the Malliavin derivative DF of F is defined as the process $\{D_t F,\, t \geq 0\}$ of $L^2(\Omega \times \mathbb{R}_+)$ with values in $L^2(\mathbb{R}_+)$ which we denote by H :

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^\infty h_1(t)dW(t), \ldots, \int_0^\infty h_n(t)dW(t)\right) h_i(t), \quad t \geq 0 \; a.s.$$

We also define the norm on $\mathscr{C}$

$$\|F\|_{1,2} = \left(\mathbf{E}(F^2)\right)^{1/2} + \left(\mathbf{E}(\int_0^\infty (D_t F)^2 dt)\right)^{1/2},$$

Then $\mathbb{D}^{1,2}$ denotes the banach space which is the completion of $\mathscr{C}$ with respect to the norm $\| \ \|_{1,2}$. The derivative operator D (also called the gradient operator) is a closed linear operator defined in $\mathbb{D}^{1,2}$ and its values are in $L^2(\Omega \times \mathbb{R}_+)$.

The next result is the chain rule for the derivation.

Property P1. *Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives and $F = (F_1, \ldots, F_n)$ a random vector whose components belong to $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and :*

$$D_t \phi(F) \quad = \quad \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) D_t F_i \,, \qquad t \geq 0 \ \ a.s.$$

In the case of a Markov diffusion process, the Malliavin derivative operator is closely related to the derivative of the process with respect to the initial condition.

Property P2. *Let $\{X(t), t \geq 0\}$ be an $\mathbb{R}^n$ valued Itô process whose dynamics are driven by the stochastic differential equation :*

$$dX(t) \quad = \quad b(X(t))\, dt + \sigma(X(t))\, dW(t) \ ,$$

where b and σ are supposed to be continuously differentiable functions with bounded derivatives. Let $\{Y(t), t \geq 0\}$ be the associated first variation process defined by the stochastic differential equation :

$$dY(t) \quad = \quad b'(X(t))\, Y(t)\, dt \ + \sum_{i=1}^{n} \sigma_i'(X(t))\, Y(t)\, dW^i(t), \qquad Y(0) \ = \ I_n,$$

where I_n is the identity matrix of $\mathbb{R}^n$, primes denote derivatives and σ_i is the i-th column vector of σ. Then the process $\{X(t), t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by :

$$D_s X(t) \quad = \quad Y(t) Y(s)^{-1} \sigma(X(s)) 1_{\{s \leq t\}}, \qquad s \geq 0 \ \ a.s.$$

Hence, if $\psi \in C_b^1(\mathbb{R}^n)$ then we have

$$D_s \psi(X_T) \ = \ \nabla\psi(X_T) Y(T) Y(s)^{-1} \sigma(X(s)) 1_{\{s \leq T\}}, \qquad s \geq 0 \ \ a.s.$$

and also

$$D_s \int_0^T \psi(X_t)\, dt \ = \ \int_s^T \nabla\psi(X_t) Y(t) Y(s)^{-1} \sigma(X(s))\, dt \qquad a.s.$$

The divergence operator δ (also called Skorohod integral) associated with the gradient operator D exists. The following integration by parts formula defines this divergence operator.

Property P3. *Let u be a stochastic process. Then $u \in Dom(\delta)$ if for any $\phi \in \mathbb{D}^{1,2}$, we have*

$$\mathbf{E}(< D\phi, u >_H) \;:=\; \mathbf{E}(\int_0^\infty D_t\phi \; u(t)\, dt) \;\le\; C(u)\, \|\phi\|_{1,2}.$$

If $u \in Dom(\delta)$, we define $\delta(u)$ by:

$$\mathbf{E}(\phi\, \delta(u)) \;=\; \mathbf{E}(< D\phi, u >_H) \qquad \textit{for any } \phi \in \mathbb{D}^{1,2}\ .$$

The stochastic process u is said to be *Skorohod integrable* if $u \in Dom(\delta)$. One of the most important properties of the divergence operator δ is that its domain $Dom(\delta)$ contains all adapted stochastic processes which belong to $L^2(\Omega \times \mathbb{R}_+)$; for such processes, the divergence operator δ coincides with the Itô stochastic integral.

Property P4. *Let u be an adapted stochastic process in $L^2(\Omega \times \mathbb{R}_+)$. Then, we have*:

$$\delta(u) \;=\; \int_0^\infty [u(t)]^*\, dW(t)\ ,$$

Moreover, if the random variable F is $\mathcal{F}_T-$adapted and belongs to $\mathbb{D}^{1,2}$ then for any u in $dom(\delta)$, the random variable Fu will be Skohorod integrable. We have the following property.

Property P5. *Let F be an $\mathcal{F}_T-$adapted random variable which belongs to $\mathbb{D}^{1,2}$ then for any u in $dom(\delta)$ we have*:

$$\delta(Fu) \;=\; F\,\delta(u) \;-\; \int_0^T D_tF\;\, u(t)\, dt.$$

Finally, we recall the Clark-Ocone-Haussman formula.

Property P6. *Let F be a random variable which belongs to $\mathbb{D}^{1,2}$. Then we have*

$$F \;=\; \mathbf{E}(F) \;+\; \int_0^T \mathbf{E}(D_tF \mid \mathcal{F}_t)\, dW(t) \qquad a.s.$$

The latter property shows that the Malliavin derivative provides an identification of the integrator in the (local) martingale representation Theorem in a Brownian filtration framework, which plays a central role in financial mathematics. Therefore, in frictionless markets, the hedging portfolio is naturally related to the Malliavin derivative of the terminal payoff.

3 Greeks

We assume that the drift and diffusion coefficients b and σ of the diffusion process $\{X(t),\ 0 \le t \le T\}$ are continuously differentiable functions with bounded Lipschitz derivatives in order to ensure the existence of a unique strong solution. Under the above assumptions on the coefficients b and σ and using the theory of stochastic flows, we may choose versions of $\{X(t),\ 0 \le t \le T\}$ which are continuously differentiable with respect to the initial condition x for each $(t,\omega) \in [0,T] \times \Omega$ (see e.g. Protter 1990, Theorem 39 p250). We denote by $\{Y(t), 0 \le t \le T\}$ the first variation process associated to $\{X(t), 0 \le t \le T\}$ defined by the stochastic differential equation :

$$Y(0) = I_n \tag{1}$$

$$dY(t) = b'(X(t))Y(t)dt + \sum_{i=1}^{n} \sigma_i'(X(t))Y(t)dW_i(t) \tag{2}$$

where I_n is the identity matrix of $\mathbb{R}^n$, the primes denote derivatives and σ_i is the i-th column of σ. Moreover, we need another technical assumption.

Assumption 3.1 *The diffusion matrix σ satisfies the uniform ellipticity condition :*

$$\exists \varepsilon > 0, \quad \xi^*\sigma^*(x)\sigma(x)\xi \;\ge\; \varepsilon \mid \xi \mid^2 \quad \textit{for any } \; \xi, x \in \mathbb{R}^n .$$

Since b' and σ' are assumed to be Lipschitz and bounded, the first variation process lies in $L^2(\Omega \times [0,T])$, see e.g. Karatzas and Shreve (1988) Theorem 2.9 p289, and therefore Assumption 3.1 insures that the process $\left\{\sigma^{-1}(X(t))Y(t), 0 \le t \le T\right\}$ belongs to $L^2(\Omega \times [0,T])$. Moreover, if the function γ is bounded then the process $\{\sigma^{-1}\gamma(X(t)), 0 \le t \le T\}$ will belong to $L^2(\Omega \times [0,T])$ and $\sigma^{-1}\gamma$ is a bounded function.

3.1 Variations in the drift coefficient

In this section, we allow the payoff function ϕ to depend on the whole sample path of the process $\{X(t), 0 \le t \le T\}$. More precisely, let ϕ be some function mapping the set $C[0,T]$ of continuous functions on the interval $[0,T]$ into $\mathbb{R}$ and satisfying

$$\mathbf{E}\left[\phi(X(.))^2\right] < \infty . \tag{3}$$

Next, consider the perturbed process $\{X^\varepsilon(t), 0 \le t \le T\}$ defined by

$$dX^\varepsilon(t) = \left[b(X^\varepsilon(t)) + \varepsilon\gamma(X^\varepsilon(t))\right] + \sigma(X^\varepsilon(t))dW(t) , \tag{4}$$

where ε is a small real parameter and γ is a bounded function from $[0,T] \times \mathbb{R}^n$ into R^n. To simplify notations, we shall denote by $\{X(t), 0 \le t \le T\}$ the non-perturbed process corresponding to $\varepsilon = 0$. We also introduce the random variable

$$Z^\varepsilon(T) = \exp\left[-\varepsilon \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) > - \frac{\varepsilon^2}{2} \int_0^T \|\sigma^{-1}\gamma(X(t))\|^2 dt\right] . \tag{5}$$

From the boundedness of $\sigma^{-1}\gamma$, we have that $\mathbf{E}[Z^\varepsilon(T)] = 1$ for any $\varepsilon > 0$ since the Novikov condition is trivially satisfied. Now, consider the expectation

$$u^\varepsilon(x) = \mathbf{E}\left[\phi(X^\varepsilon(.)) | X^\varepsilon(0) = x\right] . \tag{6}$$

The following result then gives an expression of the derivative of $u^\varepsilon(x)$ with respect to ε in $\varepsilon = 0$.

Proposition 3.1 *The function $\varepsilon \mapsto u^\varepsilon(x)$ is differentiable in $\varepsilon = 0$, for any $x \in I\!R^n$, and we have :*

$$\frac{\partial}{\partial \varepsilon} u^\varepsilon(x)\Big|_{\varepsilon=0} \quad = \quad \mathbf{E}\left[\phi(X(.)) \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) > \;\Big|\; X(0) = x\right].$$

Proof. Since $\mathbf{E}[Z^\varepsilon(T)] = 1$, the probability measure Q^ε defined by its Radon-Nikodym derivative $dQ^\varepsilon/dP = Z^\varepsilon(T)$ is equivalent to P and we have :

$$u^\varepsilon(x) \quad = \quad \mathbf{E}^{Q^\varepsilon}\left[\tilde{Z}^\varepsilon(T)\phi(X^\varepsilon(.)) \;\big|\; X^\varepsilon(0) = x\right],$$

where $\tilde{Z}^\varepsilon(T) = \exp\left[-\varepsilon \int_0^T <\sigma^{-1}\gamma(X^\varepsilon(t)), dW^\varepsilon(t)> - \frac{\varepsilon^2}{2}\int_0^T \|\sigma^{-1}\gamma(X^\varepsilon(t))\|^2 dt\right]$ and $\{W^\varepsilon(t),\ 0 \le t \le T\}$ is defined by $W^\varepsilon(t) = W(t) + \varepsilon \int_0^t \sigma^{-1}\gamma(X^\varepsilon(t))dt$ is a Brownian motion under Q^ε, by the Girsanov Theorem. Let us observe that the joint distribution of $(X^\varepsilon(.), W^\varepsilon(.))$ under Q^ε coincides with the joint distribution of $(X(.), W(.))$ under P and therefore :

$$u^\varepsilon(x) \quad = \quad \mathbf{E}\left[Z^\varepsilon(T)\phi(X(.)) \mid X(0) = x\right].$$

Now, let us notice that we have

$$\frac{1}{\varepsilon}(Z^\varepsilon(T) - 1) \quad = \quad \int_0^T Z^\varepsilon(t) < \sigma^{-1}\gamma(X(t)), dW(t) >$$

so that

$$\frac{1}{\varepsilon}(Z^\varepsilon(T) - 1) \quad \longrightarrow \quad \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) > \qquad \text{in } L^2.$$

Therefore, by the Cauchy-Schwarz inequality and using (3), we get :

$$\begin{aligned} &\left|\frac{1}{\varepsilon}(u^\varepsilon(x) - u(x)) - \mathbf{E}\left[\phi(X(.)) \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) >\right]\right| \\ &\le\ K\mathbf{E}\left[\left(\frac{1}{\varepsilon}(Z^\varepsilon(T) - 1) - \int_0^T < \sigma^{-1}\gamma(X(t)), dW(t) >\right)^2\right] \end{aligned}$$

where K is a constant. This provides the required result. □

Remark 3.1 The same kind of arguments as in the previous proof can be used to obtain similar expressions for higher derivatives of the expectation $u^\varepsilon(x)$ with respect to ε in $\varepsilon = 0$ as a weighted expectation of the same functional; the weights being independent of the payoff functional.

Remark 3.2 The result of Proposition 3.1 does not require the Markov feature of the process $\{X(t),\ 0 \le t \le T\}$. The arguments of the proof go on even if b, σ and γ are adapted processes.

3.2 Variations in the initial condition

In this section, we provide an expression of the derivatives of the expectation $u(x)$ with respect to the initial condition x in the form of a weighted expectation of the same functional. The payoff function ϕ is now a mapping from $(\mathbb{R}^n)^m$ into $\mathbb{R}$ with

$$E\left[\phi\left(X(t_1),\ldots,X(t_m)\right)^2\right] \quad < \quad \infty.$$

for a given integer $m \geq 1$ and $0 < t_1 \leq \ldots \leq t_m \leq T$, where $\mathbf{E}^x(.) = \mathbf{E}(.|X(0) = x)$. The expectation of interest is

$$u(x) \quad = \quad \mathbf{E}^x\left[\phi(X(t_1),\ldots,X(t_m))\right], \tag{7}$$

We shall denote by ∇_i the partial derivative with respect to the i-th argument and we introduce the set Γ_m defined by :

$$\Gamma_m \;=\; \left\{a \;\in\; L^2([0,T]) \;\mid\; \int_0^{t_i} a(t)\,dt \;=\; 1 \qquad \forall i = 1,\ldots,m\right\}$$

Proposition 3.2 *Under Assumption 3.1, for any $x \in \mathbb{R}^n$ and for any $a \in \Gamma_m$, we have :*

$$\nabla u(x) \quad = \quad \mathbf{E}^x\left[\phi(X(t_1),\ldots,X(t_m))\int_0^T a(t)[\sigma^{-1}(X(t))Y(t)]^* dW(t)\right]. \tag{8}$$

Proof. (i) Assume that ϕ is continuously differentiable with bounded gradient; the general case will be proved in (ii) by density argument. We first prove that the derivative of $u(x)$ with respect to x is obtained by differentiating inside the expectation operator. Indeed, since ϕ is continuously differentiable, we have that

$$\begin{aligned}&\tfrac{1}{\|h\|}\left[\phi(X^x(t_1),\ldots,X^x(t_m)) - \phi(X^{x+h}(t_1),\ldots,X^{x+h}(t_m))\right]\\ &\qquad -\tfrac{1}{\|h\|}\left\langle \textstyle\sum_{i=1}^m \nabla_i^*\phi(X(t_1),\ldots,X(t_m))Y(t_i),h\right\rangle\end{aligned} \tag{9}$$

converges to zero a.s. as h goes to zero. The second term of the last expression is uniformly integrable in h since the partial derivatives of the payoff function ϕ are assumed to be bounded. Denoting by ψ_h the first term, it is easily seen that :

$$\|\psi_h\| \quad \leq \quad M\sum_{j=1}^k \frac{\left\|X^x(t_j) - X^{x+h}(t_j)\right\|}{\|h\|},$$

where M is a uniform bound on the partial derivatives of ϕ. The uniform integrability of the right hand side term of the last inequality follows from Protter (1990, p246) and implies the uniform integrability of (9) which then converges to zero in the sense of the $L^1(\Omega)$ norm, by the dominated convergence Theorem. This proves that :

$$\nabla^* u(x) \quad = \quad \mathbf{E}^x\left[\sum_{i=1}^m \nabla_i^*\phi(X(t_1),\ldots,X(t_m))Y(t_i)\right].$$

Now, by Property P2, the process $\{X(t);\, 0 \leq t \leq T\}$ belongs to $\mathbb{D}^{1,2}$. Besides, one can easily check that for all $i \in \{1, \ldots, m\}$ and for all $t \in [0, T]$ we have $D_t X(t_i) = Y(t_j)Y(t)^{-1}\sigma(t)\ 1_{\{t \leq t_i\}}$. This shows that :

$$Y(t_i) = \int_0^T D_t X(t_i)\, a(t)\sigma^{-1}(t)Y(t)\, dt \qquad \forall a \in \Gamma_m$$

$$\nabla^* u(x) = \mathbf{E}^x \left[\int_0^T \sum_{i=1}^m \nabla_i^* \phi(X(t_1), \ldots, X(t_m)) D_t X(t_i)\, a(t)\sigma^{-1}(t)Y(t) dt \right]$$

and by the chain rule Property P2, we obtain :

$$\nabla^* u(x) = \mathbf{E}^x \left[\int_0^T D_t \phi(X(t_1), \ldots, X(t_m)) a(t)\sigma^{-1}(t)Y(t)\, dt \right]$$

Now, for a function a fixed in Γ_m, we define the $\{\mathscr{F}(t)\}$ adapted process $\{v(t),\, 0 \leq t \leq T\}$ by :

$$v(t) = a(t)\sigma^{-1}(X(t))Y(t),$$

which belongs to $L^2(\Omega \times [0, T])$ by Assumption 3.1. Then,

$$\nabla^* u(x) = \mathbf{E}^x \left[\int_0^T D_t \phi(X(t_1), \ldots, X(t_m))\, v(t) dt \right]$$

and the result follows from a direct application of the Malliavin integration by parts, see Property P3.

(ii) We now consider the general case $\phi \in L^2$. Since the set C_K^∞ of infinitely differentiable functions with compact support is dense in L^2, there exists a sequence $(\phi_n)_n \subset C_K^\infty$ converging to ϕ in L^2. Let $u_n(x) = \mathbf{E}\left[\phi_n(X(t_1), \ldots, X(t_m))\right]$ and

$$\varepsilon_n(x) = \mathbf{E}\left[\phi_n(X(t_1), \ldots, X(t_m)) - \phi(X(t_1), \ldots, X(t_m))\right]^2 .$$

First it is clear that

$$u_n(x) \longrightarrow u(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{10}$$

Next denote by $g(x)$ the function on the right hand-side of (8). Applying (i) to function ϕ_n and using Cauchy-Schwartz inequality, we see that :

$$|\nabla u_n(x) - g(x)| \leq \varepsilon_n(x)\psi(x),$$

where $\psi(x) = \mathbf{E}\left[\int_0^T a(t)[\sigma^{-1}(X(t))Y(t)]^* dW(t)\right]^2$. By a continuity argument of the expectation operator, this proves that :

$$\sup_{x \in K} |\nabla u_n(x) - g(x)| \leq \varepsilon_n(\hat{x})\psi(\hat{x}) \quad \text{for some } \hat{x} \in K,$$

where K is an arbitrary compact subset of $\mathbb{R}^n$ which provides :

$$\nabla u_n(x) \quad \longrightarrow \quad g(x) \quad \text{uniformly on compact subsets of } \mathbb{R}^n. \qquad (11)$$

From (10) and (11), we can conclude that function u is continuously differentiable and that $\nabla u = g$.

□

3.3 Variations in the diffusion coefficient

In this section, we provide an expression of the derivatives of the expectation $u(x)$ with respect to the diffusion coefficient σ in the form of a weighted expectation of the same functional. As in the previous section, the coefficients b and σ defining the diffusion process $\{X(t),\, 0 \le t \le T\}$ are assumed to be continuously differentiable and with bounded derivatives. Also, the payoff function is assumed to be path dependent and has finite L^2 norm. We start by introducing the set of deterministic functions

$$\tilde{\Gamma}_m = \left\{ a \in L^2([0,T]) \mid \int_{t_{i-1}}^{t_i} a(t)\,dt = 1, \qquad \text{for } i = 1\ldots m \right\}.$$

Let $\tilde{\sigma} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times n}$ be continuously differentiable function with bounded derivatives. The function σ and the function $\tilde{\sigma}$ are assumed to satisfy the following condition.

Assumption 3.2 *The diffusion matrix $\sigma + \varepsilon\tilde{\sigma}$ satisfies the uniform ellipticity condition for any ε :*

$$\exists \eta > 0, \quad \xi^*(\sigma + \varepsilon\tilde{\sigma})^*(x)(\sigma + \varepsilon\tilde{\sigma})(x)\xi \;\ge\; \eta \mid \xi \mid^2 \quad \textit{for any } \xi, x \in \mathbb{R}^n.$$

In order to evaluate the Gâteaux derivative of the expectation $u(x)$ with respect to the diffusion matrix σ in the direction $\tilde{\sigma}$, we consider the process $\{X^\varepsilon(t),\, 0 \le t \le T\}$ defined by :

$$\begin{aligned} X^\varepsilon(0) &= x \\ dX^\varepsilon(t) &= b(X^\varepsilon(t))dt + \left[\sigma(X^\varepsilon(t)) + \varepsilon\tilde{\sigma}(X^\varepsilon(t))\right] dW(t). \end{aligned} \qquad (12)$$

We also introduce the $\mathbb{R}^n$ valued variation process of the process with respect to ε :

$$\begin{aligned} Z^\varepsilon(0) &= 0_n \\ dZ^\varepsilon(t) &= b'(X^\varepsilon(t))Z^\varepsilon(t)dt + \tilde{\sigma}(X^\varepsilon(t))dW(t) \\ &\quad + \sum_{i=1}^n [\sigma_i' + \varepsilon\tilde{\sigma}_i'](X^\varepsilon(t))Z^\varepsilon(t)dW_i(t), \end{aligned} \qquad (13)$$

where 0_n is the zero column vector of $\mathbb{R}^n$. As in the previous section, we simply use the notation $X(t)$, $Y(t)$ and $Z(t)$ for $X^0(t)$, $Y^0(t)$ and $Z^0(t)$. Next, consider the process $\{\beta(t),\, 0 \le t \le T\}$ defined by :

$$\beta(t) \quad = \quad Z(t)Y^{-1}(t), \qquad 0 \le t \le T \quad a.s. \tag{14}$$

This process satisfies the following regularity assumption.

Lemma 3.1 *The process $\{\beta(t);\, 0 \le t \le T\}$ belongs to $\mathbb{D}^{1,2}$.*

The process $\{Y^{-1}(t);\, 0 \le t \le T\}$ satisfies

$$\begin{aligned} Y^{-1}(0) &= I_n \\ dY^{-1}(t) &= Y^{-1}(t)\left[-b'(X(t)) + \sum_{i=1}^{n}\left[\sigma_i'(X(t))\right]^2\right]dt \\ &\quad -Y^{-1}(t)\sum_{i=1}^{n}\sigma_i'(X(t))dW^i(t). \end{aligned}$$

By Lemma 2.2.2 p104 in Nualart [9], the process $\{Y^{-1}(t);\, 0 \le t \le T\}$ belongs to $\mathbb{D}^{1,2}$. We also prove by the same argument that the process $\{Z(t);\, 0 \le t \le T\}$ is in $\mathbb{D}^{1,2}$. The required result follows from a direct application of the Cauchy-Schwarz inequality.

Proposition 3.3 *Under Assumption 3.2, for any a in $\tilde{\Gamma}_m$ we have :*

$$\frac{\partial}{\partial\varepsilon}u^\varepsilon(x)\bigg|_{\varepsilon=0} \quad = \quad E^x\left[\phi(X(t_1),\ldots,X(t_m))\delta\left(\sigma^{-1}(X)Y\tilde{\beta}_a(T)\right)\right]$$

where

$$\tilde{\beta}_a(t) \;=\; \sum_{i=1}^{m} a(t)\,(\beta(t_i) - \beta(t_{i-1}))\; 1_{\{t_{i-1}\le t\le t_i\}}$$

and where $\delta\left(\sigma^{-1}(X)Y\tilde{\beta}_a(T)\right)$ is the Skorohod integral of the anticipating process

$$\left\{\sigma^{-1}(X(t))Y(t)\tilde{\beta}_a(T)\,;\quad 0 \le t \le T\right\}\;.$$

Proof. Proceeding as in the proof of Proposition 3.2, it is clear that it suffices to prove the result for continuously differentiable function ϕ with bounded derivative; the general result follows from a density argument as in (ii) of the proof of Proposition 3.2. We first prove that the derivative of $u^\varepsilon(x)$ with respect to ε is obtained by differentiating inside the expectation operator. Considering ε as a degenerate process, we can apply Theorem 39 p250 in Protter (1990) which ensures that we can choose versions of $\{X^\varepsilon(t),\, 0 \le t \le T\}$ which are continuously differentiable with respect to ε for each $(t,\omega) \in [0,T]\times\Omega$. Since ϕ is continuously differentiable, we prove by the same arguments that we have in the sense of the L^1 norm:

$$\frac{\partial}{\partial\varepsilon}u^\varepsilon(x)\bigg|_{\varepsilon=0} \quad = \quad E^x\left[\sum_{i=1}^{m}\nabla_i^*\phi(X(t_1),\ldots,X(t_m))Z(t_i)\right]. \tag{15}$$

Using Property P2, we have $D_tX(t_i) = Y(t_j)Y(t)^{-1}\sigma(t)\; 1_{\{t\le t_i\}}$ for any $i \in \{1,\ldots,m\}$ and for any $t \in [0,T]$. Hence, for all $i \in \{1,\ldots,m\}$ we have

$$\int_0^T D_t X(t_i)\sigma^{-1}(t)Y(t)\tilde{\beta}_a(T)\,dt = \int_0^{t_i} Y(t_i)\tilde{\beta}_a(T)\,dt \tag{16}$$
$$= Y(t_i)\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k} a(t)\,(\beta(t_k)-\beta(t_{k-1}))\,dt\right)$$

Since a belongs to $\tilde{\Gamma}_m$, the right-hand side of (16) can be simplified in $Y(t_i)\beta(t_i)$ which is equal to $Z(t_i)$ according to the definition (14). This shows that :

$$\left.\frac{\partial}{\partial\varepsilon}u^{\varepsilon}(x)\right|_{\varepsilon=0} = E^x\left[\int_0^T\sum_{i=1}^{m}\nabla_i^*\phi(X)D_tX(t_i)\sigma^{-1}(X(t))Y(t)\tilde{\beta}_a(T)dt\right] \tag{17}$$

Using again Property P2, the expression (17) of the derivative of the expectation $u(x)$ can be rewritten in

$$\left.\frac{\partial}{\partial\varepsilon}u^{\varepsilon}(x)\right|_{\varepsilon=0} = \mathbf{E}^x\left[\int_0^T D_t\phi(X(t_1),\ldots,X(t_m))\,\sigma^{-1}(X(t))Y(t)\tilde{\beta}_a(T)\,dt\right]$$

Finally, we define the $\{\mathscr{F}_T\}$ adapted process $\{u(t),\ 0\le t\le T\}$ by :

$$v(t) = \sigma^{-1}(X(t))Y(t)\tilde{\beta}_a(T),$$

Since the process $\{\sigma^{-1}(X(t))Y(t);\ 0\le t\le T\}$ belongs to $L^2(\Omega\times[0,T])$ and is $\{\mathscr{F}_t\}$ adapted and since we have proved in Lemma 3.1 that $\tilde{\beta}_a(T)$ is in $\mathbb{D}^{1,2}$ (recall that a is a deterministic function) and is $\{\mathscr{F}_T\}$ adapted, we can apply the Property P5. It follows that the Skorohod integral of the product process v exists. More precisely, we have

$$\delta(v) = \tilde{\beta}_a(T)\int_0^T[\sigma^{-1}(X(t))Y(t)]^*\,dW(t) - \int_0^T D_t\tilde{\beta}_a(T)\sigma^{-1}(X(t))Y(t)\,dt$$

Then, we can apply the Malliavin integration by parts property to obtain the required result. □

Remark 3.3 The same kind of arguments as in the proof of Proposition 3.3 (resp. Proposition 3.2) can be used to obtain similar expressions for higher derivatives of the expectation u with respect to ε in $\varepsilon = 0$ (resp. with respect to the initial condition) as a weighted expectation of the same functional; the weights being independent of the payoff functional.

Remark 3.4 We can also extend our results to the case of a payoff function ϕ which is a function of the mean value of the process $\{X(t);\ 0\le t\le T\}$. We give the formula for the derivative with respect to the initial condition in dimension one. The function u is defined by

$$u(x) = \mathbf{E}^x\left[\phi\left(\int_0^T X(t)dt\right)\right]$$

In this case, we have

$$u'(x) = \mathbf{E}^x \left[\phi \left(\int_0^T X(t)dt \right) \delta \left(\frac{2Y^2(t)}{\sigma(X(t))} \left(\int_0^T Y(s)ds \right)^{-1} \right) \right]$$

4 Numerical experiments

This section presents some simple examples which illustrate the results obtained in the previous sections.

We consider the famous Black and Scholes model, i.e. a one dimensional market model which consists of a risky asset S and a non-risky one with deterministic instantaneous interest rate $r(t)$. Let $\left(\Omega, \mathscr{F}, Q, (\mathscr{F}_t), (\tilde{W}_t)\right)$ be a standard Wiener process on $\mathbb{R}$. Then, it is well known, under mild conditions on the coefficients of the SDE driving the price process, that there exists a unique equivalent probability measure P such that the $P-$dynamic of the price process is

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma\,dW(t), \quad S_0 = x. \tag{18}$$

In this framework, most problems of pricing contingent claims are solved by computing the following mathematical expectation :

$$u(0,x) = \mathbf{E}[e^{-\int_0^T r(t)\,dt}\phi(S^{0,x}(T))] \tag{19}$$

where ϕ is a payoff functional.

In practice, the hedging of the contingent claim requires the computation of the Greeks, i.e. the derivatives of the value function u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial \sigma}$, etc. By using the general formulae developed in the previous section, we are able to compute analytically the values of the different Greeks without differentiating neither the value function nor the payoff functional.

In this Black and Scholes framework, the tangent process Y follows, $P-$a.s., the stochastic differential equation

$$dY_t = r(t)Y_t\,dt + \sigma Y_t\,dW_t, \quad Y_0 = 1$$

and so, we have $xY_t = S_t, \forall\, 0 \le t \le T, P-a.s.$

In our first example, we consider a functional ϕ which depends only on the terminal value S_T of the risky asset, the so called European case. First, we can compute easily an extended rho, i.e. the directional derivative of the function u for a perturbation $\tilde{r}(t)$ of the yield $r(t)$. As was shown in the previous sections, it is a trivial application of the Girsanov Theorem. We have the following result

$$\text{rho}_{\tilde{r}(t)} = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi(S_T)\int_0^T \frac{\tilde{r}(t)}{\sigma S_t}\,dW_t\right] - \mathbf{E}\left[\int_0^T \tilde{r}(t)\,dt\, e^{-\int_0^T r(t)\,dt}\phi(S_T)\right].$$

For the delta, i.e. the first derivative w.r.t. the initial condition x, we have to compute an Itô stochastic integral $\displaystyle\int_0^T a(t)\,\frac{Y_t}{\sigma S_t}\,dW_t$ where a must satisfy, $\int_0^T a(t)\,dt = 1$. A trivial choice is $a(t) = \frac{1}{T}, \forall 0 \le t \le T$. Then we get the formula

$$\frac{\partial u}{\partial x}(0,x) = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi(S_T)\,\frac{W_T}{x\sigma T}\right].$$

A straightforward computation, using again the integration-by-parts formula, gives for the gamma (the second derivatives w.r.t. the price) the following formula

$$\frac{\partial^2 u}{\partial x^2}(0,x) = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi(S_T)\,\frac{1}{x^2\sigma T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right] \ ,$$

where we also chose $a(t) = 1/T$.

For the vega, the derivative w.r.t. the volatility parameter σ, direct application of the formula developed in the previous section again with $a(t) = 1/T$, provides :

$$\frac{\partial u}{\partial \sigma}(0,x) = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi(S_T)\,\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right].$$

To illustrate these formulae, we consider the case of a European digital option whose payoff function ϕ at time T of the form $\phi(x) = 1_{[a,b]}(x)$. We compute the values of the previous derivatives with a standard quasi Monte Carlo numerical procedure based on the use of low discrepancy sequences. More precisely, we compute the values of the Greeks delta, gamma, vega for a digital option with payoff function $\phi(x) = 1_{[100,110]}(x)$ with parameters values $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year.

As a second example, we present an application of the integration-by-parts formulas by computing the Greeks for an exotic option. We consider the case of an asian option with payoff of the form $\phi(\int_0^T S_t\,dt)$. In the Black and Scholes model, a straightforward calculus using the formula given in Remark 3.4 gives for the delta

$$\frac{\partial u}{\partial x}(0,x) = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi(\int_0^T S_t\,dt)\,\left(\frac{2}{x\sigma}\,\frac{\int_0^T Y_t\,dW_t}{\int_0^T Y_t\,dt} + \frac{1}{x}\right)\right].$$

As a third example, we are able to extend our result to more complicated payoff depending for example on the mean and terminal values of the underlying asset, like $\phi(S_T, \int_0^T S_t\,dt)$. Let us define, an "asian barrier in" option with payout $\phi(x,y) = 1_{\{y\le B\}}(x-K)_+$. We obtain for the delta the following formula

$$\frac{\partial u}{\partial x}(0,x) = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\phi\left(S_T, \int_0^T S_t\,dt\right)\,\delta(G)\right],$$

where G is the random process

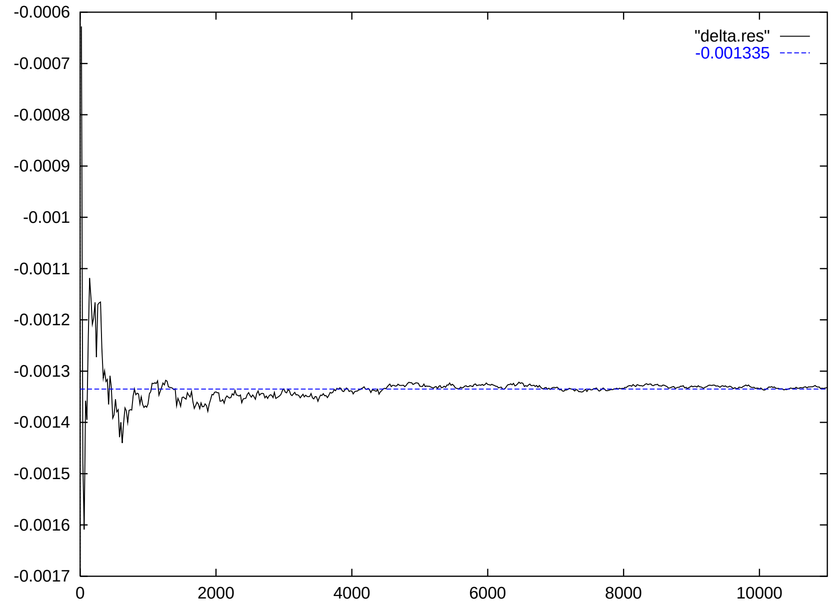

Fig. 1. Delta for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

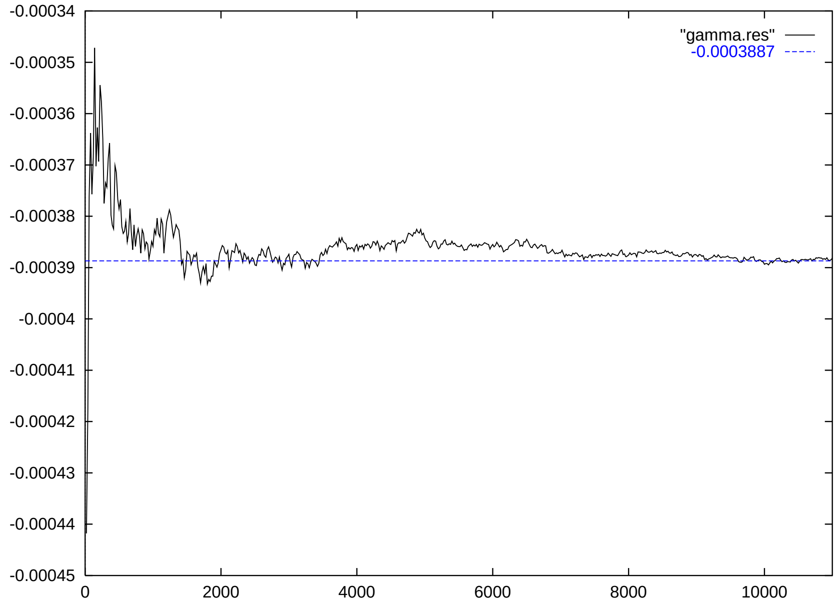

Fig. 2. Gamma for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

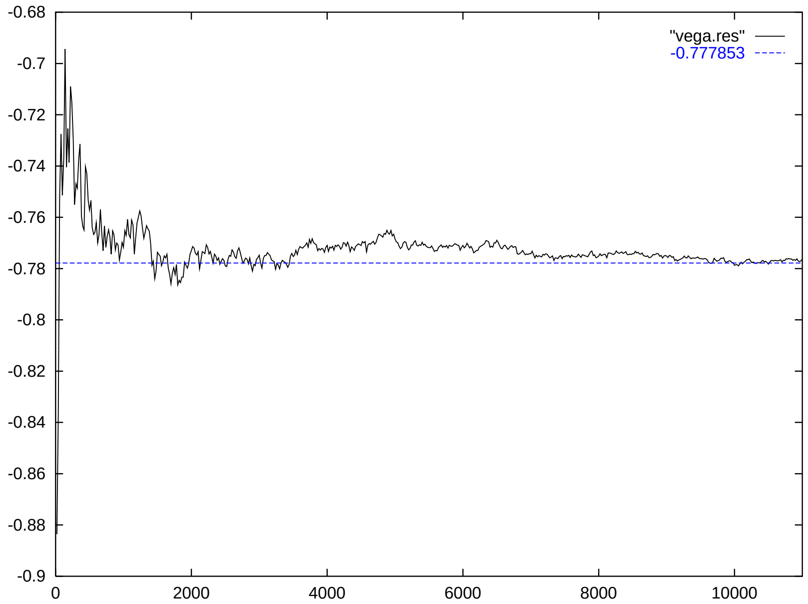

Fig. 3. Vega for a digital option with pay-off $1_{[100,110]}$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year. We use low discrepency Monte Carlo generation.

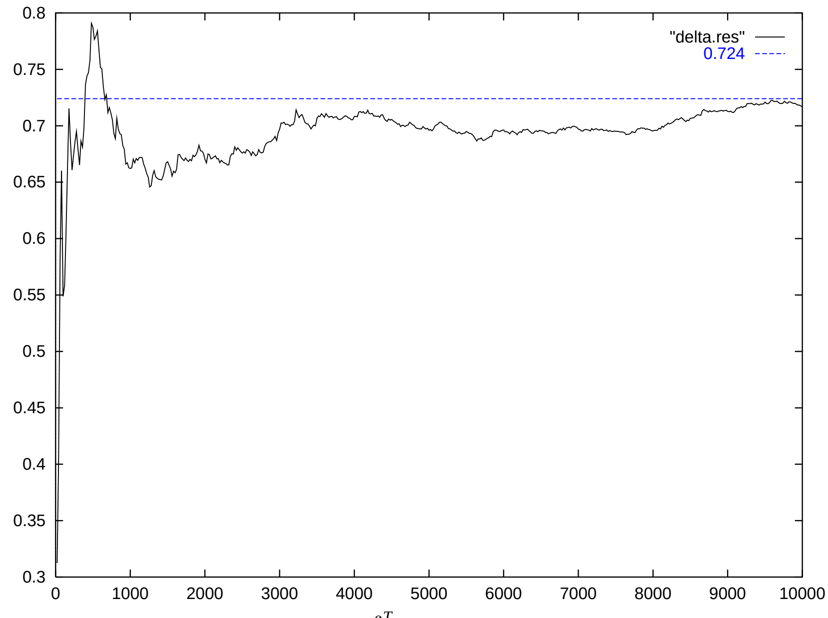

Fig. 4. Delta for a asian option with pay-off $(\int_0^T S_s\, ds - K)_+$ with $x = 100$, $r = 0.1$, $\sigma = 0.2$, $T = 1$ year, $K = 100$. We use standard Monte Carlo generation.

$$G(s) = (a + \alpha s)\frac{Y_s}{\sigma S_s} + (b + \beta s)\frac{2Y_s^2}{\sigma S_s \int_0^T S_u\, du}$$

with

$$\begin{aligned} a &= \frac{2 < s > -1}{(2 < s > -1)^2 + (2 < s^2 > -1)^2} \\ \alpha &= \frac{4 < s^2 > -2}{(2 < s > -1)^2 + (2 < s^2 > -1)^2} \\ b &= \frac{1}{2} - \frac{< s^2 > + < s > -1}{(2 < s > -1)^2 + (2 < s^2 > -1)^2} \\ \beta &= = 0 \end{aligned}$$

and $< s >= \frac{\int_0^T uS_u\, du}{\int_0^T S_u\, du}$ and $< s^2 >= \frac{\int_0^T u^2 S_u\, du}{\int_0^T S_u\, du}$.

A trivial computation in the case of the standard Wiener process ($S = W$) with $T = 1$ gives $\delta(G) = 4W_1 - 6\int_0^1 s\, dW_s$. Further analysis shows this G is optimal in the sense that it minimizes on L^2 the variance of the random variable $\phi\left(W_T, \int_0^T W_t\, dt\right)\, \delta(G)$ as we will prove in a forthcoming paper.

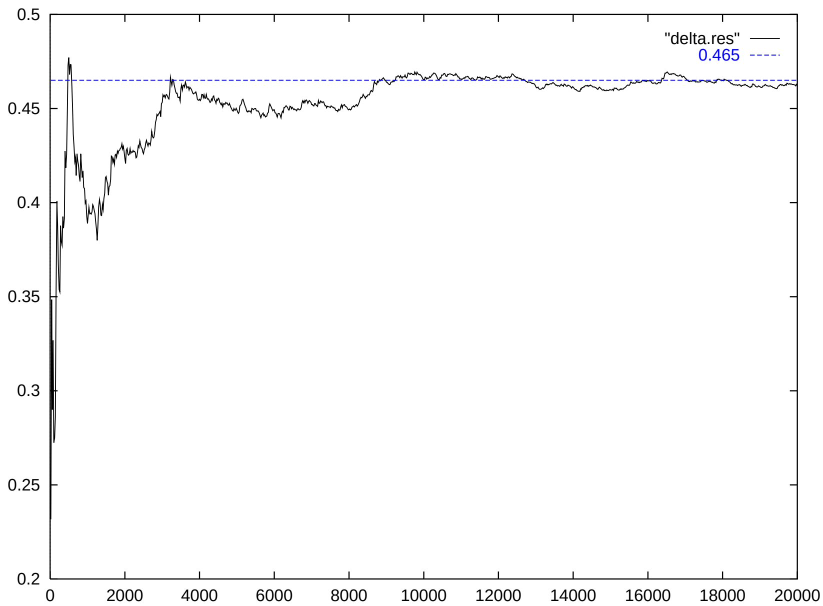

Fig. 5. Delta for a complex option with pay-off $1_{\{\int_0^1 W_s\, ds \geq B\}}(W_1 - K)_+$. We use standard Monte Carlo generation.

At this stage, we wish to observe that the Malliavin integration-by-parts which yields the above formulae, creates weights which involve powers of, say, the Brownian motion. These "global" weights in fact may slow down Monte

Carlo simulations and we now suggest a cure for this difficulty. The idea is to localize the integration-by-parts around the singularity.

In order to be more specific, let us consider the delta of a call option in the Black and Scholes model, i.e.

$$\begin{aligned}\frac{\partial}{\partial x}\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}(S_T-K)_+\right] &= \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\,1_{(S_T>K)}\,Y_T\right]\\ &= \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}(S_T-K)_+\,\frac{W_T}{x\sigma T}\right].\end{aligned}$$

The term $(S_T-K)_+W_T$ is "very large" when W_T is "large" and has a "large" variance. The idea to solve this difficulty is to introduce a localization around the singularity at K. More precisely, we set for $\delta>0$

$$\begin{aligned}H_\delta(s) &= 0,\ \text{ if } s\le K-\delta,\\ &= \frac{s-(K-\delta)}{2\delta},\ \text{ if } K-\delta\le s\le K+\delta,\\ &= 1,\ \text{ if } s\ge K+\delta\end{aligned}$$

and $G_\delta(t)=\int_{-\infty}^t H_\delta(s)\,ds$, $F_\delta(t)=(t-K)_+-G_\delta(t)$. Then, we observe that we have

$$\begin{aligned}&\frac{\partial}{\partial x}\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}(S_T-K)_+\right]\\ &\quad= \frac{\partial}{\partial x}\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}G_\delta(S_T)\right]+\frac{\partial}{\partial x}\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}F_\delta(S_T)\right]\\ &\quad= \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}H_\delta(S_T)\,Y_T\right]+\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}F_\delta(S_T)\,\frac{W_T}{x\sigma T}\right].\end{aligned}$$

Notice that F_δ vanishes for $s\le K-\delta$ and for $s\ge K+\delta$ and thus $F_\delta(S_T)W_T$ vanishes when W_T is large.

A similar idea can be used for all the Greeks. For example, we have for the gamma

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}(S_T-K)_+\right] = \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}\delta_K(S_T)\,Y_T^2\right]\\ &= \mathbf{E}\left[e^{-\int_0^T r(t)\,dt}I_\delta(S_T)\,Y_T^2\right]\\ &\quad+\mathbf{E}\left[e^{-\int_0^T r(t)\,dt}F_\delta(S_T)\,\frac{1}{x^2\sigma T}\left(\frac{W_T^2}{\sigma T}-W_T-\frac{1}{\sigma}\right)\right]\end{aligned}$$

where $I_\delta(t)=\frac{1}{2\delta}1_{|t-K|<\delta}$, $F_\delta(t)=(t-K)_+-\int_0^t\int_0^s I_\delta(u)\,du\,ds$.

The following Fig. 6 shows the efficiency of this trick by computing the gamma of a call option by global and localized Malliavin like formula (the direct integration by parts without localization is now refered to as global Malliavin).

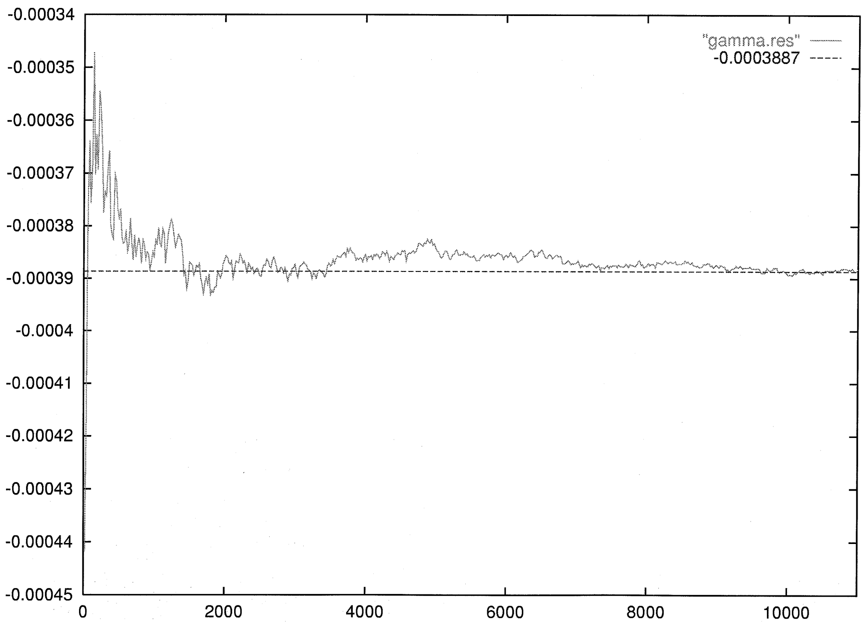

Fig. 6. Gamma of a call option computed by global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

N = 10 000	exact	MCFD	MCMALL
Delta call	0.725747	0.725639	0.725660 (loc.)
Gamma call	0.016660	0.015330	0.016634 (loc.)
Vega call	33.320063	33.250709	33.267145 (loc.)
Delta digital	-0.001335	-0.003167	-0.001335
Gamma digital	-0.000389	+0.099532	-0.000389
Vega digital	-0.777516	-0.542902	-0.778695
Delta average call	0.649078	0.660177	0.654369 (loc.)

We conclude the paper by presenting a benchmark comparing Monte Carlo simulations based on the finite difference approximation of the Greeks and our localized Malliavin calculus approach. The finite difference scheme is the following : set $u(x,\sigma) = \mathbf{E}\left[\Phi(S_T)|S_0 = x\right]$, we have the approximations

$$\begin{aligned} \text{delta} &= \frac{u(x+h,\sigma) - u(x-h,\sigma)}{2h} \\ \text{gamma} &= \frac{u(x+h,\sigma) - 2u(x,\sigma) + u(x-h,\sigma)}{h^2} \\ \text{sigma} &= \frac{u(x,\sigma+\epsilon) - u(x,\sigma-\epsilon)}{2\epsilon} \end{aligned}$$

We compare the values obtained by those two methods for a given number (10 000) of Brownian trajectories with the exact values. Of course, we use the same Brownian trajectories for the different initial conditions $x+h, x, x-h$ which

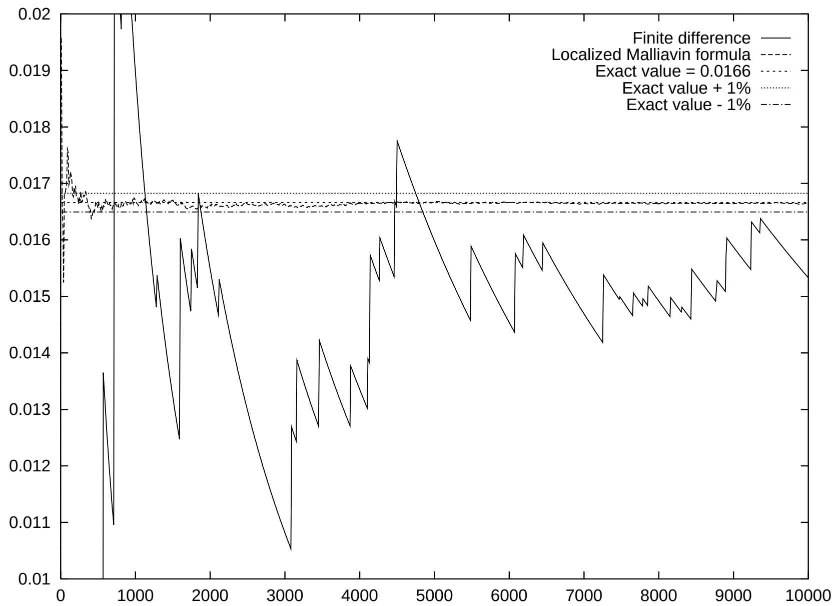

Fig. 7. Gamma of a call option computed by finite difference and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ (localization parameter). We use low discrepency sequences.

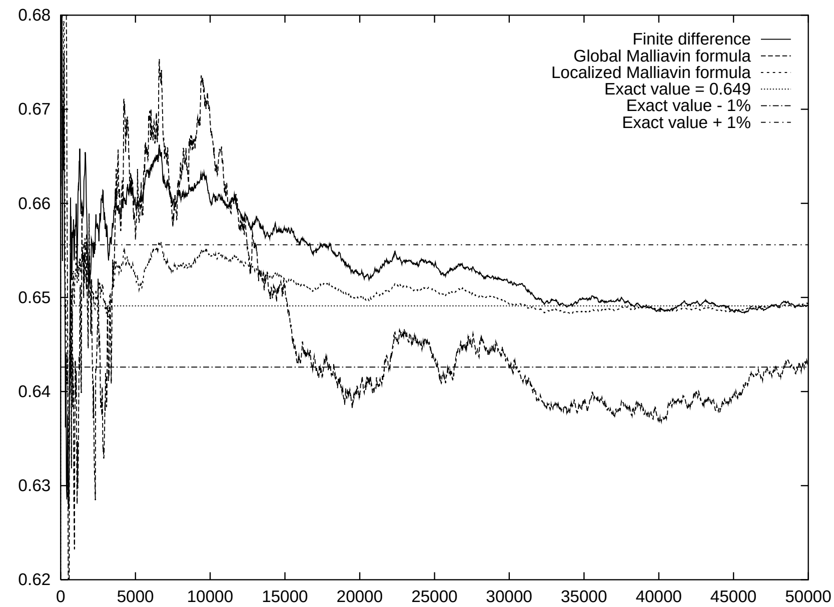

Fig. 8. Delta of an average call option computed by finite difference, global and localized Malliavin like formula. The parameters are $S0 = 100, r = 0.1, \sigma = 0.2, T = 1, K = 100$ and $\delta = 10$ for the localization parameter. We use pseudo random sequences.

gives a natural variance reduction to the finite difference method; see also the discussion in the introduction. Figures 7 and 8 give an idea of the number of paths required in order to achieve a given precision of 1%.

Acknowledgements. Most of this work was done while J.M. Lasry was Chairman of *Caisse Autonome de Refinancement*.

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