THE IMPLEMENTATION OF THE LIBOR MARKET MODEL

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Risk Training

INTEREST RATE MODELLING

Practical calibration and implementation techniques

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OVERVIEW

- Introducing the Libor Market Model
- The LMM and the Swap market Model
- Pricing swaptions with MC or approximations
- Some points on correlation modelling
- Reducing model dimension
- Calibration, diagnostics and practical cases

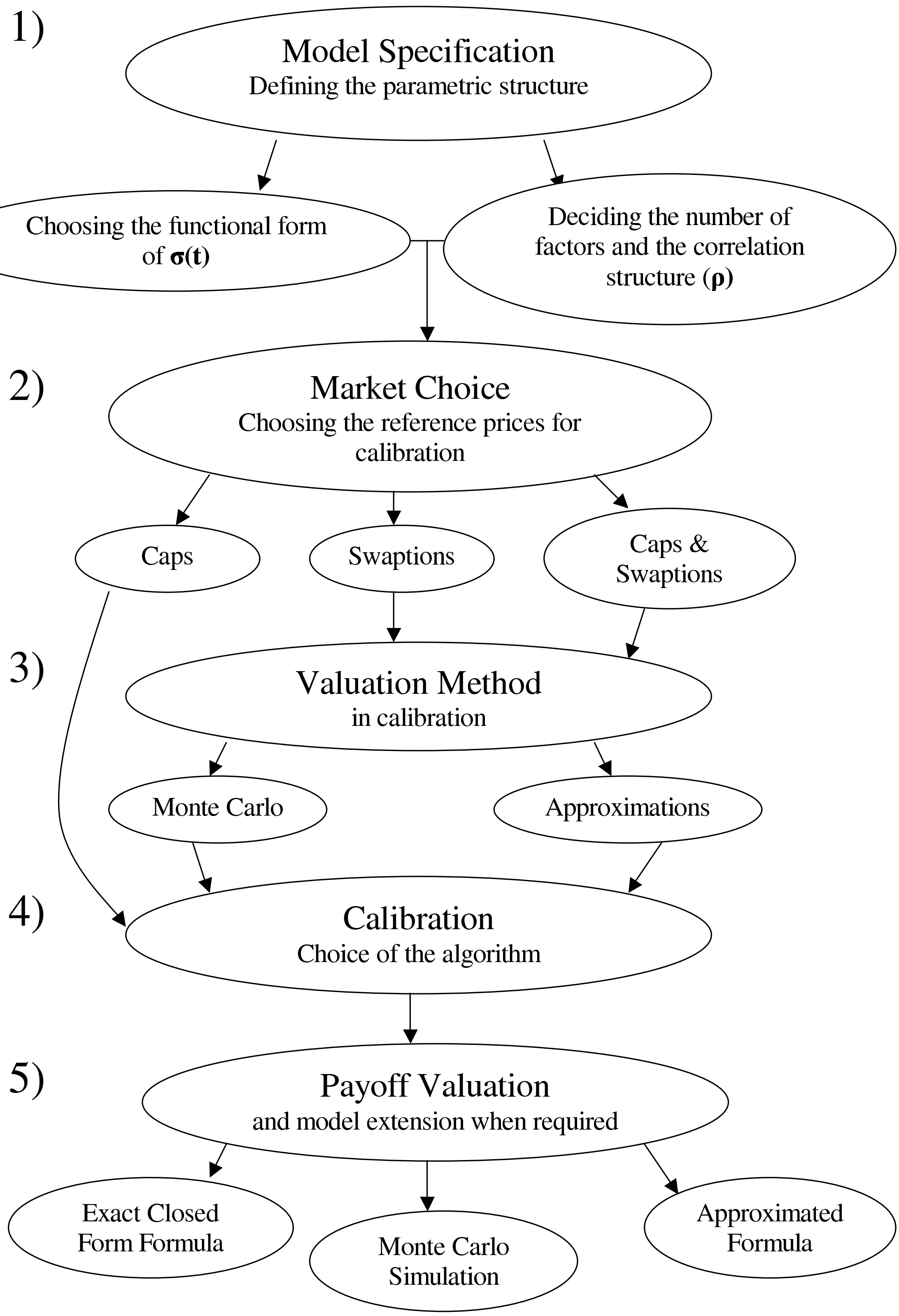
Pricing Examples:

- Finding exact formulas: In advance swaps and caps
- Finding approximations: ZC swaptions and CMS
- Tenor issues: Interpolation and Bridging techniques
- Early exercise: LSMC for the LMM

Extending the framework:

- Changing the dynamics: Libor Models for smile
- Changing the market: Market Models for credit

Implementing LMM



Interest rates

The **money-market account** $B(t)$ accrues at the **instantaneous spot rate** $r(t)$:

$$dB(t) = r(t)\, B(t)\, dt$$

so investing 1 at time 0 we have at t

$$B(t) = e^{\int_0^t r(s)\, ds}$$

With $B(t)$ define the **stochastic discount factor** from date T to date t:

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)\, ds}.$$

With $D(t,T)$ define the price at t of a security paying 1 at T, the T-maturity **zero-coupon bond**:

$$P(t,T) = \mathbb{E}_t^Q\left[D(t,T)\mathbf{1}\right] = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)\, ds}\right].$$

With $P(t,T)$ define the (simply compounded) **spot Libor rate** $L(t,T)$ with maturity T

$$[1 + L(t,T)\tau(t,T)] = \frac{1}{P(t,T)}$$

and the **forward Libor rate** $F(t;S,T)$ with expiry S and maturity T:

$$(1 + F(t;S,T)\tau(S,T)) = \frac{P(t,S)}{P(t,T)},$$

$$F(t;S,T) = \frac{1}{\tau(S,T)}\left[\frac{P(t,S)}{P(t,T)} - 1\right].$$

Interest Rate Swap

At every instant T_i in $T_{\alpha+1}, \dots, T_\beta$ the fixed leg pays

$$\tau_i K,$$

whereas the floating leg pays

$$\tau_i L(T_{i-1}, T_i).$$

The discounted payoff at t for the payer of the fixed leg is:

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K).$$

The price is

$$\begin{aligned} \text{Swap}\Pi_t &= \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i(F(t, T_{i-1}, T_i) - K) \\ &= P(t, T_\alpha) - P(t, T_\beta) - \sum_{i=\alpha+1}^{\beta} \left[P(t, T_i)\tau_i K\right]. \end{aligned}$$

The Swap rate (fair K) is

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i}.$$

A **Cap** is an interest rate swap where each exchange payment is executed only if it has positive value. The discounted payoff is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)^+.$$

A **Swaption** is an option on a swap contract as whole, with exercise date T_α. The discounted payoff is

$$D(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+.$$

$$D(t, T_\alpha)\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(S_{\alpha,\beta}(T_\alpha) - K)^+$$

Cap Market Black Formula

$$
\begin{aligned}
\textsc{Cap}\Pi &= \mathbb{E}^Q\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i(L(T_{i-1},T_i)-K)^+\right] \\
&= \sum_{i=\alpha+1}^{\beta} \underbrace{\mathbb{E}^Q\left[D(0,T_i)\tau_i(L(T_{i-1},T_i)-K)^+\right]}_{\textsc{Caplet}\Pi}.
\end{aligned}
$$

Treating $D(0,T_i)$ as a deterministic discount factor

$$
\begin{aligned}
\textsc{Caplet}\Pi &= \mathbb{E}^Q\left[D(0,T_i)\tau_i(F(T_{i-1};T_{i-1},T_i)-K)^+\right] \\
&= P(0,T_i)\tau_i\mathbb{E}^Q\left[(F(T_{i-1};T_{i-1},T_i)-K)^+\right].
\end{aligned}
$$

Assuming driftless lognormal dynamics for $F(t;T_{i-1},T_i)$

$$dF(t;T_{i-1},T_i) = \sigma F(t;T_{i-1},T_i)\,dW_t$$

leads to

$$\textsc{Caplet}\Pi = P(0,T_i)\tau_i\textsc{Black}\left(F(0;T_{i-1},T_i),K,\sigma\sqrt{T_{i-1}}\right)$$

$$\textsc{Black}\,(F,K,v) = F\ N(d_1(F,K,v)) - K\ N(d_2(F,K,v))$$

$$d_1(F,K,v) = \frac{\ln(\frac{F}{K})+\frac{1}{2}v^2}{v}, \qquad d_2(F,K,v) = \frac{\ln(\frac{F}{K})-\frac{1}{2}v^2}{v}$$

Change of Numeraire

$$(\Omega, \mathcal{F}, P, \mathbb{F})$$

Given a probability measure $Q^{N1} \sim P$ and a **numeraire** $N1_t$, Q^{N1} is the **measure associated with** $N1_t$ if the price of any tradable asset X_t expressed in terms of $N1_t$ is a martingale under Q^{N1}, i.e.

$$\frac{X_t}{N1_t} = \mathbb{E}^{N1}\left[\left. \frac{X_T}{N1_T} \right| \mathcal{F}_t \right].$$

Theorem (Geman et al.) : if Q^{N1} is a measure associated with $N1_t$, given a different numeraire $N2_t$ there exists a measure Q^{N2} equivalent to Q^{N1} such that Q^{N2} is associated to $N2_t$.

$$X_t = \mathbb{E}^{N1}\left[\left. \frac{N1_t}{N1_T} X_T \right| \mathcal{F}_t \right] = \mathbb{E}^{N2}\left[\left. \frac{N2_t}{N2_T} X_T \right| \mathcal{F}_t \right].$$

Therefore we can price a payoff using a convenient numeraire:

$$\boxed{\Pi(X_t)} = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)\,ds} \Pi(X_T) \right] = B(t)\,\mathbb{E}_t^Q\left[\frac{\Pi(X_T)}{B(T)} \right]$$
$$= \boxed{N2_t \mathbb{E}_t^{N2}\left[\frac{\Pi(X_T)}{N2_T} \right]}.$$

Two goals for **changing numeraire**:

1) $\dfrac{\Pi(X_T)}{N2_T}$ to be a **simple quantity**.

2) $N2_t X_t$ to be a tradable asset, so X_t is a **martingale**

In case $N2_t = P(t,T)$ the associated probability measure is the T**-forward measure** Q^T and

$$\begin{aligned}\Pi(X_t) &= \beta(0,t)\, E^Q \left[\frac{1}{\beta(0,T)}\Pi(X_T)\right] \\ &= P(t,T)\,\mathbb{E}^T\left[\frac{1}{P(T,T)}\Pi(X_T)\middle|\,\mathcal{F}_t\right] \\ &= P(t,T)\,\mathbb{E}^T\left[\Pi(X_T)|\,\mathcal{F}_t\right].\end{aligned}$$

Caplet Black Formula by Change of Numeraire

$$\text{CAPLET}\Pi = \mathbb{E}^Q\left[D(0,T_i)\tau_i(F(T_{i-1};T_{i-1},T_i)-K)^+\right]$$

Take numeraire $P(t,T_i)$

$$\text{CAPLET}\Pi = P(0,T_i)\mathbb{E}^{T_i}\left[\tau_i(F(T_{i-1};T_{i-1},T_i)-K)^+\right],$$

$$F(t;T_{i-1},T_i) = \frac{1}{\tau_i}\left[\frac{P(t,T_{i-1})}{P(t,T_i)}-1\right].$$

$$N2_tX_t = P(t,T_i)\frac{1}{\tau_i}\left[\frac{P(t,T_{i-1})}{P(t,T_i)}-1\right] = P(t,T_{i-1})\frac{1}{\tau_i}-\frac{1}{\tau_i}P(t,T_i),$$

which is the price of a tradable asset, so $F(t;T_{i-1},T_i)$ is a martingale. Assuming lognormality

$$dF(t;T_{i-1},T_i) = \sigma F(t;T_{i-1},T_i)\,dW_t.$$

This leads exactly to market Black pricing formula.

The Libor Market Model

The set $\{T_0, \ldots, T_M\}$ of expiry-maturity dates is the **tenor structure** with the corresponding **year fractions** $\{\tau_0, \ldots, \tau_M\}$.

We model the **simply compounded forward rates** $F_k(t)$ resetting at T_{k-1} (expiry) and with maturity T_k. Q^k is the T_k-**forward probability measure** associated to $P(t, T_k)$. We know $F_k(t)$ is a martingale under Q^k.

The **Libor Market Model** assumes for each $F_k(t)$ under Q^k:

$$\boxed{dF_k(t) = \sigma_k(t) F_k(t)\, dZ_k^k(t), \quad t \leq T_{k-1},}$$

where Z^k is an M-dimensional Wiener process, the **instantaneous correlation** is matrix ρ, and $\sigma_k(t)$ is the **instantaneous volatility** function.

If $\sigma_k(t)$ is bounded, we have a unique strong solution

$$F_k(T) = F_k(0) \exp \underbrace{\left\{ - \int_0^T \frac{\sigma_k(t)^2}{2} dt + \int_0^T \sigma_k(t) dZ_k(t). \right\}}_{\sim N\left(- \int_0^T \frac{\sigma_k(t)^2}{2} dt,\ \int_0^T \sigma_k(t)^2 dt \right)}$$

Therefore $F_k(T)$ is **lognormally distributed** under Q^k, and recalling the MGF of $N\left(\mu, \sigma^2\right)$

$$\mathbb{E}\left(e^{tx}\right) = e^{\mu t + \frac{1}{2}\sigma^2 t^2},$$

we see that

$$\mathbb{E}\left[F_k(T)\right] = F_k(0)\, e^{-\frac{1}{2}\int_0^T \sigma_k(t)^2 dt + \frac{1}{2}\int_0^T \sigma_k(t)^2 dt} = F_k(0)\,.$$

Change of Numeraire for dynamics

Start from M-dimensional X under Q^{N1} associated with $N1$

$$dX_k = \mu_k^{N1}()\,dt + \underbrace{\bar{\sigma}_k()}_{(0\;0\;\ldots\sigma_k\ldots 0\;0)} dZ^{N1},$$

with Z^{N1} M-dimensional with instantaneous correlation ρ.

The drift μ^{N2} of X under Q^{N2} associated with $N2$ is

$$\mu_k^{N2} = \mu_k^{N1} - \bar{\sigma}_k()\,\rho\text{VecDiffCoeff}\left(\ln \frac{N1}{N2}\right).$$

In the LMM, changing from Q^k to Q^i, $i < k$, we have

$$\begin{aligned} \frac{N1}{N2} &= \frac{P(t,T_k)}{P(t,T_i)} = \frac{P(t,T_k)}{P(t,T_{k-1})}\frac{P(t,T_{k-1})}{P(t,T_{k-2})}\cdots\frac{P(t,T_{i+1})}{P(t,T_i)} \\ &= \prod_{j=i+1}^{k}\frac{P(t,T_j)}{P(t,T_{j-1})} = \prod_{j=i+1}^{k}\frac{1}{1+\tau_j F_j(t)}, \end{aligned}$$

so $\frac{N1}{N2}$ is expressed in terms of rates in the LMM.

LMM Dynamics of $F_k(t)$ under Q^i, $k \neq i$

- $i < k, \quad t \le T_i$:

$$dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)$$

- $i > k, \quad t \le T_{k-1}$:

$$dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)$$

Few remarks on the tenor structure of a Libor Market Model

- **Nested rates**. The tenor structure we describe does not include nested forward rates, such as

  $$F_k(t) = F(t; T_{k-1}, T_k),\ F_{k+1}(t) = F(t; T_k, T_{k+1}),\ F(t) := F(t; T_{k-1}, T_{k+1}).$$

  In fact, assuming T_i are 6-month spaced, we have

  $$\begin{aligned} F_k(t) &= \frac{1}{0.5}\left[\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right],\ F_{k+1}(t) = \frac{1}{0.5}\left[\frac{P(t, T_k)}{P(t, T_{k+1})} - 1\right] \\ F(t) &= \frac{1}{1}\left[\frac{P(t, T_{k-1})}{P(t, T_{k+1})} - 1\right] = \frac{P(t, T_{k-1})}{P(t, T_k)}\frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \\ &= \frac{F_k(t) + F_{k+1}(t)}{2} + \frac{F_k(t)F_{k+1}(t)}{4} \end{aligned}$$

  If F_k and F_{k+1} are lognormal under their measures, F cannot be so.

- **Overlapping rates.** Tenor structure does not include overlapping rates such as

  $$F(t; T_{k-1}, T_{k+1}), \quad F(t; T_k, T_{k+2})$$

  In fact for the dynamics of $F(t; T_k, T_{k+2})$ under Q^{k+1} we need

  $$\frac{P\left(t, T_{k+2}\right)}{P\left(t, T_{k+1}\right)} = \frac{1}{1 + \tau_{k+2} F(t; T_{k+1}, T_{k+2})}.$$

  and for the dynamics of $F(t; T_{k-1}, T_{k+1})$ under Q^k we need

  $$\frac{P\left(t, T_{k+1}\right)}{P\left(t, T_k\right)} = \frac{1}{1 + \tau_{k+1} F(t; T_k, T_{k+1})}.$$

  Notice $F(t; T_k, T_{k+1})$ and $F(t; T_{k+1}, T_{k+2})$ are nested into $F(t; T_k, T_{k+2})$.

Volatility Structures

General Piecewise Constant (GPC)

$$\sigma_k(t) = \sigma_{k,\varepsilon(t)}, \text{ with } \varepsilon(t) = i \quad \text{if} \quad T_{i-2} < t \leq T_{i-1}.$$

	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\sigma_{1,1}$	-	-	$\ldots$	-
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	-	$\ldots$	-
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

Constant structure

$$\sigma_k(t) = s_k$$

Separable piecewise constant (SPC)

$$\sigma_k(t) = \sigma_{k,\varepsilon(t)} := \Phi_k \psi_{k-(\varepsilon(t)-1)}$$

Linear exponential (LE)

$$\begin{aligned} \sigma_k(t) &= \Phi_k \psi(T_{k-1} - t; a, b, c, d) \\ &= \Phi_k \left([a(T_{k-1} - t) + d]\, e^{-b(T_{k-1}-t)} + c \right). \end{aligned}$$

Cap Calibration

With

$$\textsc{Caplet}\Pi = P(0,T_i)\tau_i \textsc{Black}\left(F(0;T_{i-1},T_i),K,v_i\right),$$

$$v_i = \sigma_i\sqrt{T_{i-1}}.$$

If $\sigma_i(t)$ is the instantaneous volatility of $F_i\left(t\right)$ in the LMM, ensure

$$\boxed{\int_0^{T_{i-1}}\sigma_i(t)^2dt = v_i^2}$$

For **GPC:**

$$v_i = \sqrt{\int_0^{T_{i-1}}\sigma^2_{i,\varepsilon(t)}\,dt} = \sqrt{\sum_{j=1}^{i}\left(T_{j-1}-T_{j-2}\right)\sigma^2_{i,j}}.$$

For $\sigma_k(t) = \mathbf{s}_k$**:**

$$v_i = s_i\sqrt{T_{i-1}-t}.$$

For **SPC:**

$$v_i = \sqrt{\Phi_i^2\sum_{j=1}^{i}\tau_{j-2,j-1}\,\psi^2_{i-j+1}}\,.$$

For **LE:**

$$v_i = \sqrt{\Phi_i^2\int_0^{T_{i-1}}\left(\left[a\left(T_{i-1}-t\right)+d\right]e^{-b(T_{i-1}-t)}+c\right)^2\,dt}\,.$$

The Swap Market Model

Recall the swaption payoff

$$D(t,T_\alpha)(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}P(T_\alpha,T_i)\tau_i.$$

Consider the **numeraire**

$$C_{\alpha,\beta}\left(t\right)=\sum_{i=\alpha+1}^{\beta}P(t,T_i)\tau_i$$

and the associated **probability measure**, called the **swap measure**. Assuming lognormality of $S_{\alpha,\beta}(t)$ under this measure,

$$dS_{\alpha,\beta}(t)=\sigma_{\alpha,\beta}(t)S_{\alpha,\beta}(t)\;dW_t^{\alpha,\beta},$$

one recovers exactly the **Black formula for swaptions**:

$$\textsc{Swaption}\Pi_t=C_{\alpha,\beta}\left(t\right)\textsc{Black}\left(S_{\alpha,\beta}(t),K,\sqrt{\int_t^{T_\alpha}\sigma_{\alpha,\beta}(s)^2ds}\right).$$

Swap-Forward rates relationships

First recall that, with $k > i$

$$\frac{P(t,T_k)}{P(t,T_i)} = \frac{P(t,T_k)}{P(t,T_{k-1})}\frac{P(t,T_{k-1})}{P(t,T_{k-2})}\cdots\frac{P(t,T_{i+1})}{P(t,T_i)} = \\ = \prod_{j=i+1}^{k}\frac{P(t,T_j)}{P(t,T_{j-1})} = \prod_{j=i+1}^{k}\frac{1}{1+\tau_j F_j(t)}. \tag{1}$$

We compute the swap rate setting the swap price to zero:

$$\begin{aligned} 1)\ \ \text{SWAP}\Pi_t &= \sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i(F(t,T_{i-1},T_i)-K) \\ &= P(t,T_\alpha)-P(t,T_\beta)-\sum_{i=\alpha+1}^{\beta}[P(t,T_i)\tau_i K] = 0, \end{aligned}$$

so that

$$\boxed{S_{\alpha,\beta}(t)} = \frac{P(t,T_\alpha)-P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}P(t,T_i)\tau_i} = \frac{1-\dfrac{P(t,T_\beta)}{P(t,T_\alpha)}}{\sum_{i=\alpha+1}^{\beta}\dfrac{P(t,T_i)}{P(t,T_\alpha)}\tau_i}$$

$$\stackrel{by\ (1)}{=} \boxed{\frac{1-\prod_{j=\alpha+1}^{\beta}\dfrac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta}\tau_i\prod_{j=\alpha+1}^{i}\dfrac{1}{1+\tau_j F_j(t)}}}.$$

Swap-Forward rates relationships

...or equivalently

$$2)\quad \text{SWAP}\Pi_t = \sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i(F(t,T_{i-1},T_i) - K) = 0,$$

$$\boxed{S_{\alpha,\beta}(t)} = \frac{\sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i F(t,T_{i-1},T_i)}{\sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i}$$

$$= \frac{\sum_{i=\alpha+1}^{\beta} \frac{P(t,T_i)}{P(t,T_\alpha)}\tau_i F(t,T_{i-1},T_i)}{\sum_{i=\alpha+1}^{\beta} \frac{P(t,T_i)}{P(t,T_\alpha)}\tau_i}$$

$$= \boxed{\sum_{i=\alpha+1}^{\beta} w_i(t)\ F_i(t)}$$

$$w_i(t) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}.$$

Testing LMM-LSM Consistency

1) Assume F_k's lognormal under Q^k (hypothesis of the Libor Market Model).

2) Change the measure to see their dynamics under $Q^{\alpha,\beta}$

3) Combine them and see which distribution they imply for $S_{\alpha,\beta}(t)$ under $Q^{\alpha,\beta}$

4) Check if this distribution of $S_{\alpha,\beta}(t)$ under $Q^{\alpha,\beta}$ is lognormal (hypothesis of the Swap Market Model).

Pricing Swaptions with Montecarlo

$$
\begin{aligned}
&\textsc{Swaption}\Pi_t = \\
&= \mathbb{E}^Q\left(D(0,T_\alpha)\ (S_{\alpha,\beta}(T_\alpha) - K)^+ \ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right) \\
&= P(0,T_\alpha) E^\alpha \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right] \\
&= P(0,T_\alpha) E^\alpha \left[\underbrace{\Psi\left(F_{\alpha+1}(T_\alpha), F_{\alpha+2}(T_\alpha), \ldots, F_\beta(T_\alpha)\right)}_{\Psi(F(T_\alpha))} \right].
\end{aligned}
$$

Recall the dynamics of F_k under Q^i, $i < k$:

$$
dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{j,k} \tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)}\, dt + \sigma_k(t) F_k(t)\, dZ_k(t).
$$

Take logs and discretize

$$
d \ln\ F_k(t) = \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j} \tau_j \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)}\, dt - \frac{\sigma_k(t)^2}{2} dt + \sigma_k(t) dZ_k(t).
$$

$$
\begin{aligned}
\ln\ \bar{F}_k(t + \Delta t) \ =\ \ln\ \bar{F}_k(t) + \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j}\ \tau_j\ \sigma_j(t)\ \bar{F}_j(t)}{1 + \tau_j \bar{F}_j(t)} \Delta t \\
- \frac{\sigma_k(t)^2}{2} \Delta t + \sigma_k(t)(Z_k(t + \Delta t) - Z_k(t)).
\end{aligned}
$$

Store the $\bar{F}^n(T_\alpha)$, $n = 1, \ldots, N$ (# of scenarios) and compute

$$\textsc{Swaption}\Pi_t^{MC} = P(0, T_\alpha)\frac{1}{N}\sum_{n=1}^{N}\Psi\left[\bar{F}^n(T_\alpha)\right].$$

In the **SMM**:

$$dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)\, S_{\alpha,\beta}(t) dW^{\alpha,\beta}.$$

with $\sigma^{(\alpha,\beta)}$ deterministic. The Black Volatility is:

$$\begin{aligned} v_{\alpha,\beta}(T_\alpha) &= \sqrt{\int_0^{T_\alpha}\left(\sigma^{(\alpha,\beta)}(t)\right)^2 dt} \\ &= \sqrt{\int_0^{T_\alpha}\frac{(dS_{\alpha,\beta}(t))\,(dS_{\alpha,\beta}(t))}{S_{\alpha,\beta}(t)^2}} \end{aligned}$$

How can we find an approximation for $v_{\alpha,\beta}(T_\alpha)$ in the LMM?

Rebonato '98 industry LMM

Approximated Black Volatility

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t) \cong \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(t),$$

First Approx.

$$dS_{\alpha,\beta}(t)\, dS_{\alpha,\beta}(t) \cong \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)\, dt$$

$$\frac{dS_{\alpha,\beta}(t)dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)^2} \cong \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{\left(\sum_{i=\alpha+1}^{\beta} w_i(0)\ F_i(t)\right)^2} dt$$

$$\cong \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{\left(\sum_{i=\alpha+1}^{\beta} w_i(0)\ F_i(0)\right)^2}$$

Second Approx.

$$v_{\alpha,\beta}(T_\alpha)$$

$$\cong$$

$$\sqrt{\left(\sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\, dt\right)}$$

Correlation Modeling

Properties of correlation

1) $\rho_{i,i} = 1, \ \forall \ i$
2) $\rho_{i,j} = \rho_{j,i}, \ \forall \ i, j$
3) $x'\rho x \geq 0, \forall x (\neq 0)$
4) $|\rho_{i,j}| \leq 1 \quad \forall \ i, j$

Instantaneous correlation in the LMM can be **implied correlation** (calibrated) or **exogenous correlation** (estimated).

When **estimating correlations** for LMM, recall that rates

$$F(t; T_0, T_1), \ F(t; T_1, T_2), \ F(t; T_2, T_3) \ldots$$

have **fixed maturity**, so we need discount factors

$$P(t, T), P(t+1, T) \ldots P(t+n, T),$$

not

$$P(t, t+Z), P(t+1, t+1+Z) \ldots P(t+n, t+n+Z).$$

Historical Forward rates correlation

1	0.82	0.69	0.65	0.58	0.47	0.29	0.23	0.43	0.47	0.33	0.43	0.29	0.23	0.26	0.21	0.23	0.29	0.25
0.82	1	0.80	0.73	0.68	0.55	0.45	0.40	0.53	0.57	0.42	0.45	0.48	0.34	0.35	0.32	0.32	0.31	0.32
0.69	0.80	1	0.76	0.72	0.63	0.47	0.56	0.67	0.61	0.48	0.52	0.48	0.54	0.46	0.42	0.45	0.42	0.35
0.65	0.73	0.76	1	0.78	0.67	0.58	0.56	0.68	0.70	0.56	0.59	0.58	0.50	0.50	0.48	0.49	0.44	0.35
0.58	0.68	0.72	0.78	1	0.84	0.66	0.67	0.71	0.73	0.70	0.67	0.64	0.59	0.58	0.65	0.65	0.53	0.42
0.47	0.55	0.63	0.67	0.84	1	0.77	0.68	0.73	0.69	0.77	0.69	0.66	0.63	0.61	0.68	0.70	0.57	0.45
0.29	0.45	0.47	0.58	0.66	0.77	1	0.72	0.71	0.65	0.65	0.62	0.71	0.62	0.63	0.66	0.64	0.52	0.38
0.23	0.40	0.56	0.56	0.67	0.68	0.72	1	0.73	0.66	0.64	0.56	0.61	0.72	0.59	0.64	0.64	0.49	0.46
0.43	0.53	0.67	0.68	0.71	0.73	0.71	0.73	1	0.75	0.59	0.66	0.69	0.69	0.69	0.63	0.64	0.52	0.40
0.47	0.57	0.61	0.70	0.73	0.69	0.65	0.66	0.75	1	0.63	0.68	0.70	0.63	0.64	0.65	0.62	0.52	0.40
0.33	0.42	0.48	0.56	0.70	0.77	0.65	0.64	0.59	0.63	1	0.83	0.72	0.64	0.58	0.68	0.73	0.57	0.45
0.43	0.45	0.52	0.59	0.67	0.69	0.62	0.56	0.66	0.68	0.83	1	0.82	0.69	0.67	0.70	0.69	0.65	0.43
0.29	0.48	0.48	0.58	0.64	0.66	0.71	0.61	0.69	0.70	0.72	0.82	1	0.79	0.78	0.79	0.72	0.59	0.42
0.23	0.34	0.54	0.50	0.59	0.63	0.62	0.72	0.69	0.63	0.64	0.69	0.79	1	0.82	0.83	0.79	0.60	0.45
0.26	0.35	0.46	0.50	0.58	0.61	0.63	0.59	0.69	0.64	0.58	0.67	0.78	0.82	1	0.90	0.80	0.50	0.22
0.21	0.32	0.42	0.48	0.65	0.68	0.66	0.64	0.63	0.65	0.68	0.70	0.79	0.83	0.90	1	0.94	0.71	0.46
0.23	0.32	0.45	0.49	0.65	0.70	0.64	0.64	0.64	0.62	0.73	0.69	0.72	0.79	0.80	0.94	1	0.82	0.66
0.29	0.31	0.42	0.44	0.53	0.57	0.52	0.49	0.52	0.52	0.57	0.65	0.59	0.60	0.50	0.71	0.82	1	0.84
0.25	0.32	0.35	0.35	0.42	0.45	0.38	0.46	0.40	0.40	0.45	0.43	0.42	0.45	0.22	0.46	0.66	0.84	1

Correlations: Parametric Forms

- Schoenmakers and Coffey (2000) three-parameter form

$$\rho_{i,j} =$$

$$\exp\left[-|i-j|\left(\beta - \frac{\alpha_2}{6M-18}\left(i^2+j^2+ij-6i-6j-3M^2+15M-7\right)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{6M-18}\left(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2\right)\right)\right].$$

where the parameters are constrained to be non-negative.

- Schoenmakers and Coffey (2000) two-parameter form

$$\rho_{i,j} = \exp\left[-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty\right.\right.$$
$$\left.\left.+\eta\frac{i^2+j^2+ij-3Mi-3Mj+3i+3j+2M^2-M-4}{(M-2)(M-3)}\right)\right]$$

where $\rho_\infty = \rho_{1,M}$, with: $0 < \eta < -\ln\rho_\infty$.

- Classic, two-parameters exponential form

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp[-\beta|i-j|], \quad \beta \geq 0.$$

- Rebonato (1999b) three-parameter form

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp[-|i-j|(\beta - \alpha\,(\max(i,j)-1))].$$

Pivot forms

- Rebonato three-parameter form

$$\left(\frac{\rho_{1,M}-\rho_\infty}{1-\rho_\infty}\right)=\left(\frac{\rho_{M-1,M}-\rho_\infty}{1-\rho_\infty}\right)^{(M-1)},$$

$$\alpha=\frac{\ln\left(\frac{\rho_{1,2}-\rho_\infty}{\rho_{M-1,M}-\rho_\infty}\right)}{2-M},\qquad \beta=\alpha-\ln\left(\frac{\rho_{1,2}-\rho_\infty}{1-\rho_\infty}\right).$$

$$\rho_\infty=0.23551,\quad \alpha=0.00126,\quad \beta=0.26388.$$

- Schoenmakers and Coffey three-parameter form

$$\beta=-\ln\left(\rho_{M-1,M}\right).$$

$$\alpha_1=\frac{6\ln\rho_{1,M}}{(M-1)(M-2)}-\frac{2\ln\rho_{M-1,M}}{(M-2)}-\frac{4\ln\rho_{1,2}}{(M-2)},$$

$$\alpha_2=-\frac{6\ln\rho_{1,M}}{(M-1)(M-2)}+\frac{4\ln\rho_{M-1,M}}{(M-2)}+\frac{2\ln\rho_{1,2}}{(M-2)},$$

$$\alpha_1=0.03923,\quad \alpha_2=-0.03743,\quad \beta=0.17897.$$

- Schoenmakers and Coffey two-parameter form

$$\rho_\infty=\rho_{1,M}$$

$$\eta=\frac{(-\ln\rho_{1,2})(M-1)+\ln\rho_\infty}{2},$$

$$\rho_\infty=0.24545,\quad \eta=1.04617.$$

	Fitting	Pivot
$\sqrt{\text{MSE}}$	0.108434	0.173554
$\sqrt{\text{MSE\%}}$	0.25949	0.30890

LMM representations for correlations

Replace

$$\boxed{\mathrm{d}Z \backsim N(0, \rho\, \mathrm{d}t)}$$

by

$$C\,\mathrm{d}Y, \quad \text{with} \quad \mathrm{d}Y \backsim N\left(0, I\,\mathrm{d}t\right), \quad CC' = \rho$$

so that

$$\boxed{C\,\mathrm{d}Y \backsim N\left(0, \rho\,\mathrm{d}t\right)}$$

and the representation of the dynamics becomes

$$\mathrm{d}F_k(t) = F_k(t)\bar{\sigma}_k(t)\,\mathrm{d}Y\left(t\right).$$

with

$$\bar{\sigma}_k(t) = \sigma_k(t) C_{(k)},$$

where $C_{(k)}$ is the k-th row of C (use Cholesky decomposition, or spectral decomposition).

The rank of the correlation matrix

When $rank\ (\rho) = r < M$, there exists $\underset{M\times r}{B}$ of rank r such that

$$\rho = BB'$$

so that

$$dZ \sim BdY,$$

where Y is r-dimensional independent multivariate normal. The actual number of independent stochastic factors in the model is r.

Given ρ, define Δ the diagonal matrix of its eigenvalues λ_i in decreasing order and X the orthogonal matrix of the corresponding eigenvectors. Then we have the spectral decomposition

$$\rho X = X\Delta.$$

$$\rho = X\Delta X^{-1} = X\Delta X'.$$

Define Λ the diagonal matrix $diag\left(\sqrt{\lambda_i}\right)$ so that

$$\Delta = \Lambda\Lambda,$$

Then

$$\rho = X\Lambda\Lambda X' = (X\Lambda)(X\Lambda)' = CC'$$

If $rank\,(\rho) = r < M$, then eliminating the null columns of C one obtains $\underset{M\times r}{B}$ such that $\rho = BB'$.

Rank Reduction Methods:

Eigenvalue zeroing with normalization:

If $rank\,(\rho) = M$, then eliminating the columns of C associated to the $(M - r)$ smallest eigenvalues of ρ returns a matrix $\underset{M\times r}{\tilde{B}}$ such that

$$\tilde{\rho} = \tilde{B}\tilde{B}'$$

is the best reduced rank approximation of ρ according to Frobenius norm.

In order to recover a correlation matrix one needs to rescale, obtaining $\bar{\rho}$

$$\bar{\rho}_{ij} = \left(\frac{\tilde{B}_i}{\sqrt{\tilde{B}_i\tilde{B}_i'}}\right)\left(\frac{\tilde{B}_j'}{\sqrt{\tilde{B}_i\tilde{B}_i'}}\right).$$

Rank Reduction Methods:

Optimization on angles parameterization (Rebonato and Jackel '99)

Correlations are parameterized through trigonometric functions of a matrix Θ of angles θ. The parametric form for an $M \times M$ correlation matrix of rank $r \leq M$ is

$$A = B^r \left(B^r\right)'$$

where B^r is $M \times r$ with i^{th} row

$$\begin{aligned} b^r_{i,1} &= \cos\theta_{i,1}, \\ b^r_{i,k} &= \cos\theta_{i,k}\sin\theta_{i,1}\cdots\sin\theta_{i,k-1}, \quad 1 < k < r, \\ b^r_{i,r} &= \sin\theta_{i,1}\cdots\sin\theta_{i,r-1}. \\ i &= 1,\ldots,M. \end{aligned}$$

Rank reduction can be achieved by the unconstrained optimization

$$\min_{\Theta} \left\|\rho - B^r\left(\Theta\right)\left(B^r\left(\Theta\right)\right)'\right\|$$

Rank Reduction Methods:

Eigenvalues zeroing by iteration (Morini and Webber '03)

1. SET $s = 1$ AND $\rho^1 = \rho$;
2. REDUCE THE RANK:

$$\underbrace{X^s \begin{bmatrix} \lambda_1^s & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^s & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_r^s & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_M^s \end{bmatrix} X^{s\prime}}_{\rho^s} \Rrightarrow \underbrace{X^s \begin{bmatrix} \lambda_1^s & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^s & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_r^s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} X^{s\prime}}_{\tilde{\rho}^s}$$

3. RECOVER UNIT DIAGONAL:

$$\underbrace{\begin{bmatrix} \tilde{\rho}_{11}^s & \tilde{\rho}_{12}^s & \tilde{\rho}_{13}^s & \tilde{\rho}_{14}^s & \dots & \tilde{\rho}_{1M}^s \\ \tilde{\rho}_{21}^s & \tilde{\rho}_{22}^s & \tilde{\rho}_{23}^s & \tilde{\rho}_{24}^s & \dots & \tilde{\rho}_{2M}^s \\ \tilde{\rho}_{31}^s & \tilde{\rho}_{32}^s & \tilde{\rho}_{33}^s & \tilde{\rho}_{34}^s & \dots & \tilde{\rho}_{3M}^s \\ \tilde{\rho}_{41}^s & \tilde{\rho}_{42}^s & \tilde{\rho}_{43}^s & \tilde{\rho}_{44}^s & \dots & \tilde{\rho}_{4M}^s \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\rho}_{M1}^s & \tilde{\rho}_{M2}^s & \tilde{\rho}_{M3}^s & \tilde{\rho}_{M4}^s & \dots & \tilde{\rho}_{MM}^s \end{bmatrix}}_{\tilde{\rho}^s} \Rrightarrow \underbrace{\begin{bmatrix} 1 & \tilde{\rho}_{12}^s & \tilde{\rho}_{13}^s & \tilde{\rho}_{14}^s & \dots & \tilde{\rho}_{1M}^s \\ \tilde{\rho}_{21}^s & 1 & \tilde{\rho}_{23}^s & \tilde{\rho}_{24}^s & \dots & \tilde{\rho}_{2M}^s \\ \tilde{\rho}_{31}^s & \tilde{\rho}_{32}^s & 1 & \tilde{\rho}_{34}^s & \dots & \tilde{\rho}_{3M}^s \\ \tilde{\rho}_{41}^s & \tilde{\rho}_{42}^s & \tilde{\rho}_{43}^s & 1 & \dots & \tilde{\rho}_{4M}^s \\ \vdots & \vdots & \vdots & \vdots & [I] & \vdots \\ \tilde{\rho}_{M1}^s & \tilde{\rho}_{M2}^s & \tilde{\rho}_{M3}^s & \tilde{\rho}_{M4}^s & \dots & 1 \end{bmatrix}}_{\rho^{s+1}}$$

4. IF STOPPING CONDITION IS TRUE, STOP. ELSE SET $s = s + 1$ AND GO TO 2.

Terminal correlation

$$\frac{E^\gamma\left[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))\right]}{\sqrt{E^\gamma[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))^2]}\sqrt{E^\gamma[(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))^2]}}.$$

Basic approximation:

$$\widetilde{TC}_{ij}(T) = \frac{\int_0^T \sigma_i(t)\sigma_j(t)\rho_{i,j}\, dt}{\sqrt{\int_0^T \sigma_i^2(t)\, dt}\,\sqrt{\int_0^T \sigma_j^2(t)\, dt}}.$$

Term structure of volatility

At time T_j

T_{j+1}	$\longmapsto$	$\frac{1}{T_{j+1} - T_j} \int_{T_j}^{T_{j+1}} \sigma_{j+2}^2(t)dt$
T_{j+2}	$\longmapsto$	$\frac{1}{T_{j+2} - T_j} \int_{T_j}^{T_{j+2}} \sigma_{j+3}^2(t)dt$
$\vdots$		$\vdots$
T_{M-1}	$\longmapsto$	$\frac{1}{T_{M-1} - T_j} \int_{T_j}^{T_{M-1}} \sigma_M^2(t)dt$

CALIBRATION GOALS

- A small calibration error $\left\|\Pi^M(\Theta) - \Pi\right\|$

- Computational efficiency

- Financial reasonableness:

1) Regular instantaneous and terminal correlations
2) Smooth and stable evolution of the term structure of volatilities
3) Implied structures realistic and consistent with market patterns
4) Some stability of calibrated parameters

1) JOINT CAPS-SWAPTIONS MINIMIZATION CALIBRATION WITH SPC STRUCTURE

CALIBRATION ERRORS

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.71	0.90	1.67	4.93	3.00	3.25	2.81	0.83	0.11
2y	-2.43	-3.48	-1.54	-0.70	0.70	0.01	-0.22	-0.45	0.49
3y	-3.84	1.28	-2.44	-0.69	-1.18	0.21	1.51	1.57	-0.01
4y	1.87	-2.52	-2.65	-3.34	-2.17	-0.44	-0.11	-0.63	-0.38
5y	1.80	4.15	-1.40	-1.89	-1.74	-0.79	-0.34	-0.07	1.28
7y	-0.33	2.27	1.47	-0.97	-0.77	-0.65	-0.57	-0.15	0.19
10y	-0.02	0.61	0.45	-0.31	0.02	-0.03	0.01	0.23	-0.30

INSTANTANEOUS CORRELATION

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
10y	0.73	0.89	0.513	0.594	0.985	0.589	0.150	0.757	0.989	1.000
11y	0.98	0.89	0.995	1.000	0.725	1.000	0.883	0.976	0.708	0.596
12y	0.89	0.73	0.981	0.958	0.494	0.959	0.981	0.870	0.473	0.337
13y	0.85	0.66	0.958	0.926	0.410	0.928	0.995	0.820	0.387	0.247
14y	0.77	0.56	0.916	0.873	0.295	0.876	1.000	0.743	0.271	0.126
15y	0.87	0.69	0.968	0.940	0.444	0.942	0.991	0.841	0.422	0.283
16y	0.77	0.55	0.912	0.869	0.286	0.872	0.999	0.737	0.263	0.117
17y	0.55	0.29	0.757	0.691	0.000	0.695	0.948	0.513	-0.024	-0.172
18y	0.61	0.36	0.803	0.742	0.073	0.746	0.969	0.575	0.049	-0.099
19y	1.00	0.93	0.981	0.995	0.789	0.994	0.833	0.993	0.774	0.672

TERMINAL CORRELATIONS

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19
10y	1.00	0.56	0.27	0.19	0.09	0.21	0.08	-0.10	-0.06	0.37
11y	0.56	1.00	0.61	0.75	0.67	0.68	0.64	0.44	0.42	0.50
12y	0.27	0.61	1.00	0.42	0.71	0.53	0.48	0.43	0.40	0.42
13y	0.19	0.75	0.42	1.00	0.36	0.71	0.50	0.41	0.43	0.34
14y	0.09	0.67	0.71	0.36	1.00	0.32	0.67	0.43	0.40	0.36
15y	0.21	0.68	0.53	0.71	0.32	1.00	0.28	0.59	0.39	0.33
16y	0.08	0.64	0.48	0.50	0.67	0.28	1.00	0.22	0.62	0.30
17y	-0.10	0.44	0.43	0.41	0.43	0.59	0.22	1.00	0.17	0.36
18y	-0.06	0.42	0.40	0.43	0.40	0.39	0.62	0.17	1.00	0.07
19y	0.37	0.50	0.42	0.34	0.36	0.33	0.30	0.36	0.07	1.00

EVOLUTION OF THE TERM STRUCTURE OF VOLATILITY

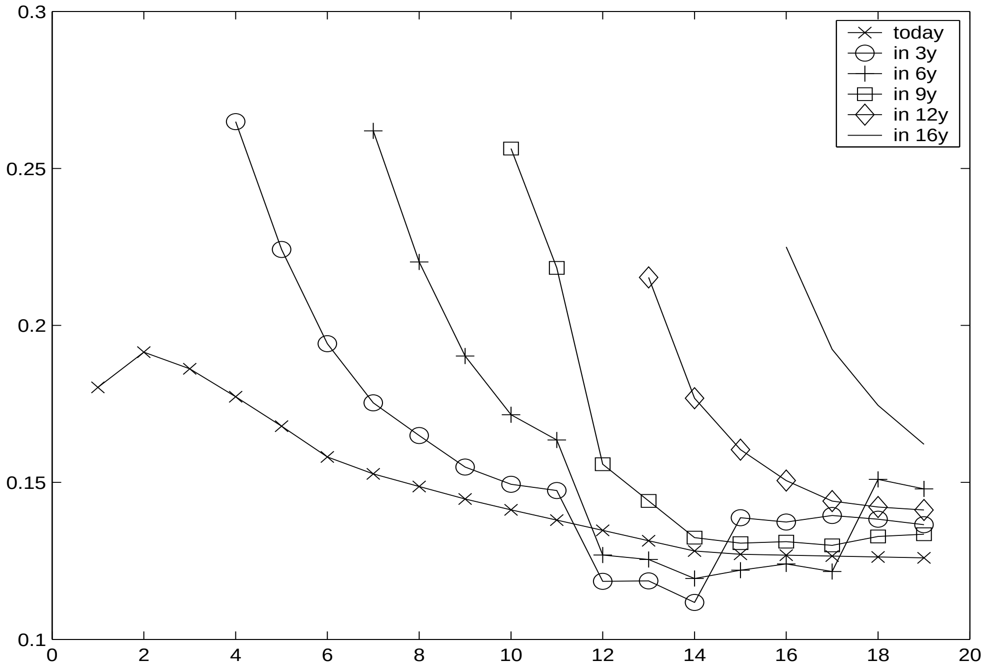


Figure 1:

2) JOINT CAPS-SWAPTIONS MINIMIZATION CALIBRATION WITH LE STRUCTURE AND MORE CONSTRAINTS ON CORRELATIONS

CALIBRATION ERRORS

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	2.28	-3.74	-3.19	-4.68	2.46	1.50	0.72	1.33	-1.42
2y	-1.23	-7.67	-9.97	2.10	0.49	1.33	1.56	-0.44	1.88
3y	2.23	-6.20	-1.30	-1.32	-1.43	1.86	-0.19	2.42	1.17
4y	-2.59	9.02	1.70	0.79	3.22	1.19	4.85	3.75	1.21
5y	-3.26	-0.28	-8.16	-0.81	-3.56	-0.23	-0.08	-2.63	2.62
7y	0.10	-2.59	-10.85	-2.00	-3.67	-6.84	2.15	1.19	0.00
10y	0.29	-3.44	-11.83	-1.31	-4.69	-2.60	4.07	1.11	0.00

INSTANTANEOUS CORRELATION

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	1.000	0.384	0.998	0.388	0.977	0.776	0.642	0.990	0.298	0.890
2y	0.384	1.000	0.447	1.000	0.573	-0.284	0.955	0.249	0.996	0.763
3y	0.998	0.447	1.000	0.451	0.989	0.731	0.693	0.978	0.363	0.919
4y	0.388	1.000	0.451	1.000	0.576	-0.280	0.956	0.253	0.995	0.766
5y	0.977	0.573	0.989	0.576	1.000	0.623	0.791	0.937	0.496	0.967
6y	0.776	-0.284	0.731	-0.280	0.623	1.000	0.015	0.858	-0.370	0.403
7y	0.642	0.955	0.693	0.956	0.791	0.015	1.000	0.526	0.923	0.921
8y	0.990	0.249	0.978	0.253	0.937	0.858	0.526	1.000	0.160	0.816
9y	0.298	0.996	0.363	0.995	0.496	-0.370	0.923	0.160	1.000	0.701
10y	0.890	0.763	0.919	0.766	0.967	0.403	0.921	0.816	0.701	1.000

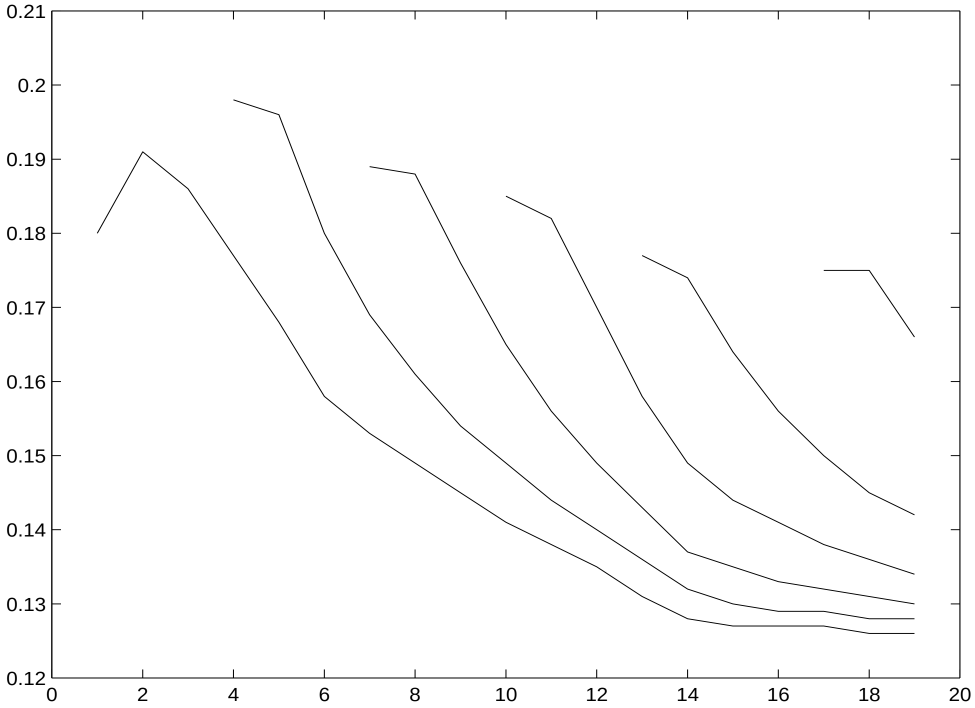


Figure 2:

The swaption cascade calibration

The analytical swaption cascade calibration assumes

- exogenous correlation;
- general piecewise constant volatility (GPC):

$$\sigma_k(t) = \sigma_{k,\varepsilon(t)}, \qquad \varepsilon(t) = i \quad \text{if} \quad T_{i-2} < t \leq T_{i-1}.$$

	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
$F_1(t)$	$\sigma_{1,1}$	-	-	$\ldots$	-
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	-	$\ldots$	-
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

The swaption analytical cascade calibration is based on inverting industry approximate formula for Black swaption volatility ${v_{\alpha,\beta}}^2$.

With GPC the formula reads

$$\sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{T_\alpha \; S_{\alpha,\beta}(0)^2} \sum_{h=0}^{\alpha}(T_h - T_{h-1})\sigma_{i,h+1}\sigma_{j,h+1},$$

$$w_i(t) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}.$$

Black implied volatilities of ATM swaptions on February 1, 2002:

	1	2	3	4	5	6	7	8	9	10
1	**17.90**	**16.50**	**15.30**	**14.40**	**13.70**	**13.20**	**12.80**	**12.50**	**12.30**	**12.00**
2	**15.40**	**14.20**	**13.60**	**13.00**	**12.60**	**12.20**	**12.00**	**11.70**	**11.50**	11.30
3	**14.30**	**13.30**	**12.70**	**12.20**	**11.90**	**11.70**	**11.50**	**11.30**	11.10	10.90
4	**13.60**	**12.70**	**12.10**	**11.70**	**11.40**	**11.30**	**11.10**	10.90	10.80	10.70
5	**12.90**	**12.10**	**11.70**	**11.30**	**11.10**	**10.90**	10.80	10.60	10.50	10.40
6	***12.50***	***11.80***	***11.40***	***10.95***	***10.75***	*10.60*	*10.50*	*10.40*	*10.35*	*10.25*
7	**12.10**	**11.50**	**11.10**	**10.60**	10.40	10.30	10.20	10.20	10.20	10.10
8	***11.80***	***11.20***	***10.83***	*10.40*	*10.23*	*10.17*	*10.10*	*10.10*	*10.07*	*10.00*
9	***11.50***	***10.90***	*10.57*	*10.20*	*10.07*	*10.03*	*10.00*	*10.00*	*9.93*	*9.90*
10	**11.20**	10.60	10.30	10.00	9.90	9.90	9.90	9.90	9.80	9.80

Cascade Calibration Algorithm

1. Fix s, dimension of the swaption matrix;
2. Set $\alpha = 0$;
3. Set $\beta = \alpha + 1$;
4. Solve in $\sigma_{\beta,\alpha+1}$

$$A_{\alpha,\beta}\sigma^2_{\beta,\alpha+1} + B_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C_{\alpha,\beta} = 0,$$

$A_{\alpha,\beta} = w_\beta(0)^2 F_\beta(0)^2 (T_\alpha - T_{\alpha-1}),$

$B_{\alpha,\beta} = 2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j}\ (T_\alpha - T_{\alpha-1})\ \sigma_{j,\alpha+1},$

$C_{\alpha,\beta} = \sum_{i,j=\alpha+1}^{\beta-1} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}\ \sum_{h=0}^{\alpha} (T_h - T_{h-1})\ \sigma_{i,h+1}\ \sigma_{j,h+1}$

$+2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j}\ \sum_{h=0}^{\alpha-1} (T_h - T_{h-1})\ \sigma_{\beta,h+1}\ \sigma_{j,h+1}$

$+w_\beta(0)^2 F_\beta(0)^2 \sum_{h=0}^{\alpha-1} (T_h - T_{h-1})\ \sigma^2_{\beta,h+1} - T_\alpha\ S_{\alpha,\beta}(0)^2\ v^2_{\alpha,\beta}.$

5. Set $\beta = \beta + 1$. If $\beta \leq s$, go to point 4.
   Else set $\alpha = \alpha + 1$;
6. If $\alpha < s$ go to point 3. Else stop.

Extended Cascade Calibration Algorithm

Points 4 and 5 are modified as follows:

4. In case $\beta = s + \alpha$ set

$$\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1} \text{ for } \quad \beta = s + \alpha.$$

The equation to solve is now

$$A^*_{\alpha,\beta}\sigma^2_{\beta,\alpha+1} + B^*_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C^*_{\alpha,\beta} = 0,$$

$$A^*_{\alpha,\beta} = w_\beta(0)^2 F_\beta(0)^2 (T_\alpha - T_{\alpha-1}) + w_\beta(0)^2 F_\beta(0)^2 \sum_{h=0}^{\alpha-1}(T_h - T_{h-1}),$$

$$B^*_{\alpha,\beta} = 2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j} \ (T_\alpha - T_{\alpha-1}) \ \sigma_{j,\alpha+1}$$

$$+2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j} \ \sum_{h=0}^{\alpha-1}(T_h - T_{h-1}) \ \sigma_{j,h+1},$$

$$C^*_{\alpha,\beta} = \sum_{i,j=\alpha+1}^{\beta-1} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j} \ \sum_{h=0}^{\alpha}(T_h - T_{h-1}) \ \sigma_{i,h+1} \ \sigma_{j,h+1}$$

$$-T_\alpha \ S_{\alpha,\beta}(0)^2 \ v^2_{\alpha,\beta}.$$

5. Replace condition $\beta \leq s$ with $(\beta - \alpha) \leq s$.

Cascade Calibration: main features

1. Correlation matrix is an exogenous input;
2. Calibration is fast, through closed form formulas;
3. If industry formula is used for pricing, market prices are fit exactly;
4. Solution is unique given correlation;
5. Induces one-to-one relation between model $\sigma's$ and market volatilities;

Cascade Calibration: problems

0.180	-	-	-	-	-	-	-	-	-
0.155	0.204	-	-	-	-	-	-	-	-
0.129	0.156	0.233	-	-	-	-	-	-	-
0.118	0.104	0.166	0.244	-	-	-	-	-	-
0.109	0.099	0.097	0.161	0.248	-	-	-	-	-
0.113	0.073	0.078	0.101	0.162	0.263	-	-	-	-
0.104	0.098	0.050	0.074	0.113	0.163	0.263	-	-	-
0.094	0.105	0.094	0.032	0.086	0.097	0.168	0.273	-	-
0.106	0.079	0.086	0.082	0.068	0.054	0.092	0.176	0.285	-
0.101	0.092	0.058	0.103	0.151	**-0.032**	0.039	0.085	0.163	0.278
0.097	0.092	0.079	0.043	0.030	0.209	**-0.038**	0.075	0.095	0.185
0.082	0.083	0.083	0.071	0.045	0.062	0.156	**-0.010**	0.073	0.091
0.074	0.074	0.074	0.074	0.080	0.058	0.094	0.123	**-0.016**	0.061
0.070	0.070	0.070	0.070	0.070	0.101	0.051	0.082	0.120	**-0.021**
0.073	0.073	0.073	0.073	0.073	0.073	0.100	0.043	0.062	0.118
0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.074	0.055	0.033
0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.071	0.070
0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.068
0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066

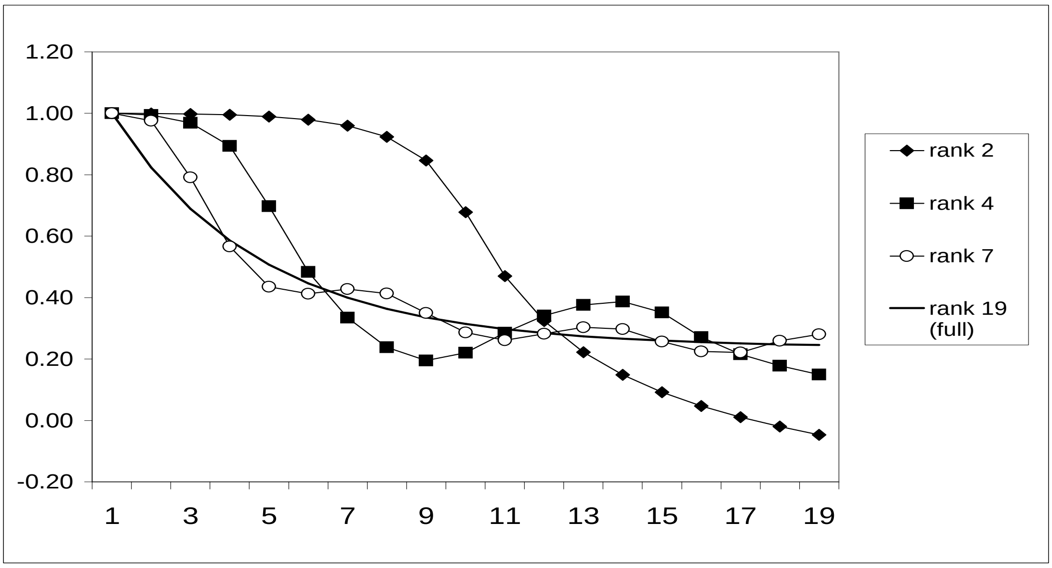
Calibration with Re 3 parameter pivot correlation at rank 7

0.179									
0.153	0.155								
0.144	0.129	0.154							
0.144	0.134	0.105	0.156						
0.140	0.122	0.112	0.112	0.154					
0.143	0.134	0.103	0.101	0.106	0.153				
0.143	0.127	0.143	0.088	0.097	0.086	0.144			
0.146	0.153	0.128	0.078	0.070	0.098	0.093	0.145		
0.157	0.109	0.155	0.160	0.067	0.007	0.101	0.081	0.107	
0.136	0.152	0.126	0.123	0.121	0.108	**-0.04**	0.120	0.077	0.067

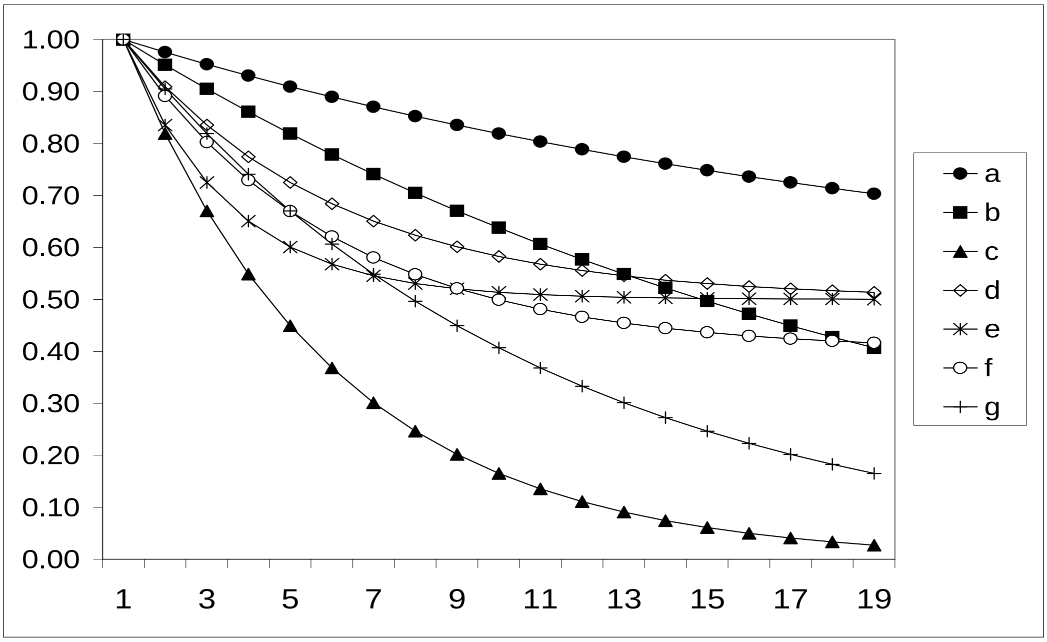
Calibration with Re 3 parameter pivot correlation at rank 3

0.179									
0.152	0.156								
0.131	0.130	0.165							
0.123	0.132	0.120	0.164						
0.128	0.123	0.120	0.118	0.153					
0.141	0.128	0.098	0.101	0.108	0.162				
0.144	0.115	0.122	0.082	0.102	0.106	0.159			
0.147	0.137	0.106	0.065	0.071	0.110	0.114	0.159		
0.156	0.098	0.136	0.131	0.054	0.031	0.119	0.111	0.139	
0.134	0.147	0.117	0.106	0.095	0.086	0.007	0.138	0.102	0.122

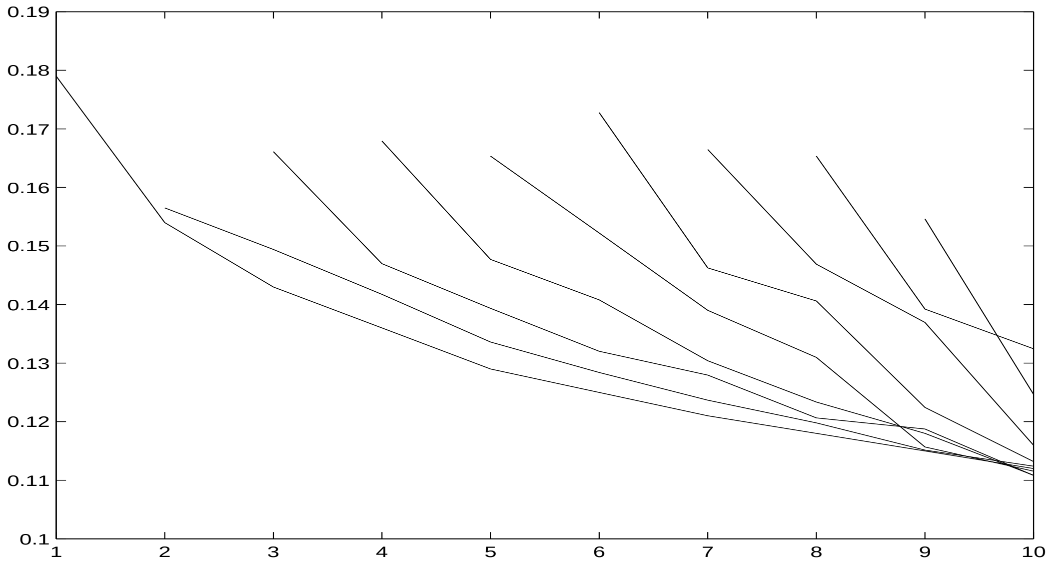
First columns of correlation matrices of different ranks



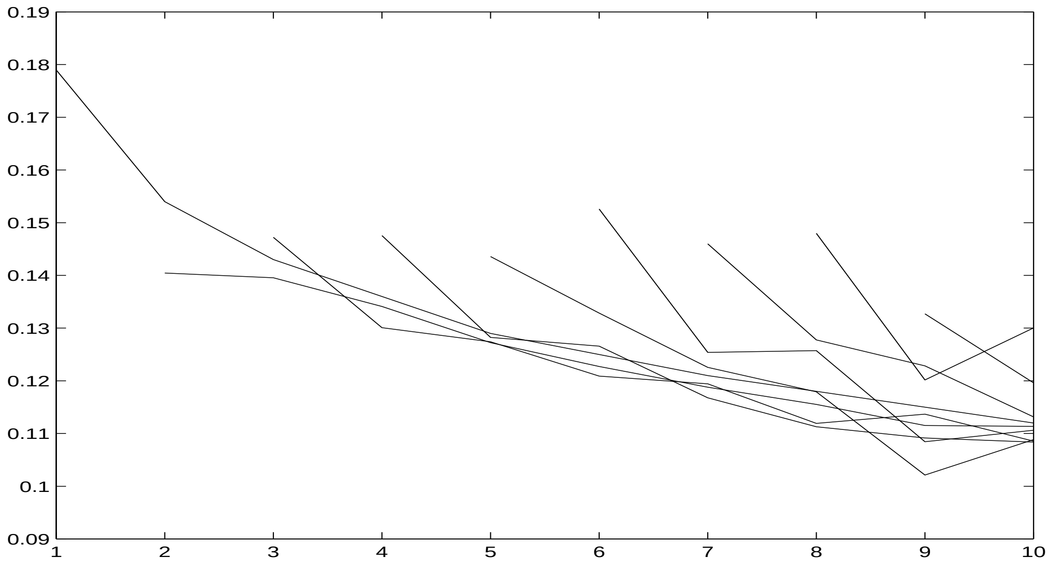
First columns of various configurations of exponential structure



Evolution of the term structure of volatility



TSV with correlation at rank 2



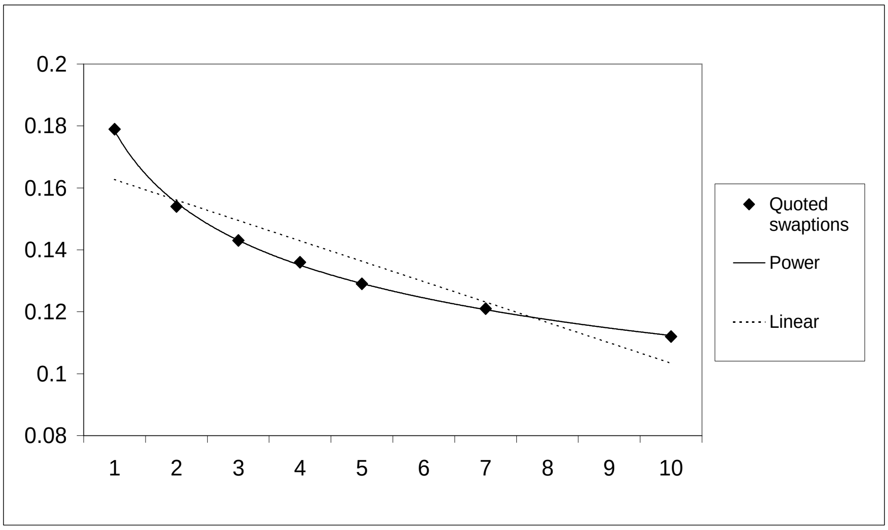
TSV with correlation at rank 10

Terminal correlations with S&C 2 parameter instantaneous correlation matrix at rank 2

	10	11	12	13	14	15	16	17
10	1.000	0.928	0.895	0.920	0.855	0.846	0.928	0.924
11	0.928	1.000	0.863	0.909	0.933	0.881	0.901	0.923
12	0.895	0.863	1.000	0.916	0.908	0.910	0.878	0.939
13	0.920	0.909	0.916	1.000	0.944	0.931	0.956	0.926
14	0.855	0.933	0.908	0.944	1.000	0.954	0.923	0.928
15	0.846	0.881	0.910	0.931	0.954	1.000	0.937	0.958
16	0.928	0.901	0.878	0.956	0.923	0.937	1.000	0.957
17	0.924	0.923	0.939	0.926	0.928	0.958	0.957	1.000

Terminal correlations with S&C 2 parameter instantaneous correlation matrix at rank 10

	10	11	12	13	14	15	16	17
10	1.000	0.887	0.806	0.792	0.708	0.690	0.757	0.734
11	0.887	1.000	0.822	0.837	0.825	0.746	0.749	0.753
12	0.806	0.822	1.000	0.877	0.841	0.820	0.758	0.801
13	0.792	0.837	0.877	1.000	0.919	0.877	0.881	0.806
14	0.708	0.825	0.841	0.919	1.000	0.932	0.878	0.840
15	0.690	0.746	0.820	0.877	0.932	1.000	0.915	0.914
16	0.757	0.749	0.758	0.881	0.878	0.915	1.000	0.934
17	0.734	0.753	0.801	0.806	0.840	0.914	0.934	1.000


0.2
0.18
0.16
0.14
0.12
0.1
0.08
1
2
3
4
5
6
7
8
9
10
Quoted swaptions
Power
Linear

Endogenous Interpolation Cascade Calibration Algorithm (Brigo and Morini 2003)

1. Fix s, dimension of the swaption matrix including non quoted maturities, and set $K :=$

$$\{k \in \mathbb{Z} : 0 \leq k < s - 1, v_{k,y} \text{ missing}, k < y \leq k + s\}$$

2. Set $\alpha = 0$;

3. a. If $\alpha \in K$, set

$$\sigma_{j,m+1} = \sigma_{j,m} = \ldots = \sigma_{j,\alpha+1} =: \sigma_j, \tag{2}$$
$$\alpha + 1 \leq j \leq s,$$
$$m = \min\{i \in \mathbb{Z} : \alpha < i < s,\ i \notin K\}.$$

   Set $\gamma = \alpha$ and $\alpha = m$.

   b. If $\alpha \notin K$, set $\gamma = \alpha$.

   Set $\beta = \alpha + 1$.

4. a. If $\gamma \in K$, solve in σ_β (CCA) adjusting for (2).

   b. If $\gamma \notin K$, solve in $\sigma_{\beta,\alpha+1}$ equation (CCA).

5. Set $\beta = \beta + 1$. If $\beta < s + \gamma$ go to point 4. If $\beta = s + \gamma$, set

$$\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1}$$

   and solve in $\sigma_{\beta,\alpha+1}$ (RCCA). If $\beta < s + \alpha$, repeat point 5, else set $\alpha = \alpha + 1$.

6. If $\alpha < s$, go to point 3, else stop.

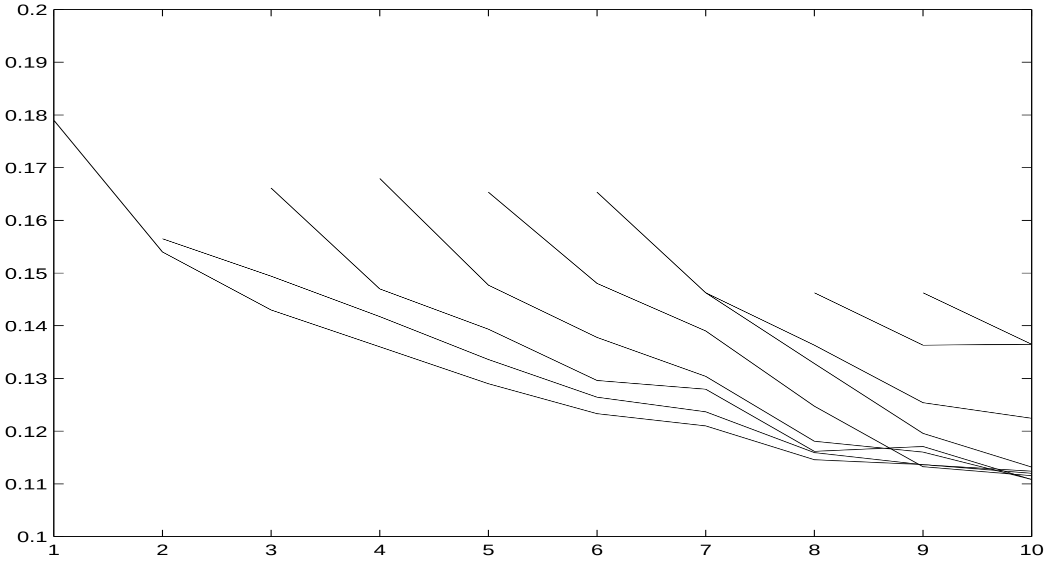
February 1, 2002, with historical correlation at full rank

0.179									
0.167	0.140								
0.153	0.138	0.138							
0.142	0.148	0.130	0.122						
0.135	0.131	0.134	0.135	0.109					
0.142	0.135	0.106	0.118	0.112	0.109				
0.155	0.126	0.145	0.098	0.130	0.087	0.087			
0.150	0.141	0.118	0.099	0.103	0.142	0.142	0.087		
0.130	0.092	0.136	0.153	0.095	0.122	0.122	0.142	0.087	
0.109	0.127	0.116	0.116	0.130	0.088	0.088	0.112	0.112	0.112
0.123	0.123	0.115	0.112	0.166	0.115	0.115	0.118	0.118	0.118
0.111	0.111	0.111	0.165	0.056	0.147	0.147	0.081	0.081	0.081
0.118	0.118	0.118	0.118	0.107	0.102	0.102	0.083	0.083	0.083
0.117	0.117	0.117	0.117	0.117	0.145	0.145	0.097	0.097	0.097
0.127	0.127	0.127	0.127	0.127	0.127	0.127	0.106	0.106	0.106
0.104	0.104	0.104	0.104	0.104	0.104	0.104	0.135	0.135	0.135
0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114
0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120
0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166

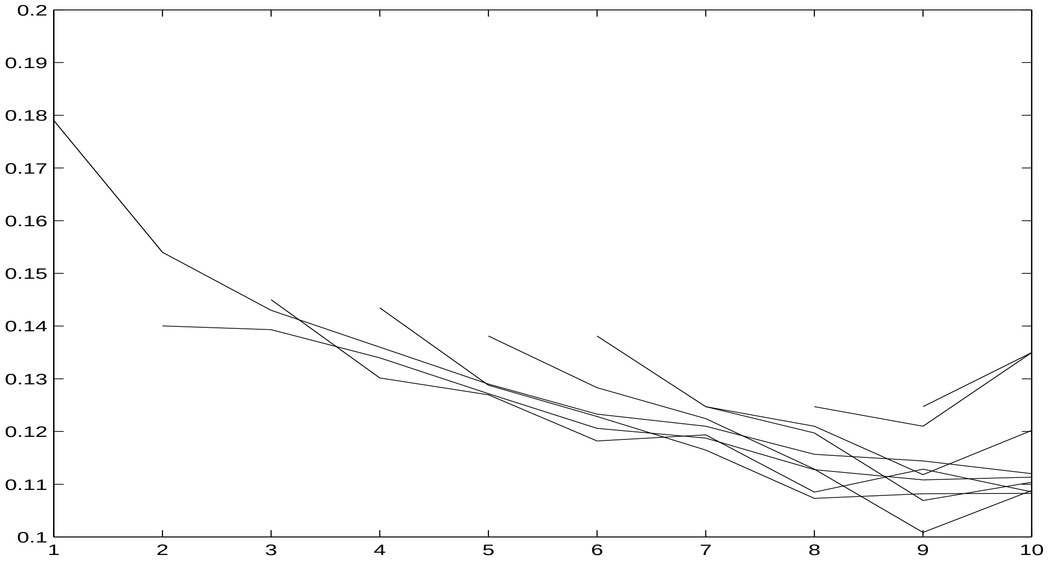
May 16, 2000, Brigo and Mercurio (2001)

0.180									
0.155	0.204								
0.129	0.156	0.233							
0.118	0.104	0.166	0.244						
0.109	0.099	0.097	0.161	0.248					
0.113	0.073	0.078	0.101	0.162	0.248				
0.104	0.098	0.050	0.074	0.113	0.219	0.219			
0.094	0.105	0.094	0.032	0.086	0.137	0.137	0.219		
0.106	0.079	0.086	0.082	0.068	0.075	0.075	0.137	0.219	
0.101	0.092	0.058	0.103	0.151	0.013	0.013	0.194	0.194	0.194
0.092	0.092	0.079	0.043	0.030	0.061	0.061	0.135	0.135	0.135
0.083	0.083	0.083	0.071	0.049	0.113	0.113	0.058	0.058	0.058
0.074	0.074	0.074	0.074	0.080	0.078	0.078	0.045	0.045	0.045
0.070	0.070	0.070	0.070	0.070	0.073	0.073	0.045	0.045	0.045
0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077
0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.049	0.049	0.049
0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071
0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.069	0.069
0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066

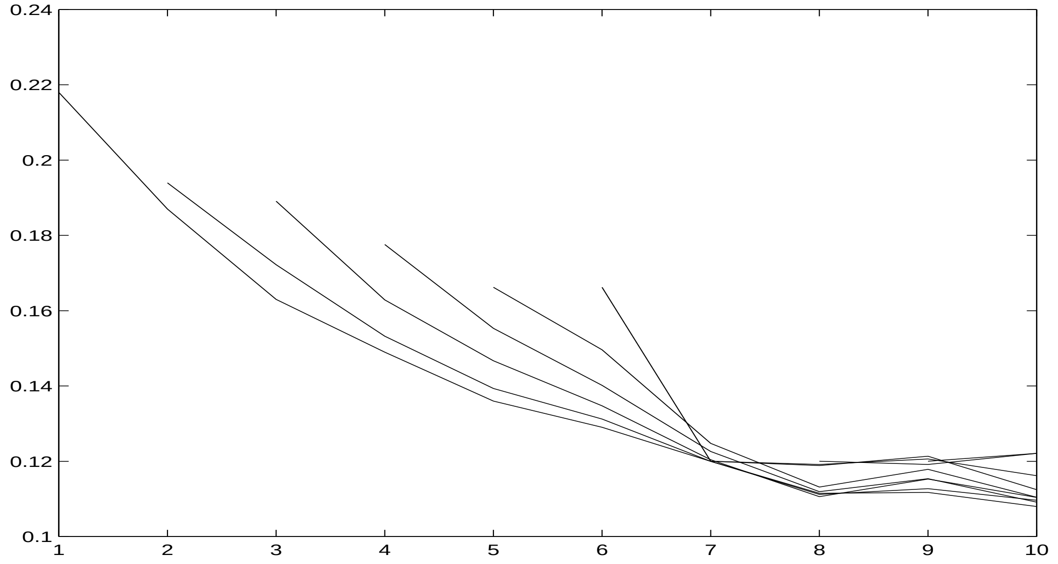
February 1, 2002, parametric correlation at rank 2



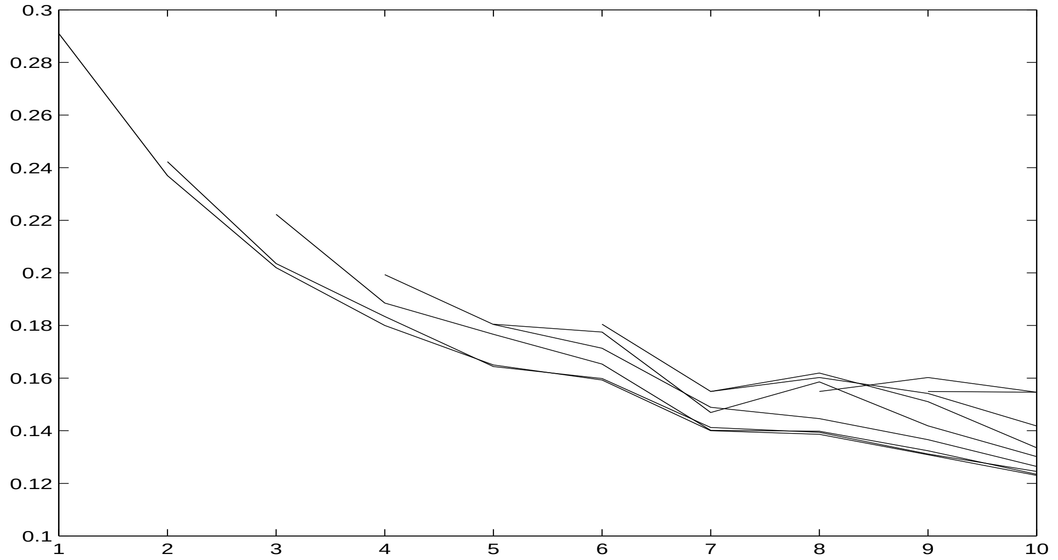
February 1, 2002, parametric correlation at rank 19



December 10, 2002, historical correlation at rank 2



October 10, 2003, historical correlation at rank 19



ENDOGENOUS INTERPOLATION

	1	2	3	4	5	6	7	8	9	10
1	0.291	0.252	0.228	0.209	0.197	0.185	0.173	0.165	0.159	0.153
2	0.237	0.209	0.193	0.178	0.167	0.158	0.152	0.147	0.143	0.140
3	0.202	0.185	0.172	0.161	0.150	0.144	0.139	0.135	0.132	0.130
4	0.180	0.168	0.157	0.147	0.138	0.133	0.129	0.126	0.123	0.122
5	0.165	0.155	0.143	0.135	0.129	0.125	0.122	0.119	0.117	0.116
6	*0.159*	**0.143**	**0.135**	**0.129**	**0.123**	**0.120**	**0.117**	**0.114**	**0.112**	**0.111**
7	0.140	0.134	0.127	0.120	0.116	0.113	0.110	0.108	0.107	0.106
8	*0.139*	*0.130*	**0.122**	**0.117**	**0.114**	**0.110**	**0.107**	**0.105**	**0.104**	**0.102**
9	*0.131*	**0.123**	**0.117**	**0.113**	**0.109**	**0.105**	**0.103**	**0.102**	**0.100**	**0.099**
10	0.123	0.116	0.111	0.106	0.102	0.100	0.099	0.097	0.096	0.096

STABILITY

Average $|\Delta\sigma|$	Feb-02	Dec-02	Oct-03
Feb-02	0.000	0.018	0.029
Dec-02	0.018	0.000	0.023
Oct-03	0.029	0.023	0.000

Max $|\Delta\sigma|$	Feb-02	Dec-02	Oct-03
Feb-02	0.000	0.066	0.113
Dec-02	0.066	0.000	0.109
Oct-03	0.113	0.109	0.000

Average $|\Delta v|$	Feb-02	Dec-02	Oct-03
Feb-02	0.000	0.009	0.029
Dec-02	0.009	0.000	0.020
Oct-03	0.029	0.020	0.000

Max $|\Delta v|$	Feb-02	Dec-02	Oct-03
Feb-02	0.000	0.039	0.112
Dec-02	0.039	0.000	0.073
Oct-03	0.112	0.073	0.000

MC TESTS

1 February 2002, 5×6 swaption and Rebonato 3 par. rank 2 correlation:

MC volatility	MC inf	MC sup	Approximation
0.108612	0.108112	0.109112	0.109000

Upwardly shifted volatilities:

	MC volatility	MC inf	MC sup	Approx.
Cal. vol's $\times 1.2$	0.130261	0.129644	0.130878	0.130800
Cal. vol's $+0.2$	0.300693	0.298988	0.302399	0.303820
	MC vol (no a.v.)	MC inf	MC sup	Approx.
Cal. vol's $+0.2$	0.301934	0.299010	0.304861	0.303820

10 December 2002, 10×10 swaption, historically estimated correlation at full rank 19:

	MC volatility	MC inf	MC sup	Approx.
Cal. vol's	0.094937	0.094512	0.095363	0.095000
Cal. vol's $\times 1.2$	0.113853	0.113333	0.114372	0.114000
Cal. vol's $+0.2$	0.273425	0.272064	0.274788	0.277094
	MC vol (no a.v.)	MC inf	MC sup	Approx.
Cal. vol's $+0.2$	0.273317	0.270865	0.275774	0.277094

Upwardly shifted forward rates:

MC volatility	MC inf	MC sup	Approximation
0.097716	0.097301	0.098130	0.098048

INCLUDING CAP MARKET - Annualization

With $0 < S < T < U$, all six-months spaced, consider an $S \times 1$ swaption and S and T-expiry six-month caplets associated with semi-annual forward rates $F_1(t)$ and $F_2(t)$ with correlation ρ (infracorrelation).

Approximation for constant volatility forward rates:

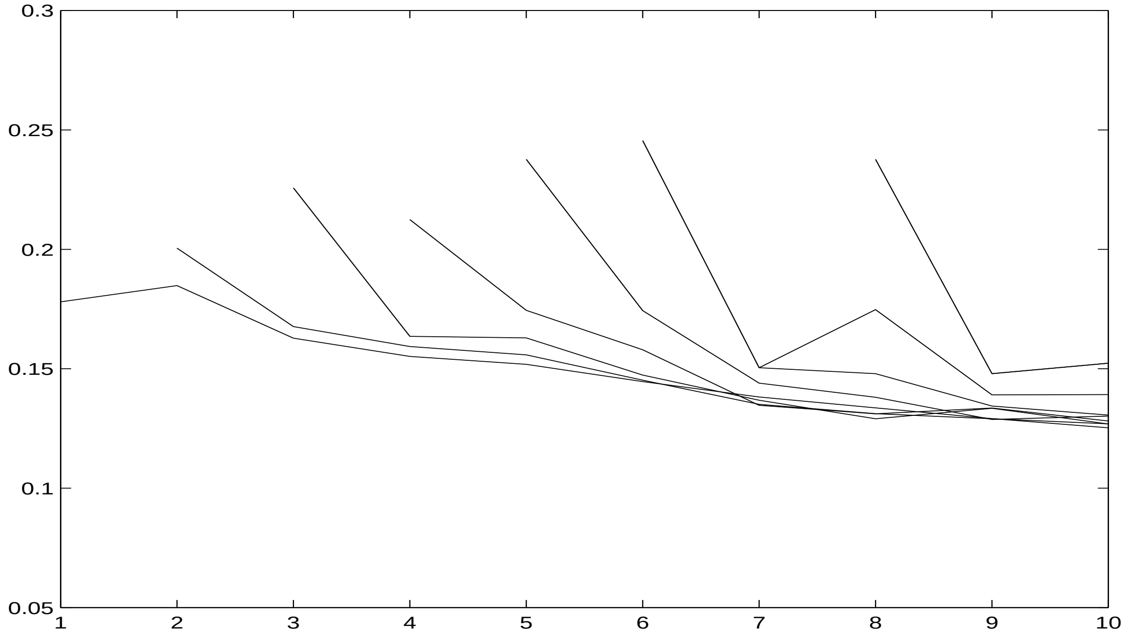
$$v_{\rm sw}^2 \approx$$

$$u_1^2(0) v_{S-caplet}^2 + u_2^2(0) v_{T-caplet}^2 + 2\rho u_1(0) u_2(0) v_{S-caplet} v_{T-caplet},$$

$$u_1(t) = \frac{1}{F(t)} \left(\frac{F_1(t)}{2} + \frac{F_1(t) F_2(t)}{4} \right)$$
$$u_2(t) = \frac{1}{F(t)} \left(\frac{F_2(t)}{2} + \frac{F_1(t) F_2(t)}{4} \right)$$

Annualized caplet volatilities									
0.178	0.185	0.163	0.155	0.152	0.145	0.138	0.134	0.129	0.125

Implied evolution of term structure of volatility



Infra-correlations									
1.022	0.388	0.543	0.536	0.444	0.493	0.533	0.560	0.586	0.598

Endogenous Interpolation Cascade Calibration - Conclusions

1. Calibration is fast, through closed form formulas;
2. If industry formula is used, market prices are fit exactly;
3. Solution is unique given correlation;
4. One-to-one correspondence model-market volatilities;
5. Exogenous correlation matrix: regular instantaneous and terminal correlations;
6. Parameters satisfy required constraints - appear stable over time - are consistent with market patterns - imply acceptably regular evolution of the TSV;

Pricing with the Libor Market Model

- Pricing with **exact formulas**

Example: In-Advance Swaps and Caps

- Pricing with **approximations**

Example for a tailored approximation: Zero-Coupon Swaptions

Example for a general approximation: CMS

- Pricing with **non-standard-tenor rates**:

Example: Accrual Swaps

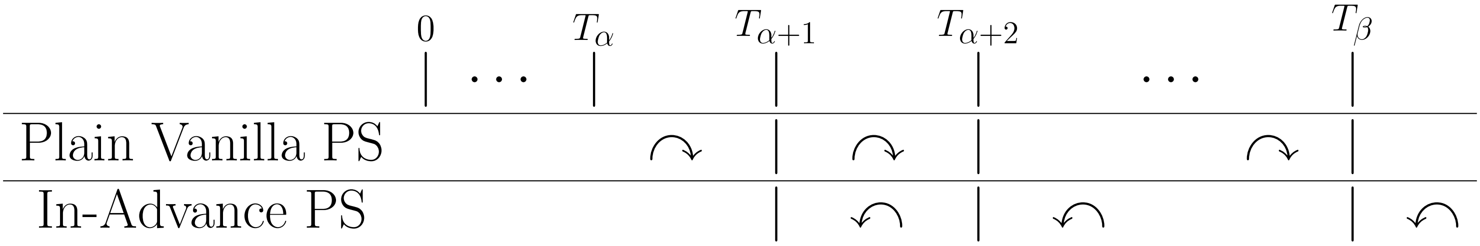
Pricing with exact formulas: In-Advance Swap

Plain-Vanilla Swap payoff

$$\sum_{i=\alpha+1}^{\beta} D\left(0, T_i\right) \tau_i (F(T_{i-1}; T_{i-1}, T_i) - K),$$

In-Advance Swap payoff

$$\sum_{i=\alpha+1}^{\beta} D\left(0, T_i\right) \tau_{i+1} (F(T_i; T_i, \boxed{T_{i+1}}) - K).$$



Plain-Vanilla Swap Payments at T_i

$$\begin{aligned} &\tau_i (F(T_{i-1}; T_{i-1}, T_i) - K) = \\ = \; &\tau_i \left(\frac{1}{\tau_i} \left(\frac{1}{P\left(T_{i-1}, T_i\right)} - 1 \right) - K \right) = \\ = \; &\frac{1}{P\left(T_{i-1}, T_i\right)} - 1 - K\tau_i. \end{aligned}$$

We can replicate this payoff with a portfolio of bonds:

$$= P\left(0, T_{i-1}\right) - P\left(0, T_i\right) - P\left(0, T_i\right) K\tau_i.$$

In-Advance Swap Payments at T_i

$$\tau_{i+1}(F(T_i, T_i, T_{i+1}) - K\tau_{i+1}) =$$
$$= \frac{1}{P(T_i, T_{i+1})} - 1 - K\tau_{i+1}.$$

Using Change of Numeraire, the price of a plain vanilla swap is

$$\mathbb{E}^Q \left[\sum_{i=\alpha+1}^{\beta} D(0, T_i)\, \tau_i(F(T_{i-1}; T_{i-1}, T_i) - K) \right] =$$
$$= \sum_{i=\alpha+1}^{\beta} \mathbb{E}^Q \left[D(0, T_i)\, \tau_i(F(T_{i-1}; T_{i-1}, T_i) - K) \right] =$$
$$\overset{by\ CoN}{=} \sum_{i=\alpha+1}^{\beta} P(0, T_i)\, \mathbb{E}^i \left[\tau_i(F(T_{i-1}; T_{i-1}, T_i) - K) \right] =$$
$$= \sum_{i=\alpha+1}^{\beta} P(0, T_i)\, \tau_i(F(0; T_{i-1}, T_i) - K).$$

The price of an in-advance swap is

$$\mathbb{E}^Q \left[\sum_{i=\alpha+1}^{\beta} D(0, T_i)\, \tau_{i+1}(F(T_i; T_i, T_{i+1}) - K) \right] =$$
$$= \sum_{i=\alpha+1}^{\beta} P(0, T_i)\, \mathbb{E}^i \left[\tau_i(F(T_i; T_i, T_{i+1}) - K) \right].$$

Law of Iterated Expectation

$$\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}_S\right] \mid \mathcal{F}_T\right] = \mathbb{E}\left[X \mid \mathcal{F}_T\right], \quad T < S$$

Proposition 1 The current value of the amount X_T at T is equivalent to the current value of the accrued amount $\dfrac{X_T}{P(T,S)}$ at $S > T$:

$$\boxed{\mathbb{E}_t\left[\tfrac{D(t,S)}{P(T,S)} X_T\right]} = \mathbb{E}_t\left[\mathbb{E}_T\left[\frac{D(t,S)}{P(T,S)} X_T\right]\right] =$$

$$= \mathbb{E}_t\left[\mathbb{E}_T\left[\frac{D(t,T)\,D(T,S)}{P(T,S)} X_T\right]\right] =$$

$$= \mathbb{E}_t\left[\frac{D(t,T)\,X_T}{P(T,S)} \mathbb{E}_T\left[D(T,S)\right]\right] =$$

$$= \mathbb{E}_t\left[\frac{D(t,T)\,X_T}{P(T,S)} P(T,S)\right] = \boxed{\mathbb{E}_t\left[D(t,T)\,X_T\right]}$$

$\underline{\text{IAS}\Pi} =$

$$= \mathbb{E}\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F(T_i;T_i,T_{i+1}) - K)\right] =$$

$$\overset{by\ def\ of\ F}{=} \sum_{i=\alpha+1}^{\beta} \mathbb{E}\left[D(0,T_i)(\frac{1}{P\left(T_i,T_{i+1}\right)} - 1 - \tau_{i+1}K)\right] =$$

$$= \sum_{i=\alpha+1}^{\beta} \mathbb{E}\left[\frac{D(0,T_i)}{P\left(T_i,T_{i+1}\right)} - D(0,T_i)\left(1 + \tau_{i+1}K\right)\right] =$$

$$\overset{by\ Prop\ 1}{=} \sum_{i=\alpha+1}^{\beta} \mathbb{E}\left[\frac{D(0,T_{i+1})}{P\left(T_i,T_{i+1}\right)^2} - D(0,T_i)\left(1 + \tau_{i+1}K\right)\right] =$$

$$= \sum_{i=\alpha+1}^{\beta} \left[\mathbb{E}\left(\frac{D(0,T_{i+1})}{P\left(T_i,T_{i+1}\right)^2}\right) - \mathbb{E}\left(D(0,T_i)\left(1 + \tau_{i+1}K\right)\right)\right] =$$

$$\overset{by\ CoN}{=} \sum_{i=\alpha+1}^{\beta} \left[P(0,T_{i+1})\mathbb{E}^{i+1}\left(\frac{1}{P\left(T_i,T_{i+1}\right)^2}\right)\right.$$
$$\left. - P(0,T_i)\left(1 + \tau_{i+1}K\right)\right].$$

In terms of forward rates:

$$\underline{\text{IAS}\Pi} = \sum_{i=\alpha+1}^{\beta} \Big[P(0,T_{i+1})\mathbb{E}^{i+1}\left(1+2F(T_i;T_i,T_{i+1})\tau_{i+1} + \right.$$
$$\left. + \boxed{F(T_i;T_i,T_{i+1})^2}\tau_{i+1}^2\right) - P(0,T_i)\left(1+\tau_{i+1}K\right)\Big]$$

From the **dynamics** of $F(t;T_i,T_{i+1})$

$dF(t;T_i,T_{i+1}) = \sigma_{i+1}(t)F(t;T_i,T_{i+1})\,dZ_{i+1}(t)$

via **Ito formula**

$dF(t;T_i,T_{i+1})^2 = \sigma_{i+1}^2(t)F(t;T_i,T_{i+1})^2\,dt$

$+2\sigma_{i+1}(t)F(t;T_i,T_{i+1})^2\,dZ_{i+1}(t)$

with **solution**

$F(T;T_i,T_{i+1})^2$:

$$F(0;T_i,T_{i+1})^2 \exp \underbrace{\left\{ \int_0^T -\sigma_{i+1}^2(s)\,ds + \int_0^T 2\sigma_{i+1}(s)\,dZ_{i+1}(s) \right\}}_{N\left(\int_0^T -\sigma_{i+1}^2(s)\,ds\ ,\ \int_0^T 4\sigma_{i+1}^2(s)\,ds\right)},$$

Via **MGF** of X$\sim N\left(\mu,\sigma^2\right)$

$E\left[e^{tX}\right] = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

we find

$E_0\left[F(T;T_i,T_{i+1})^2\right] = F(0;T_i,T_{i+1})^2 e^{\int_0^T \sigma_{i+1}^2(s)\,ds}.$

<u>IASΠ</u>=

$= \sum_{i=\alpha+1}^{\beta} \left[P(0,T_{i+1}) \mathbb{E}^{i+1} \left(1 + 2F(T_i;T_i,T_{i+1})\tau_{i+1} + \right.\right.$

$\left.\left. + F(T_i;T_i,T_{i+1})^2 \tau_{i+1}^2 \right) - P(0,T_i)\left(1+\tau_{i+1}K\right) \right] =$

$$\boxed{\begin{array}{l} = \displaystyle\sum_{i=\alpha+1}^{\beta} \left[P(0,T_{i+1}) \left(1 + 2\tau_{i+1} F(0;T_i,T_{i+1}) + \right.\right. \\ \left.\left. + \tau_{i+1}^2 F(0;T_i,T_{i+1})^2 e^{\int_0^{T_i} \sigma_{i+1}^2(s)\,ds} \right) - P(0,T_i)\left(1+\tau_{i+1}K\right) \right] \end{array}}$$

<u>In-advance cap</u> paying and resetting at $T_{\alpha+1}, \ldots, T_\beta$, with unit notional amount and strike K. The payoff discounted at $t = 0$ is

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i) - K)^+$$

The price is

$$\text{IACΠ} = E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i) - K)^+ \right]$$

Applying the same reasoning seen for in-advance swaps we get:

$$\begin{aligned} \text{IACΠ} &= E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\left(\frac{1}{P(T_i,T_{i+1})} - 1 - \tau_{i+1}K\right)^+ \right] \\ &= E\left[\sum_{i=\alpha+1}^{\beta} \left(\frac{D(0,T_i)}{P(T_i,T_{i+1})} - D(0,T_i)\left(1+\tau_{i+1}K\right)\right)^+ \right] \end{aligned}$$

$$
\begin{aligned}
\text{IAC}\Pi &= \mathbb{E}\left[\sum_{i=\alpha+1}^{\beta}\left(\frac{D(0,T_{i+1})}{P\left(T_i,T_{i+1}\right)^2} - \frac{D(0,T_{i+1})}{P\left(T_i,T_{i+1}\right)}\left(1+\tau_{i+1}K\right)\right)^+\right] \\
&= E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_{i+1})\Big(\left(1+\tau_{i+1}F_{i+1}(T_i)\right)^2\right. \\
&\quad \left. - \left(1+\tau_{i+1}F_{i+1}(T_i)\right)\left(1+\tau_{i+1}K\right)\Big)^+\right] \\
&= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1})\tau_{i+1}E^{i+1}\left[\left(1+\tau_{i+1}F_{i+1}(T_i)\right)\right. \\
&\quad \left.\left(F_{i+1}(T_i)-K\right)^+\right] \\
&= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1})\tau_{i+1}\Big\{E^{i+1}\left[\left(F_{i+1}(T_i)-K\right)^+\right] \\
&\quad + \tau_{i+1}E^{i+1}\left[\Big(F_{i+1}(T_i)^2 - F_{i+1}(T_i)K\Big)^+\right]\Big\}
\end{aligned}
$$

First expectation can be computed by Black formula, second by Margrabe formula for options to exchange one asset for another.

Pricing with Approximations: ZeroCoupon Swaption

The payoff of a Zero-Coupon Swap is

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\,\tau_i F(T_{i-1};T_{i-1},T_i) - D(0,T_\beta)\,\tau_{\alpha,\beta}K.$$

The price is

$$\begin{aligned}
&\sum_{i=\alpha+1}^{\beta} P(0,T_i)\,\tau_i F(0;T_{i-1},T_i) - P(0,T_\beta)\,\tau_{\alpha,\beta}K = \\
= &\sum_{i=\alpha+1}^{\beta} P(0,T_i)\,\tau_i\left(\frac{1}{\tau_i}\left(\frac{P(0,T_{i-1})}{P(0,T_i)} - 1\right)\right) - P(0,T_\beta)\,\tau_{\alpha,\beta}K \\
&= P(0,T_\alpha) - P(0,T_\beta) - P(0,T_\beta)\,\tau_{\alpha,\beta}K
\end{aligned}$$

Zero-Coupon Swaption:

$$D(0,T_\alpha)\left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\,\tau_i F(T_\alpha;T_{i-1},T_i) - P(T_\alpha,T_\beta)\,\tau_{\alpha,\beta}K\right]^+$$

$$\begin{aligned}
&= D(0,T_\alpha)\left[1 - P(T_\alpha,T_\beta) - P(T_\alpha,T_\beta)\,\tau_{\alpha,\beta}K\right]^+ \\
&= D(0,T_\alpha)\,\tau_{\alpha,\beta}P(T_\alpha,T_\beta)\left[F(T_\alpha;T_\alpha,T_\beta) - K\right]^+
\end{aligned}$$

$F(t; T_\alpha, T_\beta)$ and $F(t; T_i, T_{i+1})$ rates:

$$F(t; T_\alpha, T_\beta) = \frac{1}{\tau_{\alpha,\beta}} \left(\frac{P(t, T_\alpha)}{P(t, T_\beta)} - 1 \right),$$

$$\frac{P(t, T_\alpha)}{P(t, T_\beta)} = \frac{P(t, T_\alpha)}{P(t, T_{\alpha+1})} \frac{P(t, T_{\alpha+1})}{P(t, T_{\alpha+2})} \cdots \frac{P(t, T_{\beta-1})}{P(t, T_\beta)}$$

$$1 + \tau_{\alpha,\beta} \underbrace{F(t; T_\alpha, T_\beta)}_{F(t)} = \prod_{i=\alpha+1}^{\beta} \left(1 + \tau_i \underbrace{F(t; T_{i-1}, T_i)}_{F_i(t)} \right).$$

Approximations: what for?

When we have driftless lognormal dynamics

$$dX = \nu(t) X \, dW,$$

roughly one finds Black input V^2 as

1) $\dfrac{dX}{X} = \nu(t) \; dW$

2) $\dfrac{dX dX}{X^2} = \nu(t) \nu(t) \; dW \; dW = \nu(t)^2 \; dt$

3) $V^2 = \int_0^T \nu(t)^2 \; dt$

When dynamics is different, we may find at 2) a stochastic

$$\tilde{\nu}(t, X(t, \omega))^2 \; dt.$$

Then, to use Black, take the approximation

$$\tilde{V}^2 = \int_0^T \tilde{\nu}\left(t, \boxed{X(0)}\right)^2 \; dt.$$

$$\underline{ZC\ Swaption\ Payoff} = D\left(0,T_\alpha\right)\tau_{\alpha,\beta}P\left(T_\alpha,T_\beta\right)\left[F(T_\alpha)-K\right]^+$$

$$1+\tau_{\alpha,\beta}F(t) = \prod_{i=\alpha+1}^{\beta}\left(1+\tau_i F_i\left(t\right)\right).$$

$$\ln\left(1+\tau_{\alpha,\beta}F(t)\right) = \sum_{i=\alpha+1}^{\beta}\ln\left(1+\tau_i F_i\left(t\right)\right).$$

$$d\ln\left(1+\tau_{\alpha,\beta}F(t)\right) = \sum_{i=\alpha+1}^{\beta}d\ln\left(1+\tau_i F_i\left(t\right)\right).$$

Apply Ito's formula to each addend

$$\begin{aligned} g(t,F_i) &= \ln\left(1+\tau_i F_i\left(t\right)\right), \\ dg(t,F_i) &= g'_{F_i}\, dF_i + \textit{drift part}, \end{aligned}$$

$$d\ln\left(1+\tau_{\alpha,\beta}F(t)\right) = \sum_{i=\alpha+1}^{\beta}\frac{\tau_i\ dF_i\left(t\right)}{\left(1+\tau_i F_i\left(t\right)\right)} + \left(\cdots\right)\ dt.$$

Apply Ito's formula to the left-hand side

$$d\ln\left(1+\tau_{\alpha,\beta}F(t)\right) = \frac{\tau_{\alpha,\beta}\ dF\left(t\right)}{\left(1+\tau_{\alpha,\beta}F\left(t\right)\right)} + \left(\cdots\right)\ dt$$

$$\Downarrow$$

$$dF\left(t\right) = \frac{\left(1+\tau_{\alpha,\beta}F\left(t\right)\right)}{\tau_{\alpha,\beta}}d\ln\left(1+\tau_{\alpha,\beta}F(t)\right)\left(\cdots\right)\ dt.$$

Putting together

$$dF(t) = \frac{(1+\tau_{\alpha,\beta}F(t))}{\tau_{\alpha,\beta}} \sum_{i=\alpha+1}^{\beta} \frac{\tau_i \; dF_i(t)}{(1+\tau_i F_i(t))} + (\cdots)\; dt$$

$$\frac{dF(t)\, dF(t)}{F(t)\, F(t)} =$$

$$= \left(\frac{(1+\tau_{\alpha,\beta}F(t))}{\tau_{\alpha,\beta}F(t)}\right)^2 \sum_{i=\alpha+1}^{\beta}\sum_{j=\alpha+1}^{\beta} \frac{\tau_i \; dF_i(t)}{(1+\tau_i F_i(t))}\frac{\tau_j \; dF_j(t)}{(1+\tau_j F_j(t))}$$

$$= \left(\frac{(1+\tau_{\alpha,\beta}F(t))}{\tau_{\alpha,\beta}F(t)}\right)^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i\tau_j F_i(t)\, F_j(t)\, \sigma_i(t)\, \sigma_j(t)\, \rho_{ij}}{(1+\tau_i F_i(t))\,(1+\tau_j F_j(t))} dt,$$

since

$$\begin{aligned} dF_i(t)\, dF_j(t) &= \sigma_i(t)\, \sigma_j(t)\, F_i(t)\, F_j(t)\, dZ_i dZ_j = \\ &= \sigma_i(t)\, \sigma_j(t)\, F_i(t)\, F_j(t)\, \rho_{ij}\; dt. \end{aligned}$$

Freeze rates at their 0 value, finding

$$\left(V_{\alpha,\beta}^{ZC}\right)^2 = \left(\frac{1+\tau_{\alpha,\beta}F(0)}{\tau_{\alpha,\beta}F(0)}\right)^2 .$$

$$\sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i\tau_j F_i(0)\, F_j(0)\, \rho_{ij}}{(1+\tau_i F_i(0))\,(1+\tau_j F_j(0))} \int_0^{T_\alpha} \sigma_i(t)\, \sigma_j(t)\, dt$$

$\underline{\text{ZCSwaption}\Pi} = P(0,T_\beta)\tau_{\alpha,\beta}\text{Black}(F(0;T_\alpha,T_\beta),K,V_{\alpha,\beta}^{ZC}).$

Testing the approximation for Zero-Coupon Swaption volatility

The approximated volatility turns out very close to the volatility computed via MC, confirming the approximation is accurate. But consider also:

- $T_\alpha = 2$ years, $T_\beta = 19$ years

$$\text{Monte Carlo ZC Swaption Volatility } \frac{V_{ZC}^{MC}}{T_\alpha} = 0.1410$$

$$\text{Approximate ZC Swaption volatility } \frac{V_{ZC}^{App}}{T_\alpha} = 0.1455$$

$$\text{Approximate PV Swaption volatility } \frac{V_{PV}^{App}}{T_\alpha} = 0.0997$$

Monte Carlo 98% window [0.1404 0.1416]

- $T_\alpha = 10$ years, $T_\beta = 19$ years

$$\text{Monte Carlo ZC Swaption Volatility } \frac{V_{ZC}^{MC}}{T_\alpha} = 0.1081$$

$$\text{Approximate ZC Swaption volatility } \frac{V_{ZC}^{App}}{T_\alpha} = 0.1114$$

$$\text{Approximate PV Swaption volatility } \frac{V_{PV}^{App}}{T_\alpha} = 0.0897$$

Monte Carlo 98% window [0.1076 0.1086]

A relationship between the approximations V_{PV}^{App} and V_{ZC}^{App}

$$V_{PV}^{App} = \sum_{i,j=\alpha+1}^{\beta} \lambda_i \lambda_j \rho_{ij} \int_0^{T_\alpha} \sigma_i(t)\,\sigma_j(t)\;dt, \quad \lambda_i = \frac{w_i(0)\,F_i(0)}{S_{\alpha\beta}(0)},$$

$$w_i(0) = \frac{\tau_i P(0,T_i)}{\sum_{k=\alpha+1}^{\beta} \tau_k P(0,T_k) F_k(0)}, \quad S_{\alpha\beta}(0) = \sum_{i=\alpha+1}^{\beta} w_i(0)\,F_i(0)$$

$$V_{ZC}^{App} = \sum_{i,j=\alpha+1}^{\beta} \mu_i \mu_j \rho_{ij} \int_0^{T_\alpha} \sigma_i(t)\,\sigma_j(t)\;dt, \quad \mu_i = \frac{(1+\tau_{\alpha,\beta}F(0))}{\tau_{\alpha,\beta}F(0)} \frac{\tau_i F_i(t)}{(1+\tau_i F_i(t))},$$

Using the relations between forward-rates and discount factors

$$\lambda_i = \frac{\tau_i P(0,T_i) F_i(0)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) F_i(0)} = \frac{P(0,T_{i-1}) - P(0,T_i)}{P(0,T_\alpha) - P(0,T_\beta)}$$

$$\mu_i = \frac{(1+\tau_{\alpha,\beta}F(0))}{\tau_{\alpha,\beta}F(0)} \frac{\tau_i F_i(t)}{(1+\tau_i F_i(t))} = \frac{\dfrac{P(0,T_\alpha)}{P(0,T_\beta)}}{\dfrac{P(0,T_\alpha)}{P(0,T_\beta)} - 1} \frac{\dfrac{P(0,T_{i-1})}{P(0,T_i)}}{\dfrac{P(0,T_{i-1})}{P(0,T_i)} - 1}$$

$$= \frac{P(0,T_\alpha)P(0,T_{i-1}) - P(0,T_i)}{P(0,T_\alpha) - P(0,T_\beta)P(0,T_{i-1})} = \frac{P(0,T_\alpha)}{P(0,T_{i-1})}\lambda_i.$$

$$\frac{P(0,T_\alpha)}{P(0,T_{i-1})} \geqslant 1 \underset{for\ \rho\geq 0}{\Longrightarrow} V_{ZC}^{App} \geqslant V_{PV}^{App}$$

A general Libor Market Model approximation

$F_k(t)$ under the associated T_k-forward measure has lognormal distribution. But when we need joint dynamics, for $F_k(t)$ under Q^i, $i \neq k$, the dynamics is

$$dF_k(t) = \mu_{i,k}(t, \mathrm{F}(t))F_k(t)\,dt + \sigma_k(t)F_k(t)\,dZ_k(t)\,.$$

$$\mu_{i,k}(t, \mathrm{F}(t)) = \begin{cases} -\sigma_k(t)\sum_{j=k+1}^{i} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} & k < i \\ \sigma_k(t)\sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} & k > i \end{cases}.$$

$\mu_{i,k}(t, \mathrm{F}(t))$ is stochastic, no known transition densities, we need discretization and simulation.

How can we approximate $\mu_{i,k}(t, \mathrm{F}(t))$ with a deterministic function? We can use typical **"freezing the drift"**, fixing forward rates at current time in dynamics:

$$dF_k(t) \simeq \underbrace{\mu_{i,k}(t, \mathrm{F}(0))}_{\mu_{i,k}(t)}F_k(t)\,dt + \sigma_k(t)F_k(t)\,dZ_k(t). \quad \text{(Appr.1)}$$

having a GBM with time-varying coefficients:

$$F_k(t) =$$

$$F_k(0)\exp\underbrace{\left[\int_0^t \mu_{i,k}(s)\,ds - \int_0^t \frac{1}{2}\sigma_k^2(s)\,ds + \int_0^t \sigma_k(s)\,dZ_k(s).\right]}_{\sim N\left(\int_0^t \mu_{i,k}(s)\,ds - \int_0^t \frac{1}{2}\sigma_k^2(s)ds,\ \int_0^t \sigma_k^2(s)\,ds\right)}$$

so $F_k(t)$ is lognormal. Recalling the MGF of X$\sim N\left(\mu, \sigma^2\right)$

$$\mathbb{E}\left(e^{tX}\right) = e^{\mu t + \frac{1}{2}\sigma^2 t^2},$$
$$\mathbb{E}\left[F_k(t)\right] = F_k(0)\,e^{\int_0^t \mu_{i,k}(t)\,dt}$$

When dealing with **swap rates** we can, similarly, simplify the computation going from

$$S_{\alpha,\beta}(t) = \sum_{k=\alpha+1}^{\beta} w_i(\{F_j\,(t)\}_{j=\alpha+1,\dots,\beta})\ F_k(t)$$

$$w_i(t, \{F_j\,(t)\}_{j=\alpha+1,\dots,\beta}) \;=\; \frac{\tau_i \prod_{j=\alpha+1}^{i} \dfrac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \dfrac{1}{1+\tau_j F_j(t)}}.$$

to

$$S_{\alpha,\beta}(t) \simeq \sum_{k=\alpha+1}^{\beta} w_i(\{F_j\,(0)\}_{j=\alpha+1,\dots,\beta})\ F_k(t) \qquad \text{(Appr.2)}$$

Example: Constant maturity swaps

In a basic constant-maturity swap at time T_{i-1}, $n \geq i \geq 1$, institution A pays to B the c-year swap rate resetting at time T_{i-1} in exchange for a fixed rate K:

$$\boxed{A} \quad \begin{array}{|llr|} \hline \rightarrow & S_{i-1,i-1+\mathbf{c}}(T_{i-1})\,\tau_i & \rightarrow \\ \hline \leftarrow & K\,\tau_i & \leftarrow \\ \hline \end{array} \quad \boxed{B}$$

The net value of the contract to B at time 0 is

$$E\left(\sum_{i=1}^{n} D(0,T_{i-1})\tau_i(S_{i-1,i-1+\mathbf{c}}(T_{i-1}) - K)\right)$$

$$= \sum_{i=1}^{n} \tau_i P(0,T_{i-1})\left\{E^{i-1}\left[S_{i-1,i-1+\mathbf{c}}(T_{i-1})\right] - K\right\}. \quad \text{by CoN}$$

We need only compute

$$\boxed{E^{i-1}\left[S_{i-1,i-1+c}(T_{i-1})\right]}\,, \quad i = 1,\ldots,n$$

We may use Monte Carlo, or resort first to **Appr.2**

$$S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(T_\alpha)$$

and then compute $E^\alpha\left[S_{\alpha,\beta}(T_\alpha)\right]$ using **Appr.1**

$$E^\alpha\left[S_{\alpha,\beta}(T_\alpha)\right] \quad \approx$$

$$\approx \sum_{i=\alpha+1}^{\beta} w_i(0)E^\alpha\left[F_i(T_\alpha)\right] \approx \sum_{i=\alpha+1}^{\beta} w_i(0)e^{\int_0^{T_\alpha}\mu_{\alpha,i}(t)dt}F_i(0)$$

Tenor issues (extending to more rates)

Example: Accrual Swap

$$L^{\tau}(t) = F(t; t, t + \tau),$$

A pays at t_i : $L^{\tau}(t_i)$

B pays at t_i : $(\alpha L^{\tau}(t_i) + spreadQ)\ g_i$

$$g_i = \frac{\#\ \text{trading days in}\ [t_{i-1}, t_i]\ \text{when}\ L^{\tau}_{\min} \le L^{\tau}(t_i) \le L^{\tau}_{\max}.}{\#\ \text{trading days in}\ [t_{i-1}, t_i]}$$

Starting from $L^{\tau}(t)$, as time goes on we need

$$\begin{aligned} L^{\tau}(t+1) &= F(t+1; t+1, t+1+\tau), \\ L^{\tau}(t+2) &= F(t+2; t+2, t+2+\tau), \\ L^{\tau}(t+3) &= F(t+3; t+3, t+3+\tau), \ldots \end{aligned}$$

In the LMM, starting from $F_k(t) = F(t; T_{k-1}, T_k)$, as time goes on we consider

$$\begin{aligned} F_k(t+1) &= F(t+1; T_{k-1}, T_k), \\ F_k(t+2) &= F(t+2; T_{k-1}, T_k), \\ F_k(t+3) &= F(t+3; T_{k-1}, T_k), \ldots \end{aligned}$$

Naive Interpolation

Take $\tau = T_i - T_{i-1}$, $i = 1, \ldots, M$. Indicate $L^{\tau}(t)$ by maturity:

$$\begin{aligned} L^{\tau}(t) &= F(t; t, t+\tau) = F(t; t, U) \\ &= F(U-\tau; U-\tau, U) =: L_u \end{aligned}$$

Maturity U is in $[T_k, T_{k+1}]$. We have

$$\begin{aligned} L_k &= F(T_{k-1}; T_{k-1}, T_k) \\ L_{k+1} &= F(T_k; T_k, T_{k+1}). \end{aligned}$$

Interpolate:

$$L_u = \frac{L_{k+1} - L_k}{\tau}(U - T_k) + L_k.$$

or interpolate between

$$F(T_{k-1}; T_{k-1}, T_k) \text{ and } F(t_i; T_k, T_{k+1}).$$

Drift Interpolation

Recall

$$dF(t; T_{j-1}, T_j) = \sigma_j(t) F_j(t) [Drift\ Part]_j^i dt + \sigma_j(t) F_j(t) dZ_j^i.$$

For $F(t; U-\tau, U)$ with maturity U in $[T_k, T_{k+1}]$ take:

$\sigma_U(t)$ interpolating the parameters of $\sigma_k(t)$ and $\sigma_{k+1}(t)$,

$$dZ_U dZ_Y = \rho_{k,j}\, dt \quad \text{if} \;\; T_{k-1} < U \le T_k,\; T_{j-1} < Y \le T_j,$$

and, since

$$\begin{aligned}
[Drift\ Part]_k^k \;&=\; 0 \\
[Drift\ Part]_{k+1}^k \;&=\; \frac{\tau_{k+1}\, F_{k+1}(t)\sigma_{k+1}(t)}{1+\tau_{k+1}\, F_{k+1}(t)},
\end{aligned}$$

take for $F\,(t; U-\tau, U)$ under Q^k:

$$\begin{aligned}
[Drift\ Part]_U^k \;&=\; \frac{(U-T_k)}{\tau_{k+1}} \frac{\tau_{k+1}\, F_{k+1}(t)\sigma_{k+1}(t)}{1+\tau_{k+1}\, F_{k+1}(t)} \\
&=\; \frac{(U-T_k)\;\, F_{k+1}(t)\sigma_{k+1}(t)}{1+\tau_{k+1}\, F_{k+1}(t)}.
\end{aligned}$$

Bridging Technique

Suppose we need $L^\tau(t)$ at $\mathbf{s}_1 = T_{k-1}, \mathbf{s}_2, \ldots, \mathbf{s}_{l-1}, \mathbf{s}_l = T_k$.
In the LMM we have the rates

$$\begin{aligned} L^\tau(T_{k-1}) &= F_k\left(T_{k-1}\right) \\ L^\tau(T_k) &= F_{k+1}\left(T_k\right) \end{aligned}$$

and can generate by simulation

$$L_j^\tau(T_k),\ L_j^\tau(T_{k-1}), \quad j = 1, \ldots, N$$

Assume $L^\tau(t)$ follows a geometric brownian motion of parameters μ and σ between T_{k-1} and T_k,

$$\ln \frac{L^\tau(T_k)}{L^\tau(T_{k-1})} = \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma Z\left(\tau\right)$$

Find estimates $\hat{\mu}$ and $\hat{\sigma}$ and invert

$$\ln \frac{L_j^\tau(T_k)}{L_j^\tau(T_{k-1})} = \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right)\tau + \hat{\sigma} Z_j\left(\tau\right)$$

to find $Z_j\left(\tau\right) = f\left(\frac{L_j^\tau(T_k)}{L_j^\tau(T_{k-1})}, \hat{\mu}, \hat{\sigma}, \tau\right)$.

The realizations

$$Z_j\left(s_h - s_1\right), Z_j\left(s_{h+1} - s_1\right), \ldots$$

are replaced by realizations of

$$\tilde{Z}_j\left(t - s_1\right) = W\left(t - s_1\right) - \frac{t - s_1}{\tau}\left(W\left(\tau\right) - Z_j\left(\tau\right)\right),$$

Notice $\tilde{Z}_j\left(0\right) = Z_j\left(0\right) = 0$, and $\tilde{Z}_j\left(\tau\right) = Z_j\left(\tau\right)$. Compute

$$L_j^\tau(s_{h+1}) =$$

$$\boxed{L_j^\tau(s_h)\exp\left[\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)(s_{h+1}-s_h)+\hat{\sigma}(\tilde{Z}_j(s_{h+1})-\tilde{Z}_j(s_h))\right].}$$

Pricing the Accrual Swap

Having found, by one of the above methods, simulated paths for $L^\tau(t)$ under a forward measure, one can price the accrual swap:

$$\begin{aligned}
\text{AS}\Pi &= \\
&= E\left\{\sum_{i=1}^n D(0,T_i)\tau_i\left[(\alpha\ L(T_{i-1})+Q)\quad\cdot\right.\right.\\
&\left.\left.\cdot\ \frac{\sum_{g\in\mathcal{G}\cap[T_{i-1},T_i)}\mathbf{1}\{L_1\le L(g)\le L_2\}}{\#\{\mathcal{G}\bigcap[T_{i-1},T_i)\}}-L(T_{i-1})\right]\right\}\\
&= P(0,T_n)\sum_{i=1}^n \tau_i E^n\left\{\frac{1}{P(T_i,T_n)}\left[(\alpha\ L(T_{i-1})+Q)\right.\right.\cdot\\
&\left.\left.\cdot\ \frac{\sum_{g\in\mathcal{G}\cap[T_{i-1},T_i)}1\{L_1\le L(g)\le L_2\}}{\#\{\mathcal{G}\bigcap[T_{i-1},T_i)\}}-L(T_{i-1})\right]\right\}
\end{aligned}$$

where $\mathcal{G}$ is the set of trading days.

LONGSTAFF & SCHWARTZ MONTE CARLO SCHEME

1. Ordinary Monte Carlo simulation of underlying d-dimensional variable X ($\bar{X}_k^j$ is the j^{th} path of X until time t_k, $\quad j = 1, 2, ...M, \quad k = 0, 1, 2, ...n$)

2. Computing final time payoff for each path j:

$$\begin{aligned} CV^j(t_n) &= EV\left(t_n; \bar{X}_n^j\right) = \text{final payoff} \\ j &= 1, 2, ...M \end{aligned}$$

3. Set $k = n$ and restart backwards:

4. Calculate the immediate excercise value

$$EV\left(t_{k-1}; \bar{X}_{k-1}^j\right) \quad j = 1, 2, ...M$$

selecting only scenarios in $I_{k-1} := \left\{ j | EV\left(t_{k-1}; \bar{X}_{k-1}^j\right) > 0 \right\}$;

5. Considering only these scenarios regress discounted continuation value on functions ϕ_h of the underlying, to estimate combinators λ_h in the relationship:

$$DiscFac^j(t_{k-1}, t_k) * CV^j(t_k) = \sum_{h=1}^{i_{k-1}} \lambda_h(t_{k-1}) \, \phi_h\left(t_{k-1}; \bar{X}_{k-1}^j\right) \qquad \forall j \in I_{k-1}$$

6. For each scenario:
 if

$$EV\left(t_{k-1};\bar{X}^{j}_{k-1}\right) > \sum_{h=1}^{i_{k-1}} \lambda_h\left(t_{k-1}\right)\phi_h\left(t_{k-1};\bar{X}^{j}_{k-1}\right)$$

then

$$CV^{j}\left(t_{k-1}\right) = EV\left(t_{k-1};\bar{X}^{j}_{k-1}\right)$$

else

$$CV^{j}\left(t_{k-1}\right) = DiscFac^{j}(t_{k-1},t_k) * CV^{j}\left(t_k\right)$$

7. Replace k with $k-1$ and restart form point 4, until $k=0$;

8. Now average on all cash-flows discounted at current time, finding LSMC price.

Extending to other dynamics: Libor Models for Smile

- **Smile and skew: what?** In lognormal model the volatility is a characteristic of the underlying, it is not contract dependent. Therefore, given different option prices for different strikes

K_1	K_2	$\ldots$	K_n
Π_1	Π_2	$\ldots$	Π_n

inverting Black we should find

K_1	K_2	$\ldots$	K_n
σ	σ	$\ldots$	σ

When instead we find different values

K_1	K_2	$\ldots$	K_n
σ_1	σ_2	$\ldots$	σ_n

we say we have a volatility smile. The **volatility smile** is the curve

$$K \longmapsto \sigma(K).$$

Usually **skew** is used for decreasing curve, **smile** for curve with minimum

- **Smile and skew in interest rate markets: where and when?**

In the caplet market, in 1996 appearance of skews. In 1998, after Russian crisis, appearance of marked hockey-stick-shaped smiles.

- **Smile and skew in interest rate markets: why?**

No general consensus. According to Rebonato (2002):

1) Smile and skew in the interest rate market are due to causes different from those acting in other markets

2) Smile and skew in the interest rate market are due each to a different cause:

Skews: reaction of rates to shocks is not fully proportional to the level of the rate.

Smile: stochastic volatility.

- **Smile and skew: so what?**

Expression of a non-lognormal distribution. Rather than adapting a lognormal model one can use an alternative model with different assumptions.

Dealing with skews: Shifted Lognormal Libor Model (SLLM)

Assume that

$$\begin{aligned} F_k(t) &= X_k(t) + \alpha, \\ dX_k(t) &= \beta(t) X_k(t)\, dW_t, \end{aligned}$$

under Q^k, so

$$dF_k(t) = \beta(t)(F_k(t) - \alpha)\, dW_t.$$

The distribution of $F_k(T)|\mathrm{F}_k(t)$, $t < T$, is a shifted lognormal:

$$\begin{aligned} & f_{F_k(T)|F_k(t)}(x) = \\ & \frac{1}{(x-\alpha)U(t,T)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln \frac{x-\alpha}{F_k(t)-\alpha} + \frac{1}{2}U^2(t,T)}{U(t,T)} \right)^2 \right\}, \\ & U(t,T) := \sqrt{\int_t^T \beta^2(s) ds} \end{aligned}$$

The pricing formula is

$$\begin{aligned} \text{CAPLET}\Pi_t &= \tau P(t,T_k) E^k \left[(F_k(T_{k-1}) - K)^+ | \mathcal{F}_t \right] \\ &= \tau P(t,T_k) E^k \left[(F_k(T_{k-1}) - \alpha + \alpha - K)^+ | \mathcal{F}_t \right] \\ &= \tau P(t,T_k) E^k [(\underbrace{(F_k(T_{k-1}) - \alpha)}_{X_k(T_{k-1}) \sim \text{lognormal}} - (K - \alpha))^+ | \mathcal{F}_t], \\ &= \tau P(t,T_k) \text{BLACK}(F_k(t) - \alpha, K - \alpha, U(t, T_{k-1})). \end{aligned}$$

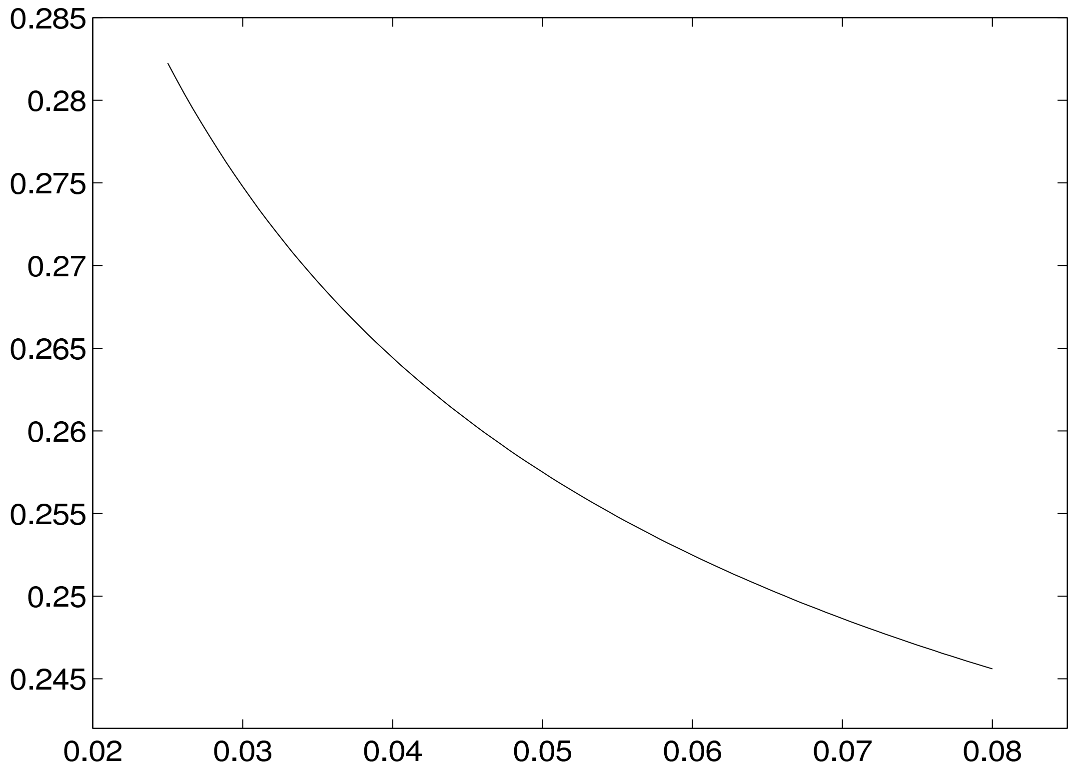
The implied Black volatility $\hat{\sigma} = \hat{\sigma}(K)$ is obtained by solving in $\hat{\sigma}$:

$$\textsc{Black}(F_k(t), K, \hat{\sigma}\sqrt{T_{k-1} - t}\,)$$

$$=$$

$$\textsc{Black}(F_k(t) - \alpha, K - \alpha, U(t, T_{k-1})).$$

$\alpha > 0 \;\Rightarrow\; \hat{\sigma}\,(K)$ increasing
$\alpha = 0 \;\Rightarrow\; \hat{\sigma}\,(K)$ flat (classic lognormal assumption)
$\alpha < 0 \;\Rightarrow\; \hat{\sigma}\,(K)$ decreasing (typical market skew)



SLLM implied volatility curve

Dealing with (U-shaped) smiles: Lognormal Mixture Libor Model (LMLM)

Assume F_k under the forward measure Q^k has a distribution given by a mixture of lognormal densities. Marginal density is

$$f_t(x) \;:\; = \frac{d}{dx}\left[Q^k\text{- Pr}\left(F_k(t) \le x\right)\right] \; = \sum_{i=1}^{N} \lambda_i f_t^i(x),$$

$$\lambda_i \; > \; 0, \; \sum_{i=1}^{N} \lambda_i = 1$$

where the $f_t^i(y)$ are lognormal densities,

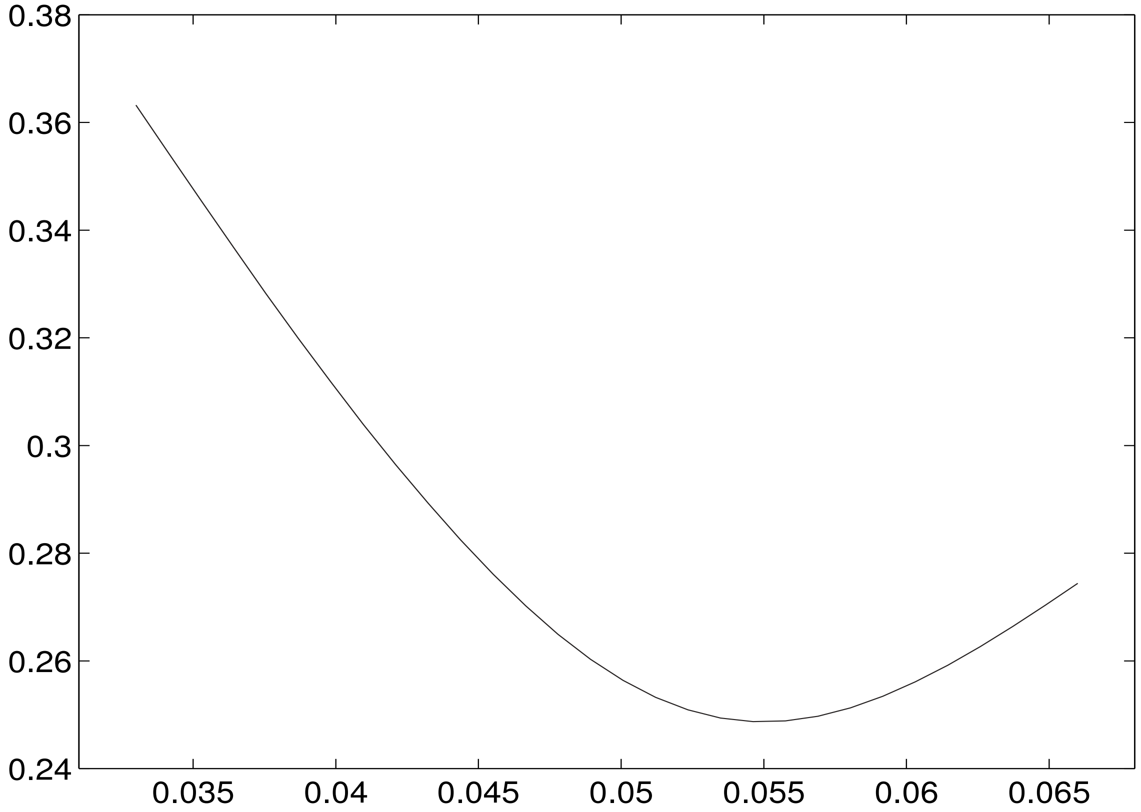
$$f_t^i(x) = \frac{1}{x v_k^i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2 v_k^i(t)^2} \left[\ln \frac{x}{F_k(0)} + \tfrac{1}{2} v_k^i(t)^2 \right]^2 \right\},$$

$$v_k^i(t) := \sqrt{\int_0^t \sigma_k^i(u)^2 du}.$$

With this assumption, caplet pricing formula at time zero is

$$\begin{aligned}
\text{CAPLET}\Pi &= \tau P(0,T_k) E^k\left[(F_k(T_{k-1}) - K)^+\right] \\
&= \tau P(0,T_k) \int_0^{+\infty} (x-K)^+ p_{T_{k-1}}(x) dx \\
&= \tau P(0,T_k) \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} (x-K)^+ p^i_{T_{k-1}}(x) dx \\
&= \tau P(0,T_k) \sum_{i=1}^{N} \lambda_i scBlack(F_k(0), K, v_k^i(T_{k-1}))
\end{aligned}$$

The model is tractable and gives realistic U-shapes for the smile, with a minimum at $K = F_k(0)$



LMLM implied volatility curve

Two possibilities for **model dynamics:**

- **local volatility** (usual source of randomness)
- **stochastic volatility** (specific source of randomness)

Local volatility: the LMLMLV

In local volatility models we assume that

$$dF_k(t) = \bar{\sigma}_k(t, F_k(t))F_k(t)\, dW(t)$$

where $\bar{\sigma}_k(t, F_k(t))$ is a deterministic function of $F_k(t)$. Under some technical conditions on $\sigma_i(t)$ Brigo and Mercurio (2000) show that with

$$\bar{\sigma}_k(t, x)^2 = \sum_{i=1}^{N} \Lambda_i(t, x)\sigma_i^2(t),$$

$$\Lambda_i(t, x) := \frac{\lambda_i f_t^i(y)}{\sum_{i=1}^{N} \lambda_i f_t^i(y)}, \quad t > 0, x > 0,$$

$F_k(t)$ has lognormal mixture marginal distribution.

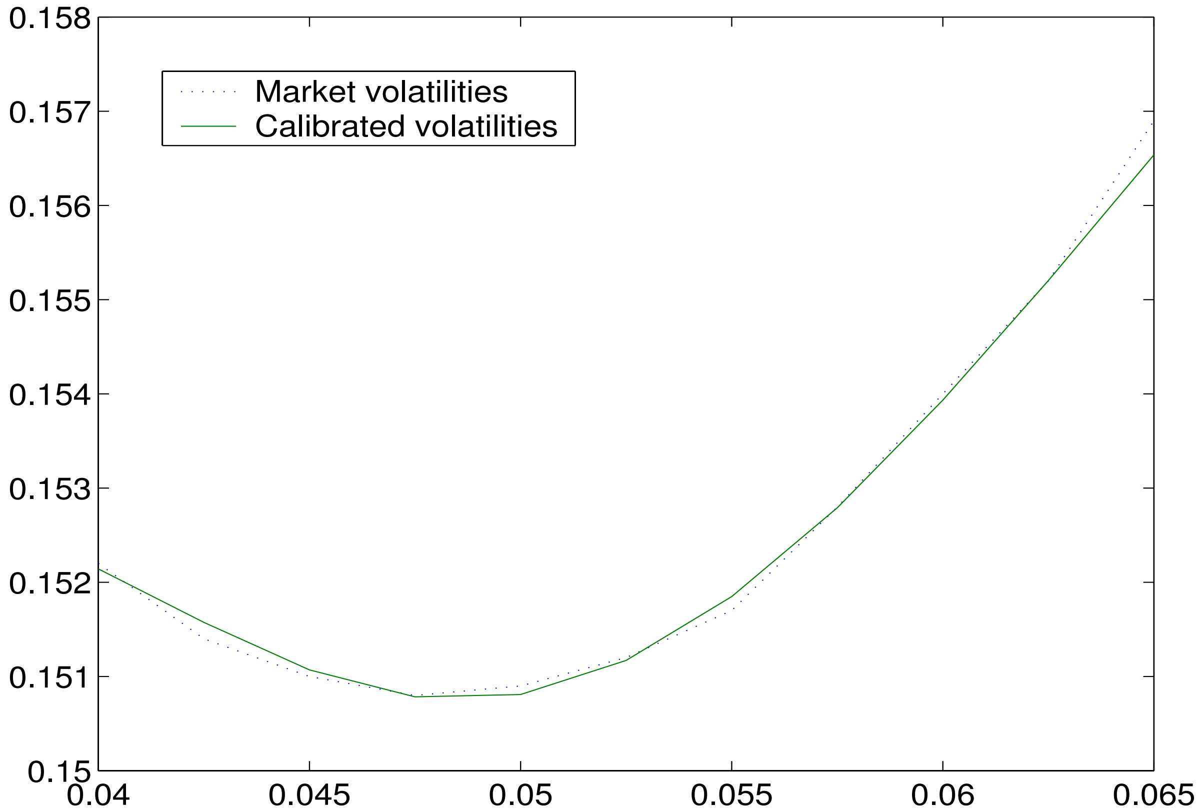
Dealing with smile/skew: Shifted Lognormal Mixture Libor Model

For smiley/skewed shapes (minimum **not** at $K = F_k(0)$) consider a **shifted mixture of lognormals**,

$$dF_k(t) = \bar{\sigma}_k\left(t, F_k(t) - \alpha_k\right)(F_k(t) - \alpha_k)dW_t.$$

with pricing formula

$$\text{C}\textsc{aplet}\Pi = \tau P(0, T_k)\sum_{i=1}^{N} \lambda_i \text{B}\textsc{lack}(F_k(0) - \alpha_k, K - \alpha_k, v_k^i(T_{k-1})).$$



SLMLM and cap market 11/14/00 implied volatility curves

Uncertain volatility: LMLMUV

The second choice for the dynamics involves a volatility-specific source of randomness. Assume

$$dF_k(t) = \sigma_k(t)F_k(t)\,dW(t)$$

where

$$\sigma_k(t) = \begin{cases} \sigma_k^1(t) & \text{with probability } \lambda_1 \\ \sigma_k^2(t) & \text{with probability } \lambda_2 \\ \vdots & \vdots \\ \sigma_k^N(t) & \text{with probability } \lambda_N \end{cases}$$

Associate scenarios with random variable I, drawn at $t = \varepsilon$ infinitesimal after 0, independent of W and which takes values in $\{1, 2, \ldots, N\}$ with probability

$$Q^k\text{-}\Pr(I = i) = \lambda_i,$$

and set

$$\sigma_k(t) = \sigma_k^I(t).$$

What is the current **caplet pricing formula** in this context?

$$
\begin{aligned}
\textsc{Caplet}\Pi &= \tau P(0,T_k) E^k[(F_k(T_{k-1}) - K\}^+] \\
&= \tau P(0,T_k) E^k[E^k[(F_k(T_{k-1}) - K)^+ | I]] \\
&= \tau P(0,T_k) \sum_{i=1}^{N} \lambda_i [E^k[(\underbrace{F_k(T_{k-1})}_{\substack{\text{lognormal} \\ \text{with } \sigma_k^i(t)}} - K)^+] \\
&= \tau P(0,T_k) \sum_{i=1}^{N} \lambda_i \textsc{Black}(F_k(0), K, v_k^i(T_{k-1})).
\end{aligned}
$$

It coincide with **LMLMLV** formula.

What differences?

- LV pros: explicit formulas for current european prices, model is complete, there is a future smile.
- LV cons: tractability does not extend, interpretation is less transparent
- UV pros: explicit formulas for current european prices, analytical tractability extends as much as in Black model, interpretation is transparent
- UV cons: there is no future smile, model is not complete.

Easy to extend this approach to **smiley/skewed shapes**. Use a different shift for each scenario:

Shifted Lognormal Libor Model with Uncertain Parameters (SLLMUP)

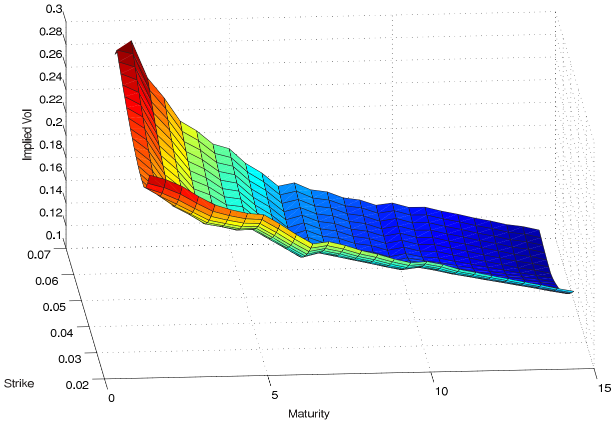
$$dF_k(t) = \sigma_k^I(t) \left(F_k(t) - \alpha_k^I \right) \, dW(t)$$

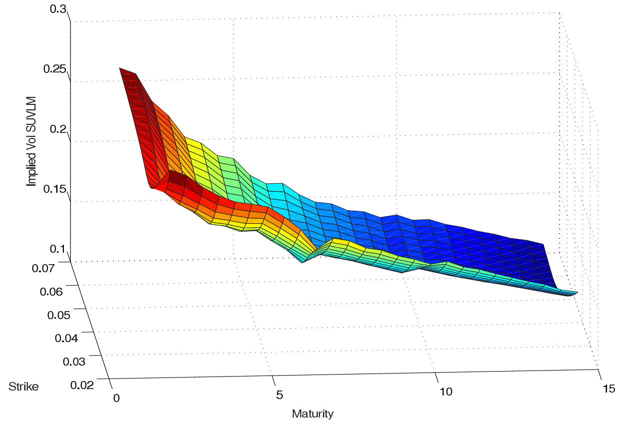
with pricing formula

$$\textsc{Caplet}\Pi = \tau P(0, T_k) \sum_{i=1}^{N} \lambda_i \textsc{Black}(F_k(0) - \alpha_k^i, K - \alpha_k^i, v_k^i(T_{k-1})).$$

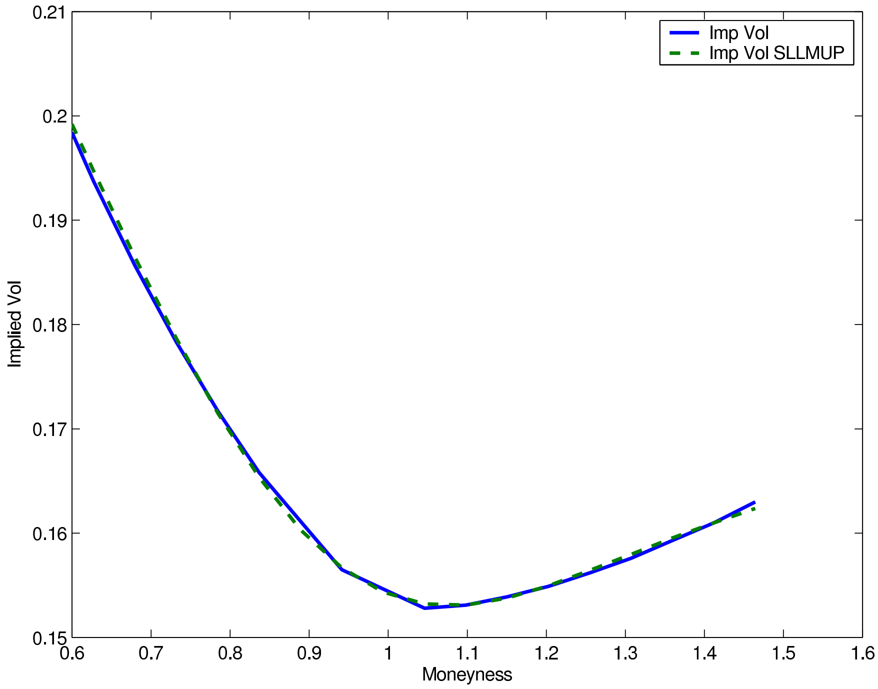
This model is simple, tractable, and can replicate market hockey stick shapes. See examples using 3 scenarios with fixed probabilities

$\lambda_1 = 0.6$ (high probability scenario)
$\lambda_2 = 0.3$ (mean probability scenario)
$\lambda_3 = 0.1$ (low probability scenario)





SUVLM and cap market 08/11/04 implied volatility surfaces



SLLMUP and 10-year maturity implied volatility curves

With this approach there are approximations for swaption volatility similar to the one found in the plain lognormal case. Is there a specification of the model allowing for automatic exact calibration to ATM swaptions while keeping caplet smile calibration? It seems there is (work in progress, Mercurio and Morini (2005)).

Other approaches to smiles/skews: constant elasticity variance model (Andersen and Andreasen (2000)), stochastic volatility as a continuous process (Rebonato (2002), Hagan et al. (2002), Pieterbarg (2003)).

Extending to different markets: CDS Market Model

Credit Default Swap (CDS)

With τ default time, α year fraction, in a CDS:

Protection. Buyer	$\rightarrow$ $R\alpha$ at $T_{k+1},\ldots,T_b$ before τ $\rightarrow$ $\leftarrow$ Protection at τ, $T_k < \tau < T_b$ $\leftarrow$	Protection Seller

Discounted Payoff of a basic CDS is

$$
\begin{aligned}
CDS_t^{Payoff}(R) = & \sum_{i=k+1}^{b} D(t,T_i)\,\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}} \\
& - \sum_{i=k+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\ge T_i\}}
\end{aligned}
$$

By expectation under the risk neutral measure Q, the price is

$$
\mathrm{CDS}\Pi_t(R) = \mathbb{E}\left[\mathbf{CDS}_t^{Payoff}(R)\,|\mathcal{F}_t\right], \tag{3}
$$

but considering subfiltrations, so that

$\mathcal{F}_t$	=general market information up to t
$\mathcal{H}_t$	=default-free market information up to t

,

we can equivalently express price as (Jeanblanc and Rutkowski)

$$
\mathrm{CDS}\Pi_t(R) = \frac{\mathbf{1}_{\{\tau>t\}}}{Q(\tau>t|\mathcal{H}_t)}\mathbb{E}\left[\mathbf{CDS}_t^{Payoff}(R)\,|\mathcal{H}_t\right] \tag{4}
$$

Set price to zero and derive R

$$
\begin{aligned}
R_{k,b}(t) &= \frac{\sum_{i=k+1}^{b} \mathbb{E}\left[D(t,T_i)\,\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{H}_t\right]}{\sum_{i=k+1}^{b} \alpha_i \mathbb{E}\left[D(t,T_i)\,\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_t\right]} \\
&= \frac{\sum_{i=k+1}^{b} \mathbb{E}\left[D(t,T_i)\,\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{H}_t\right]}{\sum_{i=k+1}^{b} \alpha_i Q(\tau>t|\mathcal{H}_t)\,\bar{P}(t,T_i)}
\end{aligned}
$$

with

$$
\bar{P}(t,T_i) = \frac{\mathbb{E}\left[D(t,T_i)\,\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_t\right]}{Q(\tau>t|\mathcal{H}_t)}.
$$

Pricing CDS options

The CDS option to enter a CDS with fixed rate K at a future time T_k has discounted payoff

$$D(t,T_k)\left[\mathrm{CDS}\Pi_{T_k}(K)\right]^+ =$$

$$D(t,T_k)\left[\mathrm{CDS}\Pi_{T_k}(K) - \underbrace{\mathrm{CDS}\Pi_{T_k}(R_{k,b}(T_k))}_{0}\right]^+$$

which, substituting the above, is

$$= \frac{\mathbf{1}_{\{\tau>T_k\}}}{Q\left(\tau>T_k|\mathcal{H}_{T_k}\right)} D(t,T_k)\cdot$$

$$\cdot\left[\sum_{i=k+1}^{b}\alpha_i Q\left(\tau>T_k|\mathcal{H}_{T_k}\right)\bar{P}(T_k,T_i)\right](R_{k,b}(T_k)-K)^+ .$$

Using pricing formula (4) and law of iterated expectation we have

$$\mathrm{CDS}\textsc{Option}\Pi = \frac{\mathbf{1}_{\{\tau>t\}}}{Q(\tau>t|\mathcal{H}_t)}\cdot$$

$$E\left[D(t,T_k)\left\{\sum_{i=k+1}^{b}\alpha_i Q\left(\tau>T_k|\mathcal{H}_{T_k}\right)\bar{P}(T_k,T_i)\right\}(R_{k,b}(T_k)-K)^+\middle|\mathcal{H}_t\right]$$

Change of numeraire

Choose a **numeraire** N so that

1) $\dfrac{Payoff\,(X_T)}{N_T}$ is simple, to ease expectation.
2) $\frac{N_t X_t}{N_t} = X_t$ is a martingale under Q^N.

A quantity satisfying 1 and 2 (under some conditions, Jamshidian 2004) is

$$\boxed{\hat{C}_{k,b}(t) = \sum_{i=k+1}^{b} \alpha_i Q(\tau > t|\mathcal{H}_t)\,\bar{P}(t, T_i)}$$

so that

$$R_{k,b}(t) = \frac{\sum_{i=k+1}^{b} \mathbb{E}\left[D(t,T_i)\,\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{H}_t\right]}{\hat{C}_{k,b}(t)}$$

and

$$\text{CDSO}\textsc{ption}\Pi = \frac{\mathbf{1}_{\{\tau>t\}}}{Q(\tau>t|\mathcal{H}_t)}\mathbb{E}\left[D(t,T_k)\,\hat{C}_{k,b}(T_k)\,(R_{k,b}(T_k)-K)^+\,|\mathcal{H}_t\right].$$

Changing numeraire according to

$$\Pi(X_t) = E_t^Q\left[D(t,T)\,\Pi(X_T)\right] = N_t \mathbb{E}^N\left[\frac{\Pi(X_T)}{N_T}\right],$$

with $N_t = \hat{C}_{k,b}(t)$ **and** $Q^N = \hat{Q}^{k,b}$ **associated to** $\hat{C}_{k,b}(t)$, we have

$$\text{CDSO}\textsc{ption}\Pi_t = \frac{\mathbf{1}_{\{\tau>t\}}}{Q(\tau>t|\mathcal{H}_t)}\hat{C}_{k,b}(t)\,\mathbb{E}^{k,b}\left[(R_{k,b}(T_k)-K)^+\,|\mathcal{H}_t\right]$$

The Market Model

Under the $\hat{Q}^{k,b}$ measure, $R_{k,b}(t)$ is a martingale. Assuming it is lognormal with volatility $\sigma_{k,\beta}$

$$dR_{k,b}(t) = \sigma_{k,\beta} R_{k,b}(t)\, dW^{k,b}(t).$$

leads to the simple Black-like formula

$$\textsc{CDSOption}\Pi_t =$$

$$= \mathbf{1}_{\{\tau > t\}} \sum_{i=k+1}^{b} \alpha_i \bar{P}(t, T_i)\, \textsc{Black}\left(R_{k,b}(t), K, \sigma_{k,\beta}\sqrt{T_k - t}\right)$$

Now compare **quotations**:

Without Market Model

	$R_{k,b}(0)$	K	Mid
Option 1	61	60	**32.5**
Option 2	43.4	43	**24.5**

With Market Model

	$R_{k,b}(0)$	K	$Mid\ implied\ \sigma$
Option 1	61	60	**62.16%**
Option 2	43.4	43	**63.71%**

One-period-CDS Market Model

$\hat{Q}^j := \hat{Q}^{j-1,j}$

$\hat{C}_j := \hat{C}_{j-1,j}$

$R_j(t) := R_{j-1,j}(t)$.

Then

$$R_j(t) = \frac{\mathbb{E}\left[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}$$

$$= \frac{\mathbb{E}\left[D(t,\mathbf{T}_j)\mathbf{1}_{\{\tau>\mathbf{T}_{j-1}\}}|\mathcal{H}_t\right] - \mathbb{E}\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}$$

$$= \frac{1}{\alpha_j}\frac{\frac{\mathbb{E}\left[D(t,\mathbf{T}_j)\mathbf{1}_{\{\tau>\mathbf{T}_{j-1}\}}|\mathcal{H}_t\right]}{Q(\tau>t|\mathcal{H}_t)} - \bar{P}(t,T_j)}{\bar{P}(t,T_j)}. \tag{5}$$

What is the relationship between $R_j(t)$ and $R_{k,b}(t)$?

$$R_{k,b}(t) = \sum_{j=k+1}^{b}\alpha_j\frac{\bar{P}(t,T_j)}{\sum_{i=k+1}^{b}\alpha_i\bar{P}(t,T_i)}R_j(t) \tag{6}$$

Completing the model

Recall that for completing the model we need to show dynamics under a general measure according to

$$\mu_k^{N2} = \mu_k^{N1} - \bar{\sigma}_k\left(\right)\rho\text{VecDiffCoeff}\left(\ln\frac{N1}{N2}\right).$$

Here

$$\frac{N1}{N2} = \frac{\hat{C}_j\left(t\right)}{\hat{C}_i\left(t\right)} = \frac{\alpha_j\bar{P}\left(t,T_j\right)}{\alpha_i\bar{P}\left(t,T_i\right)}$$

Notice in (5) that now we cannot represent $R_j\left(t\right)$ only in terms of numeraire ratios, and we cannot express numeraire ratios only in terms of $R_j\left(t\right)$'s.

Adding rates

We can use (6) to see that

$$\frac{\bar{P}(t,T_j)}{\bar{P}(t,T_i)} = \frac{\alpha_i}{\alpha_j} \prod_{k=i+1}^{j} \frac{R_{k-1}(t) - R_{k-2,k}(t)}{R_{k-2,k}(t) - R_k(t)}.$$

However, the following must hold

$$0 < \frac{\bar{P}(t,T_j)}{\bar{P}(t,T_{j-1})} < 1.$$

imposing constraints

$$\min\left(R_{j-1}(t), \tfrac{R_{j-1}(t)+R_j(t)}{2}\right) < R_{j-2,j}(t) < \max\left(R_{j-1}(t), \tfrac{R_{j-1}(t)+R_j(t)}{2}\right)$$

One should look for a complex dynamics...

Is it possible to complete the model in a financially natural way not requiring complex dynamics and simplifying computations? It requires some probability tools (Brigo and Morini (2005), work in progress).

For further reading some papers are available at

www.damianobrigo.it

www.exoticderivatives.com/MassimoMorini.html

www.fabiomercurio.it