Estimation of the Libor Market Model: Combining Term Structure Data and Option Prices

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This version: February, 2001

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Abstract

Previous empirical work on term structure models has estimated and tested these models on the basis of either interest rate data or derivative price data. In this paper, we analyze the benefits of combining these two data sets for estimating and testing multi-factor Libor market models. We use US data on interest rates and prices of caps and swaptions from 1995 to 1999. We allow for the presence of measurement error in both the interest rates and the option prices. The results on the fit of a two-factor model show that, in case of estimation based on option prices only, the model does not accurately fit the standard deviations of interest rate changes, and, in case of estimation on the basis of interest rate data, the model misprices caps and, especially, swaptions. Thus, the two-factor model cannot fit the main features of the two data sets at the same time. This result illustrates the benefit of using both interest rate data and option price data for testing term structure models. A three-factor model provides a much better fit to both the interest rate data and the option price data. In particular, the humped shape of the volatility term structure is fitted more accurately.

JEL Codes: G12, G13, E43.

Keywords: Term Structure Models; Market Models; Interest Rate Derivatives; Volatility Hump.

1 Introduction

Previous empirical work on term structure models has estimated and tested these models on the basis of interest rate data (for example, Buhler et al. (1999), Dai and Singleton (2000), De Jong (2000), Pearson and Sun (1994)), or derivative price data (Amin and Morton (1994), Flesaker (1993)). There are several potential benefits of combining these two data sets to estimate and test term structure models. First, model parameters might be estimated more precisely. In particular, for estimating multi-factor term structure models (that have a large number of parameters) using both interest rate data and option prices seems beneficial. Second, using both data sets to test term structure models will likely give stronger tests of these models. For example, it might be the case that a given model provides a reasonable fit of the main features of interest rate data, but considerably misprices interest rate options. Therefore, in this paper, we estimate and test multi-factor term structure models using both interest rate data and option price data, and investigate the benefits of using both data sets.

The models that we analyze are in the class of the Libor Market Models (Brace, Gatarek, and Musiela (1997), Miltersen, Sandmann, and Sondermann (1997), and Jamshidian (1997)). We specify a multi-factor Libor Market Model with correlated factors, where each factor has a time-homogeneous volatility function that corresponds to mean-reverting behaviour of the factor. This way, the model is related to the affine class of term structure models (Duffie and Kan (1996)), and, in particular, to the stochastic mean model of Jegadeesh and Pennacchi (1996). In the latter model, the short rate is mean reverting around a ‘shadow’ rate, that itself is (slowly) mean reverting around a constant mean. By allowing the factors to be correlated, the model is able to generate a humped shape for the term structure of interest rate volatilities.

For the empirical analysis, we use weekly US data on Libor and swap rates and prices for caps and swaptions from 1995 to 1999. The model setup explicitly allows for the presence of measurement error in both the interest rates and derivative prices. Given this model setup, moment restrictions are derived for both variances and covariances of changes in forward Libor interest rates of different forward maturities, and for the expected prices of several caps and swaptions. Estimation is performed by applying the Generalized Method of Moments (GMM, Hansen (1982)). We estimate both two-factor and three-factor models, thereby extending the analysis of De Jong, Driessen, and Pelsser (2000), where one-factor Libor Market Models are analyzed. For comparison, we also estimate the models both only on the basis of interest rate data and only on the basis of option price data.

First, we analyze whether using both interest rate and option price data leads to more accurate parameter estimates. For both the two-factor and three-factor model, we find that, when estimating the model using both interest rate and option price data, the standard errors of the parameter estimates are not always smaller than the standard errors that result when only interest rate data or option price data are used for estimation.

Second, we analyze the fit of the models on the interest rate and option price data. The results for the two-factor model show that, in case of estimation based on option prices only, the model does not accurately fit the standard deviations of forward Libor rate changes, and, in case of estimation on the basis of interest rate data, the model misprices caps and, especially, swaptions. Thus, the two-factor model cannot fit the main features of the two data sets at the same time. This result illustrates the benefit of using both interest rate data and option price data to test term structure models. In case of joint estimation, there is a trade off between the fit on the option price data and the fit on the interest rate data, but the two-factor model still poorly fits both the forward Libor rate (co)variance structure and the maturity patterns in the cap and swaption prices. In particular, the model is not capable of both fitting the humped shape of the term structure of interest rate volatilities and the cross-correlations between forward Libor rate changes.

The three-factor model provides a better fit to both the interest rate data and the option price data. Both the humped shape of the standard deviations of forward Libor rate changes, and the humped shape of the cap implied volatility curve are fitted more accurately. Still, the model slightly overprices swaptions, and the model implies correlations between forward Libor rate changes that are a bit lower than in the data.

In line with results of Dai and Singleton (2000) and De Jong (2000), we find that the correlations between the factors are significantly different from zero. These nonzero correlations are necessary to generate a hump shaped volatility curve. The results also show that allowing for measurement error in the interest rates is an important aspect of the model setup. Neglecting this measurement error structure would lead to overpricing of caps and too low standard deviations of forward Libor rate changes. However, for all models the estimate for the variance of the Libor measurement error is unrealistically large, which might be caused by too restrictive assumptions on the measurement error structure.

The remainder of this paper is organized as follows. Section 2 discusses and motivates the modeling framework. Section 3 describes the interest rate data and option price data, as well as the estimation methodology. Section 4 contains the estimation results for two-factor and three-factor models. Section 5 concludes.

2 Modeling Framework

2.1 Libor Market Model

To jointly analyze both term structure data and option price data, we choose the Libor Market Model (LMM) as modeling framework. The reason for using the LMM is threefold. First of all, the LMM is often used by financial institutions. Second, our option price data consist of implied Black (1976) volatility quotes for caps and swaptions, and the LMM implies simple Black-type pricing formulas for caps (and approximate pricing formulas for swaptions), which facilitates the estimation of the model. Third, De Jong, Driessen, and Pelsser (2000) provide evidence that the LMM outperforms the Swap Market Model (SMM) in pricing caps and swaptions. In De Jong, Driessen, and Pelsser (2000) other advantages of the market models are mentioned.

We describe the LMM formulation based on a finite number of bond prices, following Jamshidian (1997). We start with defining a finite set of dates $T_1 < T_2 < \ldots < T_N$, the so-called *tenor structure*. We also define $\delta_i = T_{i+1} - T_i$, $i = 1,..,N-1$ as the so-called daycount fractions, which are determined by the maturity of the Libor rate that is used to determine caplet payoffs and are most often equal to 3 or 6 months. Associated with each tenor date T_n is a bond that matures at this date, and its time t price is denoted with $P_n(t)$. These N bond prices, with maturities $T_1,\ldots,T_N$, determine $(N-1)$ forward Libor rates.

We analyze a multi-factor LMM with K factors. This multi-factor LMM implies that the forward Libor rate $L_n(t)$, defined by $L_n(t) = \frac{1}{\delta_n}\left(\frac{P_n(t)}{P_{n+1}(t)} - 1\right)$, satisfies the following Itô process under the true probability measure

$$dL_n(t) = L_n(t)\mu_n(t)\,dt + L_n(t)\gamma_n(t)' dW(t), \quad n=1,...,N-1 \tag{1}$$

The function $\mu_n(t)$ is the drift function of the forward Libor rate, and $\gamma_n(t)$ is a, deterministic, K-dimensional vector that is often referred to as the volatility function. $W(t)$ is a K-dimensional vector of correlated Brownian motions. The correlation between the i^{th} component and j^{th} component of $W(t)$ is denoted by ρ_{ij}. By choosing one of the N bonds as the numeraire asset, we can obtain the process of the forward Libor rates under the equivalent martingale measure associated with this numeraire choice. Under such an equivalent martingale measure, the drift of the forward Libor rates is completely determined by the volatility functions $\gamma_n(t)$, $n=1,..,N-1$, see Jamshidian (1997). For example, if we take the longest maturity bond $P_N(t)$ as the numeraire, we obtain the so-called terminal measure Q^N, under which forward Libor rates follow the process

$$dL_n(t) = L_n(t)\left(-\sum_{i=n+1}^{N-1} \frac{\delta_i L_i(t)\gamma_i(t)'\Sigma\gamma_n(t)}{1+\delta_i L_i(t)}\,dt + \gamma_n(t)' dW^*(t)\right), \quad n=1,...,N-1 \tag{2}$$

where $W^*(t)$ is a K-dimensional Brownian motion under the terminal measure, and where Σ is the instantaneous correlation matrix of this Brownian motion, so that Σ is a K by K matrix with the $(i,j)^{th}$ component equal to ρ_{ij}.

Equation (2) implies that, in order to price and hedge interest rate derivatives, only the volatility functions $\gamma_n(t)$ have to be determined. We refer to Brace, Gatarek, and Musiela (1997) and Jamshidian (1997) for the pricing formulas for caps and swaptions. Most importantly, these formulas show that, in the LMM, cap prices depend on conditional variances of forward Libor rates, whereas swaption prices both depend on conditional variances of forward Libor rates, and conditional covariances between forward Libor rates of different maturities.

2.2 Specification of Volatility Functions

De Jong, Driessen, and Pelsser (2000) show that models with a time-inhomogeneous volatility function can lead to overfitting to option prices, and also provide empirical evidence in favour of a declining volatility function instead of a constant volatility function. Furthermore, Dai and Singleton (2000) illustrate that allowing for nonzero correlations between factors in (affine) term structure models is important for accurately describing US interest rate behaviour. Therefore, we choose the following time-homogeneous specification for the volatility functions in the LMM

$$\gamma_n(t) = (\sigma_1 \exp(-\kappa_1 (T_n - t)),....,\sigma_K \exp(-\kappa_K (T_n - t)))', \quad n = 1,...,N-1 \tag{3}$$

and allow for an unrestricted correlation matrix Σ for the Brownian motions[1]. In the remainder, we will refer to the parameter σ_i as the *volatility parameter* of factor *i* and to the parameter κ_i as its *decay parameter*.

These volatility functions are very similar to the volatility functions implied by the affine term structure models of Duffie and Kan (1996). In particular, consider a *K*-factor version of the Vasicek (1977) model. In this model, the instantaneous short rate $r(t)$ is the sum of a constant and *K* factors, i.e., $r(t) = \theta + \sum_{i=1}^{K} X_i(t)$, where each factor follows a mean reverting process under the true probability measure with parameter κ_i, that determines the strength of the mean reversion, and volatility parameter σ_i

$$dX_i(t) = -\kappa_i X_i(t) dt + \sigma_i dZ_i(t) \tag{4}$$

The vector of Brownian motions $Z(t) = (Z_1(t),...,Z_K(t))'$ is assumed to have instantaneous correlation matrix Σ. It is easy to show that, if the factor risk prices are deterministic, this model implies the following process for instantaneous forward rates $f(t,T)$[2]

$$df(t,T) = drift\, dt + \sum_{i=1}^{K} \sigma_i \exp(-\kappa_i (T-t))\, dZ_i(t) \tag{5}$$

[1]The instantaneous covariance matrix of the Brownian motions has to be symmetric and positive definite. These restrictions are imposed when estimating the model parameters.

[2]*f(t,T)* is the time *t* forward rate for instantaneous lending at time *T*.

under the true probability measure. Under the equivalent martingale measure, the diffusion terms in equation (5) remain unchanged and only the drift changes. Equation (5) shows that, in the *K*-factor Vasicek model, changes in instantaneous forward rates are determined by *K* correlated factors with volatility functions $\sigma_i \exp(-\kappa_i (T-t))$. Because the factor is strictly mean reverting, this volatility function is decreasing in the time to maturity *(T-t)*. The parameter κ_i determines the decay in the volatility function: strong mean reversion for a factor implies that the volatility function declines quickly.

Equation (3) shows that we choose the same volatility functions for the LMM, which then apply to the changes in log forward Libor rates instead of instantaneous forward rates. This choice facilitates the interpretation of the decay parameter κ_i: it is linked to the mean reversion of factor *i*.

In case of a two-factor model, the Vasicek instantaneous spot rate model has a particularly interesting interpretation, since this model can be rewritten as[3]

$$\begin{aligned} dr(t) &= a(\theta(t) - r(t))dt + \sigma_r dZ^r(t) \\ d\theta(t) &= b(\theta - \theta(t))dt + \sigma_\theta dZ^\theta(t) \\ Cov(dZ^r(t), dZ^\theta(t)) &= \rho_{r\theta}\sigma_r\sigma_\theta dt \end{aligned} \tag{6}$$

This is the central tendency model of Jegadeesh and Pennacchi (1996). In such a central tendency model the short rate is mean reverting around a 'shadow rate' (the central tendency). This shadow rate itself is mean reverting around a constant mean. Jegadeesh and Pennacchi claim that this model is much more adequate in describing the dynamics of Eurodollar futures prices than a one-factor model. Also, Jegadeesh and Pennacchi show that the central tendency model is able to generate a humped structure for forward Libor rate volatilities, which is a feature of the US Libor rate data and the cap data that we analyze. One can show that, to generate a humped volatility structure, the mean reversion of $r(t)$ to the 'spread' $\theta(t) - r(t)$ has to be strong (i.e., a large value for *a* is required), the mean reversion of the shadow rate $\theta(t)$ has to be slow, and the two Brownian motions have to be sufficiently negatively correlated.

[3]This follows from choosing $X_1(t) = (r(t)-\theta) - (\theta(t)-\theta)a/(a-b)$ and $X_2(t) = (\theta(t)-\theta)a/(a-b)$.

3 Data and Estimation Method

3.1 Data

We use two data sets: one data set containing US money-market rates and swap rates and another data set containing implied Black (1976) volatilities of US caps and swaptions. For both data sets we have 232 weekly observations from January 1995 until June 1999.

We use the US money-market rates with maturities of 1, 3, 6, 9, and 12 months, and the data on US swap rates with maturities ranging from 2 to 15 years to estimate the instantaneous forward rate curve using an exponential splines specification. This way, we obtain a trade off between fit of the money-market and swap rates and smoothness of the forward rate curve. Since estimates for forward (Libor) rates are very sensitive to small differences between money market or swap rates of nearly the same maturity, a perfect fit of all underlying money market and swap rates generally leads to unrealistically high estimates for the standard deviations of historical forward (Libor) rate changes, and low correlations between these forward Libor rates. Therefore, we impose some smoothness conditions on the shape of the forward interest rate curve via the exponential splines specification. This gives us at each week instantaneous forward rates $f(t,T)$, from which we construct forward Libor rates for different forward maturities and 3-month Libor maturity. In Table 1 we give the correlation matrix of weekly changes in the logarithm of these forward Libor rates and in Figure 1 the annualized standard deviations of these changes are plotted. In line with results presented in Amin and Morton (1994), and Moraleda and Vorst (1997), Figure 1 shows that there is evidence for a humped volatility structure for forward Libor rate changes.

The derivatives data that we use are weekly quotes for the implied Black (1976) volatilities of at-the-money-forward US caps and swaptions, in total 63 instruments. The caps have maturities ranging from 1 to 10 years, and their payoffs are defined on 3-month Libor rates. The 1-year cap consists of 3 caplets with maturities of 3, 6, and 9 months, and the 10-year cap consists of 39 caplets, with maturities ranging from 3 months to 9 years and 9 months. The other caps are constructed in a similar way. The strike of each cap is equal to the corresponding swap rate with quarterly compounding. In Figure 2 we plot the time series average of the implied volatilities of the caps. Again there is evidence for a hump shaped volatility structure.

For the swaptions, the option maturities range from 1 month to 5 years, while the swap maturities range from 1 to 10 years. The strike of an at-the-money swaption is equal to the corresponding forward swap rate. These data again provide evidence for a hump shaped volatility structure.

3.2 Estimation Methodology

In this paper, we focus on estimating the diffusion parameters or volatility functions of the forward Libor rate processes. To estimate these parameters, we derive moment restrictions that can be applied to the forward Libor rate data and the cap and swaption price data, and use the Generalized Method of Moments to estimate the models. We use two sets of moment restrictions:

1. Variances of log forward Libor rate changes and covariances between log forward Libor rate changes of different forward maturities.

2. Expected cap implied volatilities and swaption implied volatilities.

We refer to estimation on the basis of the first set of moments as *interest-rate-based estimation*, and to estimation on the basis of the second set of moments as *option-based estimation*. The use of both sets of moment restrictions is referred to as *joint estimation*. All moment restrictions are formulated under the true probability measure.

The first set of moment restrictions is based on the fact that the LMM implies that (under the true probability measure)

$$Cov(\, d\ln L_i(t), d\ln L_j(t)) \;=\; \gamma_i(t)' \Sigma \gamma_j(t)\, dt, \quad i,j = 1,\dots,N-1 \tag{7}$$

By approximation, this relation holds for small time intervals Δt.[4] The use of variances and covariances for estimation can be motivated by the fact that, if we neglect the drift of Libor rates, the (conditional) distribution of log forward Libor rates is normal. In line with previous research on term structure models (for example, De Jong (2000), Duan and Simonato (1999), Duffee (1999)), we assume that the log forward Libor rate that we observe, $\ln L_i^*(t)$, is equal to the true log forward Libor rate $\ln L_i(t)$, plus a zero-expectation error term $\epsilon_i(t)$, that is independently and identically distributed over time and forward maturities, and independent of the true log forward Libor rate $\ln L_i(t)$:

$$\ln L_i^*(t) = \ln L_i(t) + \epsilon_i(t), \quad E(\epsilon_i(t)) = 0, \quad i=1,..,N-1 \tag{8}$$

For simplicity, we assume that the error terms are maturity-specific and that the error variance is constant over forward maturities

$$V(\epsilon_i(t)) = \sigma_\epsilon^2, \quad Cov(\epsilon_i(t), \epsilon_j(t)) = 0, \quad i,j=1,...,N-1, \; i \neq j \tag{9}$$

There are several reasons to include the error term in the log forward Libor rate. First of all, the forward Libor rates are estimated using the exponential splines specification, which might induce some estimation error in the forward Libor rate estimates. Second, the underlying money-market and swap data might contain measurement error due to illiquidity and time-of-the-day effects. Third, the weekly first-order autocorrelation in the log-forward Libor rate changes is, averaged over all forward maturities, equal to -0.185, whereas the higher-order autocorrelations are close to zero or even positive. This supports the model, since it is easy to show that, abstracting from the drift of forward Libor rates that is implied by the model, (1) and (8) lead to negative first-order autocorrelation for discrete-time changes in the forward Libor rate, and zero higher-order autocorrelations.

[4]This approximate relation is only exact if the drift of the log forward Libor rates is deterministic, which is in general not the case. If the market prices of interest rate risk are very volatile or if the mean reversion parameter is extremely large, the approximation error might be important.

Note that the measurement error assumption in (8) changes the variance of the forward Libor rate changes, whereas it leaves the covariances between forward Libor rates unchanged. This way, the first set of moment restrictions is given by[5]

$$
\begin{aligned}
V(\,\Delta \ln L_i^*(t)) \;&=\; \gamma_i(t)^{\prime}\Sigma\,\gamma_i(t)\,\Delta t \;+\; 2\,\sigma_\epsilon^2, \quad i = 1,\ldots,N-1 \\
Cov(\,\Delta \ln L_i^*(t),\, \Delta \ln L_j^*(t)) \;&=\; \gamma_i(t)^{\prime}\Sigma\,\gamma_j(t)\,\Delta t, \quad i,j = 1,\ldots,N-1
\end{aligned}
\qquad (10)
$$

In Section 5.4.1 similar moment restrictions are formulated for changes in instantaneous forward rates and HJM models, but without the measurement error variance. Using data on the log forward Libor rates, we can estimate the right hand sides of equation (10) and confront these estimates with the model-implied (co)variances. For estimation, we annualize the (co)variances by multiplying (10) with $1/\Delta t$ such as to obtain the same scaling for these restrictions as the implied volatilities (see below).

To derive the moment restrictions for derivative prices, we assume that the (square of the) observed implied Black volatility quote for a cap or swaption is equal to the (square of the) Black implied volatility that corresponds to the model price, plus an independent zero-expectation error term,[6] that represents measurement error in the observed implied volatility quote. For caps, we thus get

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[5]To perform GMM on variance and covariance restrictions, we add auxiliary moment restrictions of the form $E(\Delta \ln L_i^*(t)) = \alpha_i,\ i=1,\ldots,N-1$, where the α_i`s are free coefficients that are estimated along with the other parameters. Even if the true means (i.e., the $\alpha_i$`s) are equal to zero, which would be the case if forward Libor rates are stationary, Cochrane (2001) notes that, in small samples, better estimates are obtained if one uses variances and covariances instead of second moments. In our case, the sample means are very small relative to the variance of the forward Libor rates, so that imposing that the α_i`s are equal to zero hardly affects the GMM parameter estimates.

[6]The assumption that this error term has zero expectation should be interpreted as an approximate moment restriction, due to the dependence of the LMM Black volatility $IV^{C,LMM}(t,T_i)$ on the underlying forward Libor rates. If this dependence would be linear, the presence of measurement error in the forward Libor rates would not change the unconditional expectation of $IV^{C,LMM}(t,T_i)$. In reality, this dependence is, however, not linear, so that the expectation of $IV^{C,LMM}(t,T_i)$ will depend on the variance of the measurement error in the forward Libor rates (and higher-order moments of the measurement error distribution). A Taylor expansion shows that, for at-the-money-forward caps and swaptions, this is a second order effect, and we will therefore neglect this effect in estimating the model.

$$[IV^{C}(t,T_i)]^2 = [IV^{C,LMM}(t,T_i))]^2 + \eta_i(t), \quad E(\eta_i(t)) = 0 \tag{11}$$

where $IV^{C}(t,T_i)$ is the observed implied volatility for the cap with maturity T_i, and where $IV^{C,LMM}(t,T_i)$ is the implied volatility for this cap that corresponds to the cap price implied by the LMM. A similar expression results for swaptions. We take the square of the implied volatilities so that these moment restrictions are measured with the same scale as the restrictions in (7). By taking the expectation on both sides of equation (11) we obtain moment restrictions for caps and swaptions[7].

By approximation, the measurement error variance of the forward Libor rates does not enter the moment restrictions for caps and swaptions. Thus, by combining the forward Libor rate moment restrictions and the cap and swaption restrictions, the measurement error variance can be estimated precisely. Because the measurement error variance does not enter the cap and swaption moment restrictions, this measurement error variance cannot be estimated in case of option-based estimation. In this case, given the option-based parameter estimates, the measurement error variance is estimated from the forward Libor variance moment restrictions.

As noted above, in the LMM, the prices of caps depend on the conditional variances of Libor rates, whereas the prices of swaptions depend both on conditional variances of forward Libor rates, and on the conditional covariances between forward Libor rates of different forward maturities. Thus, both sets of moment restrictions involve (conditional) variances and covariances of forward Libor rates, and from both sets of moment restrictions it is possible to identify the parameters of models with multiple factors.

We use the Generalized Method of Moments (Hansen (1982)) to estimate the parameters of two- and three-factor LMMs. We select some forward maturities and option maturities for the moment restrictions, to obtain roughly the same number of moment restrictions for interest-rate-based estimation and option-based estimation. For the forward Libor rate variance restrictions, we choose the following forward maturities (in years): 0.25, 0.75, 1.25, 1.75, 2.75, 3.75, 4.75, 6.75, and 9.75, in total 9 moment restrictions. These maturities are related to the cap maturities.

[7]This expectation is taken under the true probability measure, since the option prices are observed under this measure. Of course, to calculate the option prices implied by the LMM, one uses an equivalent martingale measure.

For the covariance restrictions, we take the covariances between forward Libor rate changes with forward maturities of 0.25 years, 1.25 years, 2.75 years, 4.75 years, and 9.75 years, in total 10 moment restrictions.

For the cap moment restrictions, we use all 7 option maturities that are available in the cap data, ranging from the 1-year cap to the 10-year cap. Since there are 56 different swaptions in the data set, we select a subset of these swaptions. We choose three option maturities, 3 months, 1 year, and 5 years, and three swap maturities, 1 year, 3 years, and 5 years. Taking all combinations gives us 9 moment restrictions for the swaptions.

In the first step of GMM, we choose an identity weighting matrix. Recall that we formulated all moment restrictions such that they all refer to annualized variances and covariances, so that giving equal weights to all moment restrictions is not unreasonable. Also, the number of moment restrictions is roughly the same for the Libor variances, Libor covariances, cap volatilities, and swaption volatilities, so that none of these four sets of restrictions dominates the first-step estimation results.

It turns out that the covariance matrix of these estimated moment restrictions is close to singularity[8]. In other words, the estimated moment restrictions are highly correlated (especially the restrictions for the caps and swaptions). The efficient, second step of GMM requires that the inverse of the covariance matrix of the estimated moment restrictions is used as the weighting matrix. As shown by Hansen (1982), this is the optimal choice for a correctly specified model in the sense that it yields the lowest asymptotic variance for the GMM parameter estimates. However, as noted by Cochrane (2001), if the covariance matrix of the moment restrictions is close to singularity, using this covariance matrix as weighting matrix implies that one fits the model parameters to linear combinations of the original moment restrictions that have very large positive and negative weights on the original moment restrictions. Using these linear combinations of moment restrictions to estimate the model might be statistically optimal for a correctly specified model (that is, asymptotically), but one can question whether these extreme linear combinations are the most interesting moment restrictions from an economic point of view (see Cochrane (2001)). Focusing on these extreme linear combinations might, therefore, substantially reduce the robustness of the estimation procedure.

[8]We use the method of Newey and West (1987) to correct this covariance matrix for heteroskedasticity and autocorrelation.

We find that, when using the optimal weighting matrix, the model is essentially fitted to the differences between the moment restrictions rather than to the level of the moment restrictions, since, due to the high correlations between the estimated moment restrictions, the standard errors of these differences are much lower than the standard errors of the levels[9]. When performing two-stage GMM estimation for our two-factor and three-factor models, we find that the shape of the Libor variance term structure, the Libor covariance structure, and the cap and swaption implied volatility term structures are fitted quite accurately, whereas the level of these term structures is not fitted well. Therefore, we use in the empirical analysis in the next section only the first-stage GMM estimator, that is obtained using an identity weighting matrix. Of course, if the model is correctly specified, this estimator is still consistent and asymptotically normal, and standard errors and tests are constructed in a straightforward way. In subsequent research, we will analyze to what extent the differences between the first-stage and second-stage GMM estimates are due to misspecification of the model and, in particular, to the restrictive specification of the measurement error structure.

An alternative explanation for the current findings is that transaction costs for the options are ignored. Driessen, Melenberg, and Nijman (1999) show that, without transaction costs, portfolios with large short and long positions in near-maturity bonds are mispriced and lead to the rejection of standard term structure models. If transaction costs are included, these portfolios are no longer mispriced. One might expect a similar result here, now applied to positions in near-maturity interest rate options instead of bonds. For simplicity, we do not explicitly include transaction costs in this paper and use only first-stage GMM.

The near-singular covariance matrix of the moment restrictions also causes the GMM J-statistic, that can be used to jointly test the overidentifying restrictions, to be very large for all models that we estimate. Therefore, we will report individual t-ratios[10] for the original moment

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[9]To see this in a simple example, suppose one fits a one-parameter model to the 1-year and 2-year forward Libor variances. These Libor variance estimates are highly positively correlated. If one performs a Cholesky decomposition of the optimal weighting matrix (the inverse of the covariance matrix), one finds that the second-step GMM estimates are obtained by fitting to the 1-year forward Libor variance, and to the difference between the 2-year and 1-year forward Libor variances, where this second moment restriction has a much higher weight than the first moment restriction.

[10]For a given moment restriction, this t-ratio is defined by the ratio of the model error for this moment restriction and the standard error of this model error. This standard error is calculated using the asymptotic

restrictions to analyze the statistical accuracy of the models.

4 Empirical Results

There is much empirical evidence against one-factor term structure models. Also, given our specification for the volatility functions in equation (3), it directly follows that one-factor models cannot generate a humped volatility structure. Therefore, we focus on a two-factor and three-factor model. As explained in the previous section, we will estimate both models three times: on the basis of interest-rate-based estimation, option-based estimation, and joint estimation.

4.1 Two-Factor Results

In this subsection we focus on the two-factor results. In Table 2, we give the parameter estimates. First of all, we note that joint estimation does not always give more accurate parameter estimates than interest-rate-based and option-based estimation. For all three estimation methodologies, we find that the first factor has a relatively low decay parameter, whereas the second factor has a high decay parameter. This implies that only forward Libor rates with very short forward maturities are influenced by the second factor, and that the other forward Libor rates are driven by the first factor only, which causes these forward Libor rates to be (almost) perfectly correlated. The estimate for the correlation between the two factors is negative and close to -1[11]. This large negative correlation is needed to generate a humped volatility structure for forward Libor rates and cap implied volatilities.

In case of option-based estimation, the volatility parameters and the decay parameters are somewhat higher than for interest-rate-based estimation. On the basis of option-based estimation

covariance matrix of the estimated moment restrictions (see also Gourieroux and Monfort (1995)).

[11]In case of interest-rate based estimation, this correlation parameter is estimated at the lower bound of -0.999. In this case, for the calculation of standard errors and tests of moment restrictions the correlation parameter is treated as a constant.

we thus find a stronger decay in the volatilities as function of maturity than based on interest-rated based estimation.

Irrespective of the estimation methodology that is used, we find a large estimate for the Libor measurement error standard deviation. For example, in case of joint estimation, this measurement error standard deviation is estimated at 0.0088, implying that the forward Libor rates are measured with an error that has a standard deviation roughly equal to 88 basis points. Thus, the model implies that forward Libor rates are measured very imprecisely, and, although we argued in Section 2 that it is likely that there is some measurement error in the forward Libor rates, this amount of measurement error seems to be too large. This large estimate for the Libor measurement error variance either implies that the term structure model is misspecified, or that the assumptions on the measurement error structure are incorrect. For example, we have assumed constant variance of the measurement error across forward Libor maturities, and no correlation between measurement errors of different forward Libor rates. In future research we plan to examine whether other measurement error assumptions lead to more reasonable estimates for the size of the measurement error. In particular, one could argue that forward Libor rates with long forward maturities are measured with higher error, since these rates are more sensitive to measurement error in the underlying swap rate data.

For the three sets of parameter estimates for the two-factor model, the fit on the standard deviations of forward Libor rates is shown in Figure 1. The figure shows that, although all parameter estimates generate a humped standard deviation structure, the shape of the model-implied standard deviation structure is clearly different from the standard deviation term structure observed in the data. Also, the shape of the Libor standard deviation structure implied by option-based estimation is different from the shapes implied by the other two estimation methods: for long forward maturities, option-based estimation leads to lower forward Libor standard deviations. Note also that, if we would not include the Libor measurement error in the model, the option-based estimates would imply forward Libor standard deviations that are all much lower than the observed standard deviations. Table 3 gives the average absolute t-ratios for the individual moment restrictions. In case of interest-rate-based estimation, none of the Libor variance moment restrictions is misfitted significantly by the model. In case of option-based estimation, two out of the nine Libor variance moment restrictions are significantly misfitted by the model.

Table 1 gives the implications of the two-factor model for the cross-correlations of forward Libor rates. As mentioned above, all three sets of parameter estimates imply that the two-factor model itself yields almost perfect correlation between forward Libor rate changes of different maturities. However, the presence of the forward Libor measurement error term generates some decorrelation between forward Libor rates of different maturities. Since the Libor measurement error variance is largest in case of option-based estimation, these parameter estimates lead to the lowest correlations between Libor rates. However, Table 1 shows that none of the three sets of parameter estimates leads to a very good fit of the correlation structure. In the data, the correlation between forward Libor rates decreases when the difference between the two forward maturities increases, and all three two-factor models do not always imply such a correlation structure. This is confirmed by the t-ratios of the covariance moment restrictions in Table 3, that show that at least three of the ten covariance moment restrictions are rejected by all three two-factor models.

Figure 2 depicts the observed implied volatility term structure for caps, and the cap volatility structures implied by the two-factor model. Of course, the option-based parameter estimates lead to the best fit of these cap volatilities. Both interest-rate based estimation and joint estimation lead to cap volatility structures that are too flat. Comparing these results with the forward Libor standard deviations shows that the decay in the cap volatility term structure is larger than the decay in the term structure of forward Libor rate standard deviations. Still, due to the large variation and the large autocorrelation in cap implied volatilities over time, none of the models imply significant mispricing of the caps.

Finally, in Figure 3 we plot the fit of the two-factor models on the swaption implied volatilities. First of all, the figure shows that interest-rate-based estimation leads to too high prices for swaptions, and Table 3 shows that the pricing errors are mostly statistically significant. Still, the shape of the swaption volatility term structure that is implied by the interest-rate based estimates is very similar to the observed shape. In case of option-based estimation, the two-factor model gives a better fit of the level of swaption volatilities, but it does not accurately fit the shape of the swaption volatility term structure: option-based estimation leads to swaption volatility term structures that are too humped and that decline too much for longer maturities. Table 3 also shows that, even in case of option-based estimation, some swaptions are still significantly mispriced. As expected, in case of joint estimation, the fit on the swaption volatilities is worse than the fit in case of option-based estimation and better than the fit in case of interest-rate based estimation. In general, the two-factor model is not able to fit both the cap volatility structure, which exhibits a strong hump and is declining for long maturities, and the swaption volatility structure, which has only a small hump.

Summarizing, the two-factor model does not give a satisfactory fit of the data. First of all, there are some inconsistencies between estimation based on interest-rate data and estimation on the basis of option price data. In case of option-based estimation, the model does not accurately fit the standard deviations of forward Libor rate changes. In case of interest-rate based estimation, the model does not give a good fit of caps and, especially, swaptions. Second, even in case of joint estimation, Table 3 shows that there are still some Libor (co)variances and option prices that are significantly misfitted by the model. In particular, the two-factor model misfits the correlation structure of forward Libor rates. The reason for this is the following. To generate a humped volatility structure, the two factors need to be very highly negatively correlated, and one factor needs to have a very high decay parameter. This implies that this factor only influences very short maturity forward Libor rates, so that most forward Libor rates are essentially driven by only one factor. The two-factor model thus implies almost perfectly correlated forward Libor rates. Only due to the presence of the forward Libor measurement error structure, the model generates some decorrelation between forward Libor rates, but the model-implied correlation structure is quite different from the observed correlation structure. Therefore, in the next subsection we analyze three-factor models.

4.2 Three-Factor Results

Table 4 presents the parameter estimates for the three-factor model for the three sets of moment conditions. Again, joint estimation does not always give more accurate parameter estimates than interest-rate-based and option-based estimation. For the three estimation methods, the estimates for the volatility and decay parameters are quite similar to each other. One factor has a high decay parameter, implying a quickly declining volatility function, another factor has a very low decay parameter, implying a flat volatility function, and the third factor has an intermediate decay parameter. Using interest rate data only and different estimation methods, Dai and Singleton (2000) and De Jong (2000) find qualitatively similar results for three-factor models. The correlations between the factors that follow from option-based estimation are somewhat different from the factor-correlations implied by interest-rate-based estimation and joint estimation. Below, we discuss the implications of this difference. Finally, the Libor rate measurement error standard deviation is around 70 basis points for all three estimation methods, which is roughly the same as in the two-factor model. Therefore, adding a third factor does not 'solve' the problem of the (too) large estimate for the Libor measurement error variance.

Figure 4 shows that all three estimation strategies provide a reasonable fit of the standard deviations of forward Libor rates, and Table 3 shows that none of the forward Libor standard deviations is misfitted significantly. Table 5 gives the forward Libor correlation structures implied by the three estimation strategies. Clearly, the fit is much better than in case of the two-factor model, although the correlations are on average a bit too low. Only in case of option-based estimation, the correlations implied by the three-factor model are significantly too low (see Table 3). Thus, the correlations implicit in swaption prices are lower than the correlations in the forward Libor data.

Figure 5 presents the fit of the three-factor models on the cap volatility structure. Interest-rate based estimation leads to a reasonable fit of cap volatilities, and the pricing error is never (individually) significant. Note that, if we would not have included the Libor measurement error structure in our model, interest-rate based estimation would have led to cap volatilities that are much higher, which shows the importance of including the measurement errors. The other two estimation strategies also lead to a good fit of the cap volatility structure.

The fit on the swaptions volatilities is given in Figure 6. In case of interest-rate based estimation and joint estimation, swaptions are overpriced by the three-factor model. The reason for this is that the forward Libor rate correlations, as estimated using the interest rate data, are higher than the correlations implicit in swaption prices. Since lower correlations lead to lower swaption prices, this implies that, in case of interest-rate based estimation, swaptions are overpriced. In case of joint estimation, there is a trade off in the fit of the covariances (or, correlations) of forward Libor rates and the fit of swaption volatilities. In the end, the model parameter estimates imply forward Libor rate correlations that are somewhat lower than in the interest rate data, and swaption volatilities that are higher than the observed swaption volatilities.

Summarizing, the three-factor model is a clear improvement over the two-factor model, although the estimate for the Libor measurement error variance is still unrealistically large. The fit on all four sets of moment restrictions, Libor variances, Libor covariances, cap volatilities, and swaption volatilities, is better, and the differences between the implications of option-based estimation and interest-rate based estimation are much smaller than in case of the two-factor model. Only for the correlation structure of forward Libor rates and swaption prices, these two estimation methods yield somewhat different results. The joint estimation strategy illustrates how information in interest-rate data and option price data can be combined to accurately estimate the three-factor model, and Table 3 confirms that, in case of joint estimation, almost all moment restrictions are fitted accurately.

5 Summary and Conclusion

In this paper we examine multi-factor Libor Market Models. We specify a model with correlated factors, where each factor has a volatility function that corresponds to mean-reverting behaviour of the factor. This way, the model is related to the affine class of term structure models (Duffie and Kan (1996)), and the stochastic mean model of Jegadeesh and Pennacchi (1996).

To estimate and test such multi-factor models, we combine the information in interest rate data with the information in the prices of interest rate options. Previous empirical work on term structure models has estimated and tested models on the basis of either interest rate data, or derivative price data. In this paper, we analyze the benefits of combining these two data sets for estimating and testing term structure models. For comparison, we also estimate the models both only on the basis of interest rate data and only on the basis of option price data.

We use weekly US data on Libor and swap rates and prices for caps and swaptions from 1995 to 1999. The model setup explicitly allows for the presence of measurement error in both the interest rates and derivative prices. Given the model setup, moment restrictions are derived for both variances and covariances of changes in forward Libor rates, and for the expected prices of caps and swaptions. Estimation is performed by applying the Generalized Method of Moments (GMM, Hansen (1982)). We estimate both two-factor and three-factor models.

First, we analyze whether using both interest rate and option price data leads to more accurate parameter estimates. For both the two-factor and three-factor model, we find that, when estimating the model using both interest rate and option price data, the standard errors of the parameter estimates are not always smaller than the standard errors that result when only interest rate data or option price data are used for estimation.

Second, we analyze the fit of the models on the interest rate and option price data. The results for the two-factor model show that, in case of estimation based on option prices only, the model does not accurately fit the standard deviations of forward Libor rate changes, and, in case of estimation on the basis of interest rate data, the model misprices caps and, especially, swaptions. Thus, the two-factor model cannot fit the main features of the two data sets at the same time. This result illustrates the benefit of using both interest rate data and option price data for testing term structure models.

The three-factor model provides a better fit to both the interest rate data and the option price data. Both the humped shape of the standard deviations of forward Libor rate changes, and the humped shape of the cap implied volatility curve are fitted more accurately. Still, the model slightly overprices swaptions, and the model implies correlations between forward Libor rate changes that are a bit lower than in the data.

The results also show that allowing for measurement error in the interest rates is an important aspect of the model setup. Neglecting this measurement error structure would lead to overpricing of caps and too low standard deviations of forward Libor rate changes. However, although the three-factor model gives a reasonably good fit of both the interest rate data and option price data, the estimate for the variance of the measurement error in the forward Libor rates seems to be unrealistically large. In this paper, we have assumed that the forward Libor measurement errors are uncorrelated across forward Libor maturities, and have the same variance. It would be interesting to analyze whether more realistic estimates for the size of the measurement error result if these assumptions are relaxed or if transaction costs are taken into account.

References

Amin, K.I., and A. Morton (1994), ‘Implied Volatility Functions in Arbitrage-Free Term Structure Models’, *Journal of Financial Economics*, 35, 141-180.

Black, F. (1976), ‘The Pricing of Commodity Contracts’, *Journal of Financial Economics* 3, 167-179.

Brace, A., D. Gatarek, and M. Musiela (1997), ‘The Market Model of Interest Rate Dynamics’, *Mathematical Finance* 7, 27-155.

Buhler, W., M. Uhrig, U. Walter, and T. Weber (1999), ‘An Empirical Comparison of Forward- and Spot-Rate Models for Valuing Interest-Rate Options’, *Journal of Finance,* 54, 269-305.

Cochrane, J.H. (2001), *Asset Pricing*, Princeton University Press.

Dai, Q., and K.J. Singleton (2000), ‘Specification Analysis of Affine Term Structure Models’, *Journal of Finance*, forthcoming.

De Jong, F. (2000), ‘Time-series and Cross-section Information in Affine Term Structure Models’, *Journal of Economics and Business Statistics*, 18 (3), 300-318.

De Jong, F., J. Driessen, and A.A.J. Pelsser (2000), ‘Libor Market Models versus Swap Market Models for Pricing Interest Rate Derivatives: An Empirical Analysis’, Center Discussion Paper 2000-35, Tilburg University.

Driessen J., B. Melenberg, and Th.E. Nijman (1999), ‘Testing Affine Term Structure Models in case of Transaction Costs’, Center Discussion Paper 1999-84, Tilburg University.

Duan, J., and J.G. Simonato (1999), ‘Estimating Exponential-affine Term Structure Models by Kalman Filter’, *Review of Quantitative Finance and Accounting* 13(2), 11-135.

Duffee, G. (1999), 'Estimating the Price of Default Risk', *Review of Financial Studies*, 12, 197-226.

Duffie, D., and R. Kan (1996), 'A Yield-Factor Model of Interest Rates', *Mathematical Finance*, 64, 379-406.

Flesaker, B. (1993), 'Testing the Heath-Jarrow-Morton/Ho-Lee Model of Interest Rate Contingent Claims', *Journal of Financial and Quantitative Analysis* 28, 483-495.

Gourieroux, C., and A. Monfort (1995), *Statistics and Econometric Models*, Cambridge University Press.

Hansen, L.P. (1982), 'Large Sample Properties of Generalized Methods of Moments Estimators', *Econometrica,* 50, 1029-1054.

Jamshidian, F. (1997), 'Libor and Swap Market Models and Measures', *Finance and Stochastics* 1, 293-330.

Jegadeesh, N., and G.G. Pennacchi (1996), 'The Behavior of Interest Rates Implied by the Term Structure of Eurodollar Futures', *Journal of Money, Credit and Banking* 28, 426-451.

Miltersen, K., K. Sandmann, and D. Sondermann (1997), 'Closed Form Solutions for Term Structure Derivatives with Lognormal Interest Rates', *Journal of Finance* 52, 407-430.

Moraleda, J.M., and A.C.F. Vorst (1997), 'Pricing American Interest Rate Claims with Humped Volatility Models', *Journal of Banking and Finance* 21, 1131-1157.

Musiela, M., and M. Rutkowski (1997), *Martingale Methods in Financial Modeling*, Springer Verlag.

Newey, W.K., and K.D. West (1987), ‘A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix’, *Econometrica* 55, 703-708.

Pearson, N.D., and T.S. Sun (1994), ‘Exploiting the Conditional Density in Estimating the Term Structure: an Application of the CIR model’, *Journal of Finance,* 49, 1279-1304.

Table 1. Fit 2-Factor Models: Correlations Forward Libor Rates.

The 2-factor LMM in equations (2) and (5) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The table reports the correlations between forward Libor rate changes of different forward maturities, as implied by the 2-factor models.

	Data				Interest-Rate Estimation			
Maturity	1.25	2.75	4.75	9.75	1.25	2.75	4.75	9.75
0.25	0.895	0.727	0.704	0.576	0.863	0.854	0.839	0.786
1.25	-	0.847	0.832	0.688	-	0.882	0.866	0.811
2.75	-	-	0.958	0.632	-	-	0.858	0.803
4.75	-	-	-	0.821	-	-	-	0.789

	Option Estimation				Joint Estimation			
Maturity	1.25	2.75	4.75	9.75	1.25	2.75	4.75	9.75
0.25	0.814	0.784	0.728	0.525	0.762	0.748	0.726	0.653
1.25	-	0.806	0.749	0.540	-	0.814	0.790	0.711
2.75	-	-	0.722	0.520	-	-	0.776	0.698
4.75	-	-	-	0.483	-	-	-	0.678

Table 2. Parameter Estimates 2-Factor Model.

The 2-factor LMM in equations (2) and (5) is estimated using first-stage GMM on the basis of three sets of moments: variances and covariances of forward Libor rate changes, cap and swaption implied volatilities, and these two sets together. The table reports parameter estimates and standard errors, which are corrected for heteroskedasticity and 20th-lag autocorrelation using Newey-West (1987).

	Interest-Rate-Based Estimation	Option-Based Estimation	Joint Estimation
σ_1	0.222 (0.019)	0.246 (0.013)	0.212 (0.010)
σ_2	0.131 (0.095)	0.504 (0.017)	0.484 (0.001)
κ_1	0.064 (0.018)	0.134 (0.018)	0.066 (0.001)
κ_2	3.234 (1.166)	8.526 (0.882)	7.617 (0.716)
ρ_{12}	-0.999	-0.996 (0.052)	-0.859 (0.091)
σ_ϵ	0.0071 (0.0008)	0.0091 (0.0028)	0.0088 (0.0018)

Table 3. Average Absolute T-ratios Moment Restrictions.

For the 2-factor and 3-factor models, the t-ratios of the individual moment restrictions are calculated, correcting for heteroskedasticity and 20th-lag autocorrelation using Newey-West (1987). The table reports for each set of moments the average of the absolute value of these t-ratios, and the number of moment restrictions that is individually rejected at the 5% significance level.

	Interest-Rate-Based Estimation	Option-Based Estimation	Joint Estimation
	2-Factor Model		
Libor Variances (9)	0.919 (0)	1.853 (2)	0.988 (1)
Libor Covariances (10)	1.670 (3)	2.593 (6)	2.201 (3)
Caps (7)	0.946 (0)	0.580 (0)	0.748 (0)
Swaptions (9)	3.973 (7)	1.318 (2)	1.801 (4)
	3-Factor Model		
Libor Variances (9)	0.225 (0)	0.390 (0)	0.243 (0)
Libor Covariances (10)	0.338 (0)	1.725 (3)	0.426 (0)
Caps (7)	0.443 (0)	0.317 (0)	0.439 (0)
Swaptions (9)	2.173 (4)	0.978 (1)	1.608 (3)

Table 4. Parameter Estimates 3-Factor Model.

The 3-factor LMM in equations (2) and (5) is estimated using first-stage GMM on the basis of three sets of moment restrictions. The table reports parameter estimates and standard errors, which are corrected for heteroskedasticity and 20th-lag autocorrelation using Newey-West (1987).

	Interest-Rate-Based Estimation	Option-Based Estimation	Joint Estimation
σ_1	0.147 (0.015)	0.149 (0.045)	0.143 (0.014)
σ_2	0.908 (1.18)	0.508 (0.024)	0.687 (0.422)
σ_3	0.754 (1.17)	0.387 (0.053)	0.505 (0.448)
κ_1	0.000 (0.008)	0.003 (0.044)	0.000 (0.016)
κ_2	1.670 (0.566)	2.964 (1.01)	2.038 (0.656)
κ_3	0.969 (0.450)	0.544 (0.024)	0.876 (0.344)
ρ_{12}	-0.635 (0.083)	-0.429 (0.633)	-0.634 (0.305)
ρ_{13}	-0.974 (0.090)	-0.805 (0.192)	-0.941 (0.103)
ρ_{23}	0.529 (0.138)	-0.106 (0.354)	0.486 (0.343)
σ_ϵ	0.0067 (0.0008)	0.0070 (0.0037)	0.0072 (0.0028)

Table 5. Fit 3-Factor Models: Correlations Forward Libor Rates.

The 3-factor LMM in equations (2) and (5) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The table reports the correlations between forward Libor rate changes of different forward maturities, as implied by the 3-factor models.

	Data				Interest-Rate Estimation			
Maturity	1.25	2.75	4.75	9.75	1.25	2.75	4.75	9.75
0.25	0.895	0.727	0.704	0.576	0.636	0.538	0.479	0.465
1.25	-	0.847	0.832	0.688	-	0.856	0.782	0.762
2.75	-	-	0.958	0.632	-	-	0.839	0.828
4.75	-	-	-	0.821	-	-	-	0.831
	Option Estimation				Joint Estimation			
Maturity	1.25	2.75	4.75	9.75	1.25	2.75	4.75	9.75
0.25	0.665	0.428	0.191	0.059	0.596	0.501	0.444	0.428
1.25	-	0.797	0.608	0.484	-	0.837	0.760	0.738
2.75	-	-	0.777	0.709	-	-	0.813	0.801
4.75	-	-	-	0.798	-	-	-	0.803

Figure 1. Libor Volatilities 2-Factor Model. The 2-factor LMM (equations (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the annualized standard deviations of forward Libor rate changes, as implied by the 2-factor models.

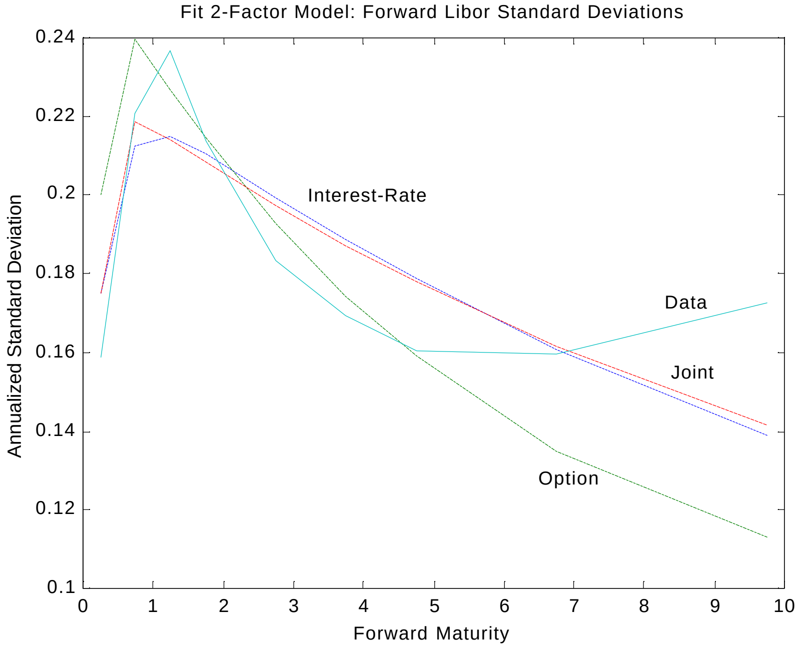

Figure 2. Cap Volatilities 2-Factor Model. The 2-factor LMM (equation (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the cap Black volatilities for different option maturities, as implied by the 2-factor models.

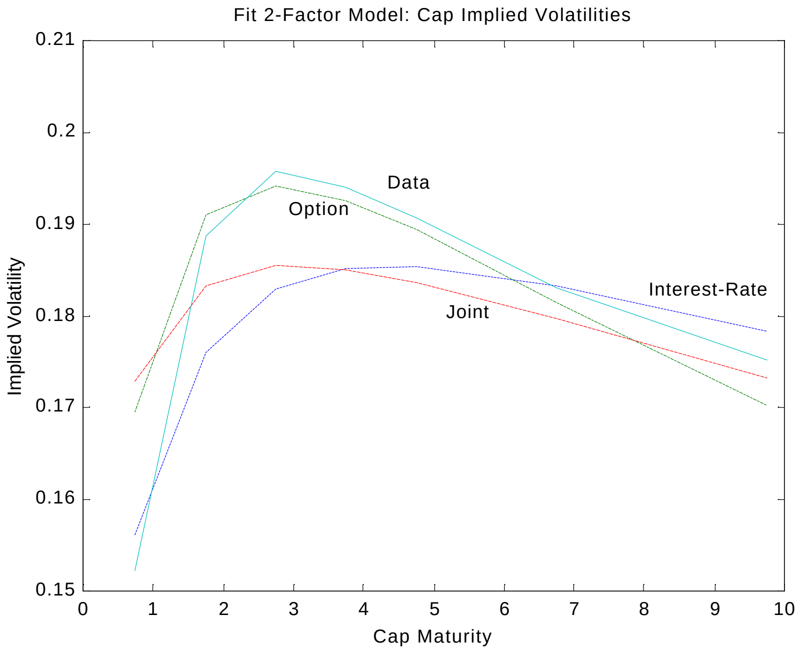

Figure 3. Swaption Volatilities 2-Factor Model. The 2-factor LMM (equations (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the swaption Black volatilities for different option maturities, as implied by the 2-factor models.

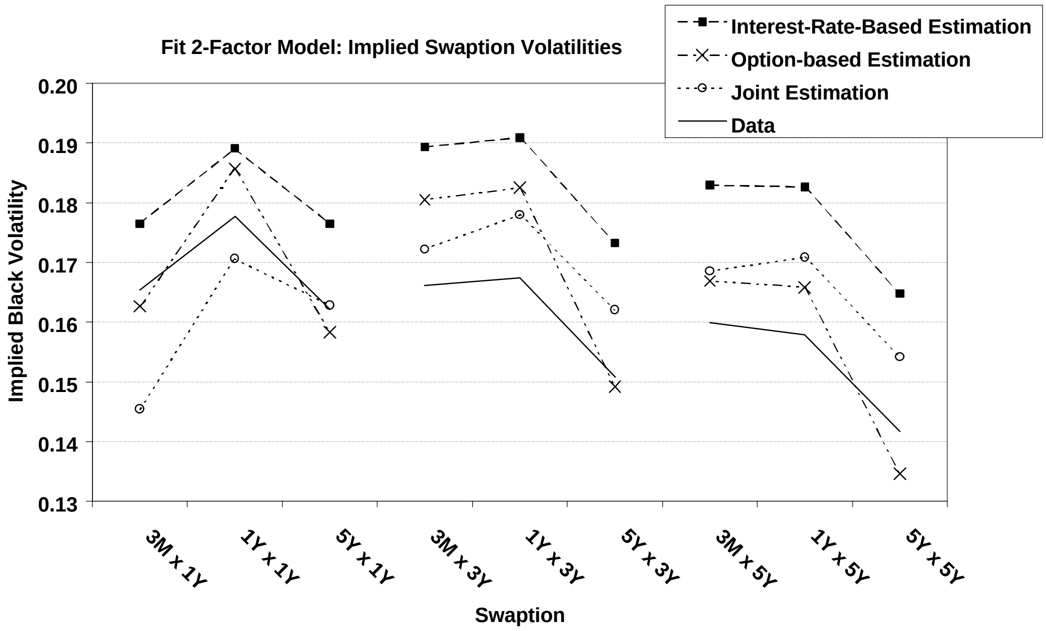

Figure 4. Libor Volatilities 3-Factor Model. The 3-factor LMM (equations (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the annualized standard deviations of forward Libor rate changes, as implied by the 3-factor models.

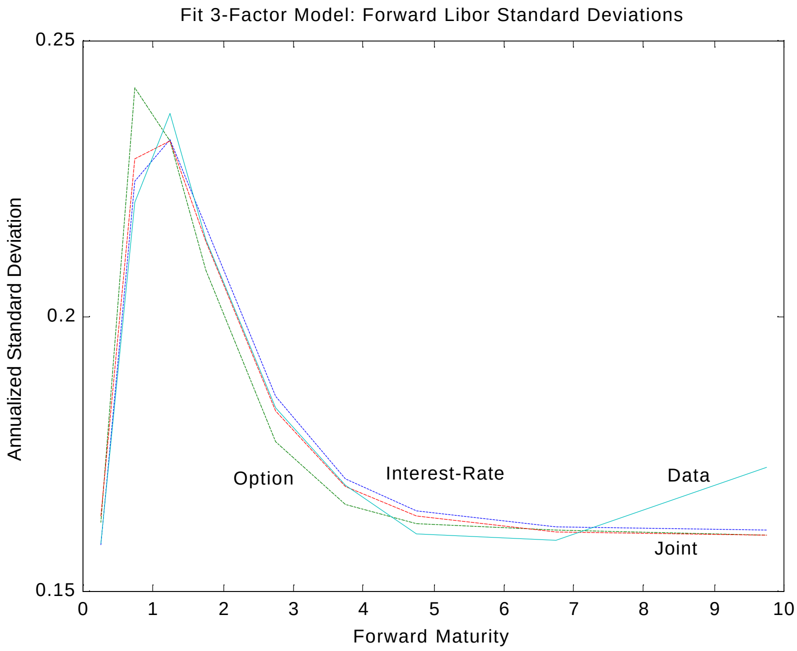

Figure 5. Cap Volatilities 3-Factor Model. The 3-factor LMM (equations (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the cap Black volatilities for different option maturities, as implied by the 3-factor models.

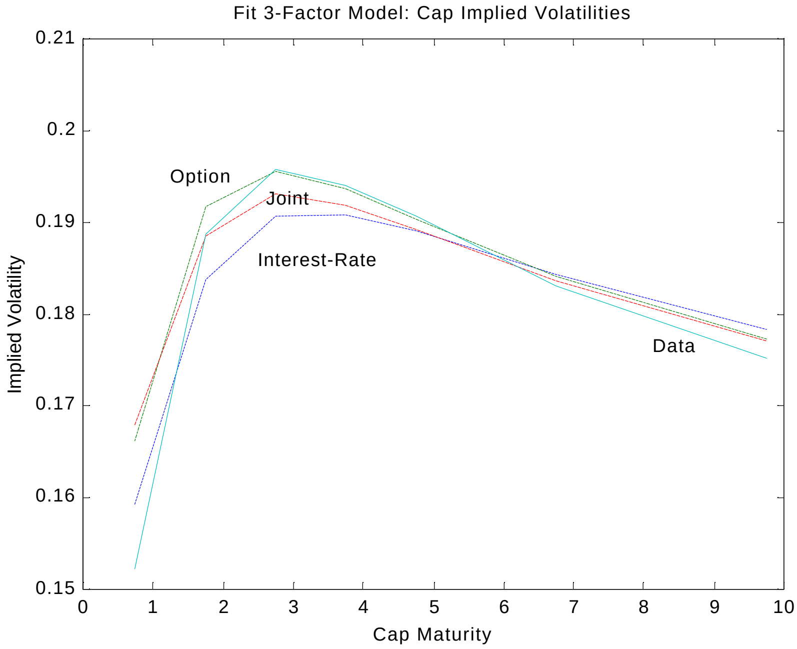

Figure 6. Swaption Volatilities 3-Factor Models. The 3-factor LMM (equations (2) and (5)) is estimated using first-stage GMM on the basis of three sets of moments as described in the text. The figure plots the swaption Black volatilities for different option maturities, as implied by the 3-factor models.