A LIBOR MARKET MODEL WITH DEFAULT RISK

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ABSTRACT. In this paper a new credit risk model for credit derivatives is presented. The model is based upon the 'Libor market' modelling framework for default-free interest rates. We model effective default-free forward rates and effective forward credit spreads as lognormal diffusion processes, and recovery is modelled as a fraction of the par value of the defaulted claim. The newly introduced survival-based pricing measures are a valuable tool in the pricing of defaultable payoffs and allow a straightforward derivation of the no-arbitrage dynamics of forward rates and forward credit spreads. The model can be calibrated to the prices of defaultable coupon bonds, asset swap rates and default swap rates for which closed-form solutions are given. For options on default swaps and caps on credit spreads, approximate solutions of high accuracy exist. This pricing formula for options on default swaps is made exact in a modified modelling framework using an analogy to the swap measure, the default swap measure.

1. INTRODUCTION

In this paper a new credit risk model is presented which uses *effective* (simply compounded) forward rates as fundamental model quantities, and not continuously compounded forward rates. This approach is motivated by the so called Libor Market Models for default-free interest rates by Miltersen / Sandmann / Sondermann (1997), Brace / Gatarek / Musiela (1997) and Jamshidian (1997).

The most important risks involved in an investment in a defaultable bond or loan are the interest-rate risk (the price risk introduced by changes in the general level of the default-free interest rates), the spread risk (ultimately caused by changes in the market's assessment of the credit quality of the obligor) and the default risk of the obligor which in turn involves recovery risk, the uncertainty about the loss given default. The aim of this paper is to provide an integrated

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framework for all these risks which is flexible enough to allow easy calibration to the market prices of traded assets and in which more exotic default contingent payoffs such as credit derivatives can be priced in a consistent manner.

There is a large number of models which can be used to capture interest-rate risk[1], among which the interest-rate models of the market-model class (either Libor-based or Swap-based) are amongst the most widely used in practice. Their popularity is due to several factors, one of which is the ease with which these models can be implemented and calibrated to market data. Instead of using a instantaneous short rate process as fundamental variable, directly observed, discretely compounded money market rates are modelled, and because Caplets and Swaptions can be priced in closed-form with the Black (1976) formula, the volatility parameters of the model can be directly calibrated to market prices of traded instruments.

In this paper we chose discretely compounded (effective) forward rates and effective forward credit spreads (or effective default intensities) as fundamental model quantities following the market models for default-free interest rates. We equip them with the continuous-time dynamics of lognormal diffusion processes. Because we assume that the credit spreads are only driven by changes in the credit quality, we thus have already implicitly modelled the default risk in the model.

An important tool to analyse this implicit default risk is a new probability measure, the T_k-survival measure $\overline{P}_k$, which is the defaultable equivalent of the T_k-forward measure P_k. The associated numeraire asset to the survival measure is the *defaultable* zero coupon bonds with maturity T_k. Using the T_k-survival measure $\overline{P}_k$ we can price all survival- and default-contingent payoffs at time T_k. Furthermore, we can easily derive necessary conditions on the dynamics of the forward rates and forward spreads that ensure absence of arbitrage in the model. In a later section of the paper a similar survival-based measure (the default swap measure) enables us to price options on default swaps in closed-form.

The only remaining risk factor is recovery risk. In this model, recovery is not modelled on the basis of defaultable zero-coupon bonds (as in most competing models) but on the basis of defaultable coupon bonds and loans. Here, recovery is a default-contingent payoff of a (possibly random) fraction of the par value of the defaulted bond. This approach for the modelling of the recovery rate was chosen because it reflects the real-world recovery mechanisms more closely than many competing models. After deriving the prices of elementary Arrow-Debreu securities for payoffs at default, we can then give the prices of *defaultable fixed and floating coupon bonds, default swaps* and *asset swaps* in the model. In a practical implementation, these securities can be used to calibrate the model.

In the last part of the paper this model setup is used to derive closed-form solutions for the prices of options on default swaps (default swaptions) and caps on credit spreads. Here, a defaultable version of the Swap Market Model by Jamshidian (1997) is introduced and applied. Options of the above mentioned type frequently occur as embedded options in other securities, e.g. prepayment or extension options in loan contracts or callability provisions in callable default swaps.

Related Literature. The literature on market models has grown substantially in recent years, and it is impossible to give a full list. Apart from the standard references (Miltersen / Sandmann / Sondermann (1997), Brace / Gatarek / Musiela (1997) and Jamshidian (1997)), the mathematical methods in E. Schlögl's (1999) multicurrency extension of the Libor market model are related to the methods in this paper. Schlögl also analyses the problems that arise when several numeraires and martingale measures have to be used in parallel. While his work concentrates

[1]For a survey and introduction see e.g. Rebonato (1998).

on the foreign exchange sector, this paper has the additional complication of default risk. The paper by Lotz and Schlögl (2000) treats the valuation of money market instruments under counterparty default risk, but they do not use the market-model framework to describe defaultable term structures of interest rates. References to techniques for the numerical implementation of market models are given in the section on implementation.

If the literature on market models is large, the literature on credit risk modelling is even larger. This paper is in the tradition of the intensity-based default risk models which all exhibit a close relationship to default-free interest rate models. Representatives of this approach are Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1997; 1999), Lando (1998) and Schönbucher (1998; 1999). In these papers the reader can also find references on the different approaches for recovery modelling: fractional recovery (Duffie/Singleton (1997; 1999)), multiple defaults with reorganisation (Schönbucher (1998)), recovery of equivalent default free bonds (Jarrow/Turnbull (1995), Madan/Unal (1998), Lando (1998)) and recovery of par (Duffie (1998)). For a survey of the different modelling approaches for default risk the reader is referred to Schönbucher (1997).

Structure of the Paper. The rest of the paper is structured as follows: After the introduction of some notation in the next section, we give a description of the no-arbitrage conditions in the continuous-time setup. This follows Heath / Jarrow / Morton (1992) for the default-free term structure of interest rates and Schönbucher (1998) for the defaultable case.

The analysis of market models of interest rates makes extensive use of the change-of-measure technique. For each pricing problem the numeraire asset and corresponding probability measure is identified. Therefore we introduce several new probability measures in the following section, where each default-free probability measure has a survival-based defaultable counterpart: T-forward measure and T-survival measure, swap-measure and default-swap measure and discrete Libor measure and discrete defaultable Libor measure. We give the changes of drift that are associated with the respective changes of measure, and identify the dynamics of defaultable and default-free forward rates, interest-rate swap rates and default swap rates under these measures.

In the next step, positive recovery is introduced. The recovery model used here is based upon the fractional recovery of par model by Duffie (1998) which has the advantage of closely adhering to real-world recovery proceedings and of recognizing the importance of the distinction between principal and coupon claims. The value of the elementary Arrow-Debreu securities under this model is derived.

In the following section some important payoffs are valued: defaultable fixed and floating coupon bonds, default swaps and asset swap packages. For independence between defaults and interest rates these are in closed-form, for non-zero correlation high-quality approximate solutions are given. The next section considers the pricing of options on default swaps. Here, we need to introduce the *default swap measure* under which the default swap rates become martingales. Using this probability measure we are able to derive option price formulae similar to the well-known Black-formula. These pricing formulae can either be used in a direct default-swap based model, or after some approximating assumptions in the Libor-based approach of the previous sections.

The paper is concluded with a discussion of the strategy to numerically implement this model for the pricing of more exotic credit derivatives.

2. Notation and Model Setup

The model is set in a filtered probability space $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, Q)$ where the filtration satisfies the usual conditions, and Q is the spot martingale measure. For convenience we assume a large but finite time-horizon $\overline{\overline{T}}$. Usually, quantities that refer to defaultable bond prices or interest rates carry an $\overline{\text{overbar}}$. All stochastic processes in this paper are adapted to $(\mathcal{F}_t)_{(t\geq 0)}$ and we omit the dependence on the state of nature ω, e.g. we write $W(t)$ for $W(t,\omega)$ etc.

2.1. The Default Model.

Assumption 1 (Defaults). *(i) The default time is given by the stopping time τ.*

(ii) Default is triggered by the first jump of a Cox process $N(t)$ which has an intensity process $\lambda(t)$.

(iii) The survival indicator function is denoted by

$$I(t) := \mathbf{1}_{\{\tau > t\}}.$$

The survival indicator function $I(t)$ is one before default and jumps to zero at the time of default.

In most parts, the model does not depend on having a Cox process triggering the default, all necessary information about the default process will be recovered from the term structure of defaultable bond prices. The Cox process properties will only be at some points when the recovery payoffs are valued.

It is well-known[2] that the survival probability from t to T in this framework is given by

$$\mathbf{E}\left[\, e^{-\int_t^T \lambda(s)ds} \,\right],$$

and $N(t) - \int_0^t \lambda(s)ds$ is a martingale.

2.2. Bond Prices and Basic Rates.

The tenor structure:
We consider payoffs that occur on a discrete set of points in time

$$0 = T_0, T_1, \ldots, T_K$$

These dates could be coupon and repayment dates for bonds or loans, fixing dates for rates and settlement dates for derivatives. The distance between two tenor dates is denoted by $\delta_k := T_{k+1} - T_k$.

The function $\kappa(t) = \min\{k \,|\, T_k > t\}$ gives the next date in the tenor structure after t. Thus $T_{\kappa(t)-1} \leq t < T_{\kappa(t)}$.

Bond prices:

(i) Default-free zero coupon bond prices at time t with maturity T_k are denoted by

$$B_k(t) = B(t, T_k).$$

(ii) Defaultable zero coupon bond prices at time t with maturity T_k are

$$I(t)\overline{B}_k(t) = I(t)\overline{B}(t, T_k).$$

These defaultable zero coupon bonds have zero recovery in default[3].

[2] For more details on point- and Cox-processes in default-risk modelling see Duffie and Singleton (1997; 1999) or Lando (1998).

[3] Note that the influence of the defaults (I (t)) and the pre-default price $\overline{B}_k(t)$ are separated. $\overline{B}_k(t)$ need not jump to zero at default because $I(t)$ already does.

(iii) The default-risk factor at time t for maturity T_k is

$$D_k(t) = D(t, T_k) = \frac{\overline{B}_k(t)}{B_k(t)}.$$

The default-risk factors D allow to separate the influence of default risk from the standard discounting with default-free interest rates. It will be shown later that $D_k(t)$ is the survival probability until T_k under the T_k-forward measure.

Forward Rates:

(i) The default-free effective forward rate over $[T_k, T_{k+1}]$ as seen from time t is

$$F_k(t) = \frac{1}{\delta_k}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right). \tag{1}$$

(ii) The defaultable effective forward rate over $[T_k, T_{k+1}]$ as seen from time t is

$$\overline{F}_k(t) = \frac{1}{\delta_k}\left(\frac{\overline{B}_k(t)}{\overline{B}_{k+1}(t)} - 1\right). \tag{2}$$

(iii) The forward credit spread over $[T_k, T_{k+1}]$ as seen from time t is

$$S_k(t) = \overline{F}_k(t) - F_k(t). \tag{3}$$

(iv) The discrete-tenor forward default intensity over $[T_k, T_{k+1}]$ as seen from time t

$$H_k(t) = \frac{1}{\delta_k}\left(\frac{D_k(t)}{D_{k+1}(t)} - 1\right). \tag{4}$$

The defaultable forward rate $\overline{F}_k(t)$ is the rate at which a lender would agree at time t to lend to the obligor over the future time-interval $[T_k, T_{k+1}]$, conditional on the obligor's survival until T_k.

From these definitions follow the following relationships between bond prices and forward rates

$$\frac{B_k}{B_{k+1}} = 1 + \delta_k F_k \qquad B_k = B_1 \prod_{j=1}^{k-1} (1 + \delta_j F_j)^{-1} \tag{5}$$

$$S_k = H_k(1 + \delta_k F_k), \tag{6}$$

and calculation rules similar to (5) apply to ($\overline{B}$ and $\overline{F}$) and (D and H).

2.3. **Dynamics.** In this subsection the volatility structure of the forward rate processes is specified. The Brownian motion W is a d-dimensional standard Q-Brownian motion, and all volatility processes are d-dimensional vector processes.

Assumption 2 (Default-Free Interest Rate Dynamics)**.** *The default-free forward rates F_k have a lognormal volatility structure*

$$\frac{dF_k}{F_k} = \mu_k^F dt + \sigma_k^F dW, \tag{7}$$

where σ_k^F are constant vectors. The drifts μ_k^F are more complicated and their full form under the respective martingale measures will be derived later on.

There are two alternatives in the specification of the dynamics of the defaultable interest-rates. Either we model the spreads S or the discrete intensities H as lognormal processes.

Assumption 3 (Defaultable Interest Rate Dynamics)**.**
Modelling Alternative (a)
The discrete default intensities H_k have a lognormal volatility structure

$$\frac{dH_k}{H_k} = \mu_k^H dt + \sigma_k^H dW, \tag{8}$$

where σ_k^H are constant vectors.
Modelling Alternative (b)
The forward credit spreads S_k have a lognormal volatility structure

$$\frac{dS_k}{S_k} = \mu_k^S dt + \sigma_k^S dW, \tag{9}$$

where σ_k^S are constant vectors.

Of the two modelling alternatives, alternative (a) with H as model primitive with lognormal volatility structure is usually more convenient, and we will use this alternative unless alternative (b) is explicitly specified.

We also define the parameters of the dynamics of the defaultable forward rates

$$\frac{d\overline{F}_k}{\overline{F}_k} = \mu_k^{\overline{F}} dt + \sigma_k^{\overline{F}} dW \tag{10}$$

but here the volatilities are not assumed to be constant. The relationships between the volatilities are

$$\sigma_k^H = \sigma_k^S - \frac{\delta_k F_k}{1+\delta_k F_k}\sigma_k^F \tag{11}$$

$$\overline{F}_k \sigma_k^{\overline{F}} = \sigma_k^F F_k + \sigma_k^S S_k = (1+\delta_k F_k) H_k \sigma_k^H + (1+\delta_k H_k) F_k \sigma_k^F. \tag{12}$$

The choice of lognormal forward rate volatilities for default-free interest rates is market standard, in this case the Black Caplet volatilities can be used directly to calibrate the model (for a thorough dicussion of the issues in the calibration of market models see Rebonato (1998) and in particular (1999b)). Directly prescribing lognormal dynamics for the *defaultable* forward rates $\overline{F}$ on the other hand is problematic because then it cannot be ensured any more that defaultable bonds are always worth less than the equivalent default-free bonds. Therefore we choose either H or S to have a lognormal volatility structure, and this potential arbitrage opportunity is ruled out.

3. Drift Restrictions for the Continuous Tenor Case

To motivate the rest of the paper we use the Heath / Jarrow / Morton (1992) framework as starting point, where continuously compounded default-free and defaultable forward rates are used to describe the term structures of interest rates

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T) \qquad\qquad \overline{f}(t,T) = -\frac{\partial}{\partial T}\ln \overline{B}(t,T).$$

In this framework the conditions for absence of arbitrage are well-known (see Heath / Jarrow / Morton (1992) and Schönbucher (1998) for the proofs):

To ensure absence of arbitrage, the dynamics of the defaultable and the default-free continuously compounded forward rates and the short credit spread must satisfy the following equations under the spot martingale measure Q

$$d\overline{f}(t,T) = \sigma^{\overline{f}}(t,T)\left(\int_t^T \sigma^{\overline{f}}(t,s)ds\right)dt + \sigma^{\overline{f}}(t,T)dW_Q, \tag{13}$$

$$df(t,T) = \sigma^{f}(t,T)\left(\int_t^T \sigma^{f}(t,s)ds\right)dt + \sigma^{f}(t,T)dW_Q, \tag{14}$$

$$\overline{f}(t,t) = \lambda(t) + f(t,t). \tag{15}$$

These conditions are sufficient to ensure absence of arbitrage in the market. The solutions to the stochastic differential equations for the bond prices are then

$$\frac{B(t,T)}{B(0,T)} = \exp\left\{\int_0^t r(s) - \frac{1}{2}\alpha^2(s,T)\,ds - \int_0^t \alpha(s,T)dW_Q(s)\right\} \tag{16}$$

$$\frac{\overline{B}(t,T)}{\overline{B}(0,T)} = \exp\left\{\int_0^t \lambda(s) + r(s) - \frac{1}{2}\overline{\alpha}^2(s,T)\,ds - \int_0^t \overline{\alpha}(s,T)dW_Q(s)\right\}. \tag{17}$$

$$\text{where} \qquad \alpha(t,T) = \int_t^T \sigma^f(t,s)ds \qquad \text{and} \qquad \overline{\alpha}(t,T) = \int_t^T \sigma^{\overline{f}}(t,s)ds. \tag{18}$$

4. Forward- and Survival Measures

There are two types of probability measures which are particularly well suited for the analysis of this model: the T_k-*forward measure* and the T_k-*survival measure*. The associated numeraire assets to these probability measures are the default-free and defaultable zero coupon bonds with maturity T_k. The probabilities under the respective measure may be regarded as *state prices* expressed in units of the numeraire.

The default-free probability measures of this section are well-known. The spot-martingale measure is described in most textbooks on quantitative finance, and the T_k forward measure is a standard tool in models of the term structure of interest rates, particularly in Gaussian term structure models and in the market models. The introduction of the T_k-forward measure goes back to Jamshidian (1987), the *survival measure* on the other hand has not appeared in the literature in this form.

4.1. **Girsanov's Theorem:** Girsanov's theorem[4] describes how the Radon-Nikodym density L of a change of probability measure determines which processes are Brownian motions under the new measure, and which form the compensator of the jump process takes under the new measure. We give a general form of this theorem which is valid for probability spaces that support marked point processes and diffusions. The marker q of the point process can be used to model uncertainty in the recovery rate.

Theorem 1 (Girsanov Theorem: Marked Point Processes)**.** *Let $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, Q)$ be a filtered probability space which supports a n-dimensional Q-Brownian motion $W_Q(t)$ and a marked point process $\mu(dq;dt)$.*
The marker q of the marked point process is drawn from the mark space $(E,\mathcal{E})$. The compensator of $\mu(dq,dt)$ is assumed to take the form $\nu_Q(dq,dt) = K_Q(dq)\lambda_Q(t)dt$ under Q. Here $\lambda_Q(t)$ is the intensity of the arrivals of the point process, and $K_Q(dq)$ is the conditional distribution of the marker on $(E,\mathcal{E})$.
Let θ be a n-dimensional predictable process and $\Phi(t,q)$ a nonnegative predictable function [5]

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[4]See Jacod and Shiryaev (1988) and Björk, Kabanov and Runggaldier (1996).

[5]In functions of the marker q (like Φ here) *predictability* means measurable with respect to the σ-algebra $\tilde{\mathcal{P}} := \mathcal{P}\otimes\mathcal{E}$. Here $\mathcal{P}$ is the σ-algebra of the predictable processes. See Jacod and Shiryaev (1988) for details.

with

$$\int_0^t \|\theta(s)\|^2 ds < \infty, \qquad \int_0^t \int_E |\Phi(s,q)|\, K_Q(dq)\, \lambda_Q(s) ds < \infty$$

for finite t. Define the process $L(t)$ by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \theta(t) dW_Q(t) + \int_E (\Phi(t,q) - 1)(\mu(dq,dt) - \nu_Q(dq,dt)).$$

Assume that $\mathbf{E}^Q\,[\; L(t)\;] = 1$ for finite t.
Then for the probability measure P with

$$dP(t) = L(t) dQ(t) \tag{19}$$

it holds that

$$dW_Q(t) - \theta(t) dt = dW_P(t) \tag{20}$$

defines W_P as P-Brownian motion and

$$\nu_P(dq,dt) = \Phi(t,q)\nu_Q(dq,dt) \tag{21}$$

is the predictable compensator of μ under P.
Define $\phi(t) := \int_E \Phi(t,q) K_Q(dq)$, and $L_E(q) := \Phi(t,q)/\phi(t)$ for $\phi(t) > 0$, $L_E(q) = 1$ otherwise. Then

$$\lambda_P(t) = \phi(t)\lambda_Q(t) \tag{22}$$

ist the intensity of the arrival rate of the marked point process under P, and

$$K_P(dq) = L_E(q) K_Q(dq) \tag{23}$$

ist the transformed conditional distribution of the marker under P.

4.2. **The Subjective Measure P.** The subjective (or historical) probability measure P gives the 'real' probabilities of the events. Because it does not take risk premia into account it cannot be used for pricing. A detailed account of the change of measure from the historical probability measure to the spot martingale measure in the case of credit risk models can be found in Schönbucher (1998), it is not repeated here. Apart from the usual change of drift in the Brownian motions, this change of measure typically results in a significantly higher default intensity λ_Q under Q which reflects the high risk premia on default risk in the market.

4.3. **The Spot Martingale Measure Q.** The spot martingale measure Q is the probability measure, under which the discounted security price processes become martingales. The numeraire to the spot-martingale measure is the continuously compounded savings account $b(t)$. Its inverse is the continuously compounded discount factor $\beta(t)$

$$\beta(t) = e^{-\int_0^t r(s)ds} \qquad b(t) = e^{\int_0^t r(s)ds}. \tag{24}$$

Under the spot-martingale measure Q, the time-t price of a random payoff X at time T_k is

$$p(t) = \mathbf{E}^Q\left[\, \frac{\beta(T_k)}{\beta(t)} X \,\middle|\, \mathcal{F}_t \right] = \mathbf{E}^Q\left[\, \frac{b(t)}{b(T_k)} X \,\middle|\, \mathcal{F}_t \right]. \tag{25}$$

Thus $\beta(t)p(t)$, i.e. the price $p(t)$ normalized with the Q-numeraire $b(t)$, is a Q-martingale, as claimed.

4.4. **The Change of Measure / Change of Numeraire Technique.** Any given price $p(t)$ under the spot numeraire $b(t)$ can be transformed to a price $p'(t)$ under a different numeraire $A(t)$ via $p'(t) := p(t)b(t)/A(t)$ (or $p'(0) = p(0)/A(0)$). Using equation (25) the price under the new numeraire is

$$p'(t) = \frac{b(t)}{A(t)}p(t) = \frac{b(t)}{A(t)}\mathbf{E}^Q\left[\left.\frac{A(T_k)}{b(T_k)}X'\;\right|\;\mathcal{F}_t\right]$$

$$= \mathbf{E}^Q\left[\left.\frac{L_A(T_k)}{L_A(t)}X'\;\right|\;\mathcal{F}_t\right] = \mathbf{E}^{P_A}\left[\,X' \mid \mathcal{F}_t\,\right] \tag{26}$$

where $X' = X/A(T_k)$ is the payoff (final value) X of the contingent claim in terms of the new numeraire asset A. In equation (26) a new pricing measure P_A is defined by the Radon-Nikodym density process $L_A(t)$.

$$\left.\frac{dP_A}{dQ}\right|_{\mathcal{F}_t} = L_A(t) := \frac{1}{A(0)}\frac{A(t)}{b(t)}. \tag{27}$$

Because $A(t)$ is the b-price of a traded asset, the process $L_A(t)$ is a nonnegative Q-martingale with initial value $L_A(0) = 1$. $L_A(t)$ is therefore a valid Radon-Nikodym density process and P_A is a well-defined probability measure.

By equation (26), prices p' in the numeraire A are P_A-martingales. Thus the calculation of the initial price p' can be reduced to the calculation of the expected final value X' under a changed probability measure P_A.

This change of measure technique would not be useful if there was no way to calculate the new expectation $\mathbf{E}^{P_A}[\,X\,]$ in (26) other than going back and evaluate $\mathbf{E}^Q[\,L_A(T_k)X\,]$. Here, Girsanov's theorem allows us to derive the dynamics of the stochastic processes under the T_k-forward measure and thus to directly evaluate expectations under P_A.

We can also see why P_A can be considered to be a set of state prices: Consider a state security p_E for state $E \in \mathcal{F}_{T_k}$. Then by equation (26), $P_A[E]$ is the A-price p'_E of a payoff of 1 units of $A(0)$ in event E.

4.5. **The T_k-Forward Measure P_k.** The T_k-forward measure is used to price payoffs that occur at time T_k. The associated numeraire to P_k is the default-free bond $B_k(t)$ that matures at T_k. Equation (25) for the price of payoff X at T_k is in this case

$$p' = \frac{p}{B_k(0)} = \mathbf{E}^Q\left[\frac{\beta(T_k)B_k(T_k)}{B_k(0)}\,X\right] =:\ \mathbf{E}^{P_k}\left[\,X\,\right]. \tag{28}$$

because $B_k(T_k) = 1$ we do not have to transform the final payoffs to the new numeraire. The Radon-Nikodym density process is

$$L_k(t) := \frac{\beta(t)B_k(t)}{B_k(0)} = \left.\frac{dP_k}{dQ}\right|_{\mathcal{F}_t}. \tag{29}$$

As required, this process is a nonnegative martingale with initial value one. By the change of measure the discount factor $\beta(T_k)$ was removed from the expectation in equation (29). This is often a crucial step in the derivation of prices for derivative securities.

Analyzing the Radon-Nikodym density (29) yields the change of drift to reach the P_k-Brownian motion $dW_k(t)$ from the Q-Brownian Motion $dW_Q(t)$:

$$dW_k(t) := dW_Q(t) + \alpha_k(t)dt. \tag{30}$$

The process $\alpha_k(t)$ is defined in equation (18) as minus the volatility of the default-free zero-coupon bond $B_k(t)$. Note that the default intensity is *not* affected by the change of measure, $\lambda_Q = \lambda_{P_k}$.

In the Libor market-model setup, the primitive quantities are the simply compounded forward rates $F_k(t)$. In terms of these rates, the $\alpha_k(t)$ are recursively related to each other through

$$\alpha_{k+1}(t) = \alpha_k(t) + \frac{\delta_k F_k(t)}{1+\delta_k F_k(t)}\sigma_k^F(t) \tag{31}$$

(see e.g. Jamshidian (1997) or Brace, Gatarek, Musiela (1997)). Thus, once the dynamics are given in one forward measure, the change to a different forward measure can be done if only the forward rates and their volatilities are known. The change from P_k to P_{k+1} is straightforward

$$\mathbf{E}^{P_k}[\,X\,] = \frac{1}{1+\delta_k F_k(0)}\,\mathbf{E}^{P_{k+1}}[\,(1+\delta_k F_k(T_k))X\,]. \tag{32}$$

4.6. **Default Probabilities under P_k:** Under the P_k-forward measure, D_k is the probability of survival until T_k:

$$D_k(0) = \frac{\overline{B}_k(0)}{B_k(0)} = \frac{1}{B_k(0)}\,\mathbf{E}^Q[\,\beta(T_k)I(T_k)\,] = \mathbf{E}^{P_k}[\,I(T_k)\,] = P_k[\tau > T_k].$$

In general

$$I(t)D_k(t) = P_k[\tau > T_k \mid \mathcal{F}_t], \tag{33}$$

which also proves that $I(t)D_k(t)$ is a P_k-martingale.

At one tenor date T_k, the P_{k+1} default probability until the next tenor date T_{k+1} is

$$\mathbf{E}^{P_k}\left[\,\mathbf{1}_{\{\tau\le T_{k+1}\}}\,\middle|\,\mathcal{F}_{T_k}\right] = (1-I(T_k)D_{k+1}(T_k)) = \frac{\delta_k I(T_k)H_k(T_k)}{1+\delta_k H_k(T_k)} = \frac{\delta_k I(T_k)S_k(T_k)}{1+\delta_k \overline{F}_k(T_k)}. \tag{34}$$

Thus, the default probability per time is the credit spread discounted with the defaultable forward rate. For small time steps ($\delta_k \to 0$) the default probability divided by the time interval δ_k converges to $H_k(T_k)$:

$$\frac{1}{\delta_k}(1-D_{k+1}(T_k)) = D_{k+1}(T_k)H_k(T_k) \to H_k(T_k) \qquad \text{as } \delta_k \to 0.$$

The default probability for the next infinitesimal small time step is known as the *default intensity*. Therefore, H_k was called the *discrete-tenor default intensity* in section 2.

4.7. **The T_k-Survival Measure $\overline{P}_k$:**

4.7.1. *Definition:* In the same way that the T_k-forward measure P_k is used to price default-free payoffs at T_k, the T_k-*survival measure* $\overline{P}_k$ is used to price *defaultable* payoffs at T_k. Assume, the payoff in equation (25) is defaultable, i.e. it is only paid if the obligor is still alive at T_k. Then it can be written as $XI(T_k)$ and equation (25) becomes

$$p' = \frac{p}{\overline{B}_k(0)} = \mathbf{E}^Q\left[\,\frac{\beta(T_k)I(T_k)\overline{B}_k(T_k)}{\overline{B}_k(0)}\,X\,\right] =: \mathbf{E}^{\overline{P}_k}[\,X\,], \tag{35}$$

where we used that $\overline{B}_k(T_k) = 1$. The Radon-Nikodym density process for the change from Q to $\overline{P}$ is

$$\overline{L}_k(t) := \frac{\beta(t)I(t)\overline{B}_k(t)}{\overline{B}_k(0)} =: \left.\frac{d\overline{P}_k}{dQ}\right|_{\mathcal{F}_t}. \tag{36}$$

This process is a nonnegative Q-martingale with initial value one, but it is not strictly positive: $\overline{L}_k(t)$ jumps to zero at default ($I(\tau)=0$). This means, that the measure $\overline{P}_k$ attaches a weight of zero to all events that involve default before T_k:

$$\overline{P}_k(\tau \le T_k) = \mathbf{E}^Q[\,\overline{L}_k(T_k)\mathbf{1}_{\{\tau\le T_k\}}\,] = 0. \tag{37}$$

Because it only attaches probability to survival events, this measure is termed the T_k-*survival measure*. The survival measure $\overline{P}_k$ is not equivalent to Q any more, but it is absolutely continuous w.r.t. Q, so Girsanov's theorem can still be applied.

There is another intuitive interpretation of the T_k survival measure: It is the measure that is reached when the T_k-forward measure is *conditioned on survival* until T_k. Consider an event $A \in \mathcal{F}_{T_k}$ and calculate its expectation conditional on $\tau > T_k$:

$$\begin{aligned}
(38)\quad \mathbf{E}^{P_k}\left[\, \mathbf{1}_{\{A\}} \,\middle|\, \tau > T_k \,\right] &= \frac{\mathbf{E}^{P_k}\left[\, \mathbf{1}_{\{A\}} I(T_k) \,\right]}{\mathbf{E}^{P_k}\left[\, I(T_k) \,\right]} = \frac{\mathbf{E}^{P_k}\left[\, \mathbf{1}_{\{A\}} I(T_k) \,\right]}{D_k(0)} \\
&= \mathbf{E}^{Q}\left[\mathbf{1}_{\{A\}} I(T_k) \frac{1}{D_k(0)} \frac{\beta(T_k)}{B_k(0)} \right] = \mathbf{E}^{Q}\left[\mathbf{1}_{\{A\}} \frac{\beta(T_k) I(T_k)}{\overline{B}_k(0)} \right] = \mathbf{E}^{\overline{P}_k}\left[\, \mathbf{1}_{\{A\}} \,\right].
\end{aligned}$$

The P_k-probability of A conditional on survival equals the probability of A under the $\overline{P}_k$-survival measure. This relationship will provide the basis of the simulation-implementation later on.

4.7.2. *Change of Drift:* Analysing $\overline{L}_k$ yields the components of the change of measure in theorem 1. The intensity factor is zero: $\phi(t) = 0$ (which again shows that under $\overline{P}$ defaults have zero probability). The change of drift for the Brownian motions is

$$d\overline{W}_k(t) := dW_Q(t) + \overline{\alpha}_k(t)dt, \tag{39}$$

where by (18) the drift correction $\overline{\alpha}_k(t)$ is minus the volatility of $\overline{B}_k(t)$. Again, the $\overline{\alpha}_k(t)$ are recursively related through

$$\overline{\alpha}_{k+1}(t) = \overline{\alpha}_k(t) + \frac{\delta_k \overline{F}_k(t) \sigma_k^{\overline{F}}(t)}{1 + \delta_k \overline{F}_k(t)} \tag{40}$$

As the defaultable forward rates $\overline{F}$ are not the primitives of our model, we would like to find a representation of $\overline{\alpha}_k(t)$ in F and H. Define $\alpha_k^D(t)$ as minus the volatility of the D_k

$$\frac{dD_k(t)}{D_k(t-)} = \ldots dt - \alpha_k^D(t) dW. \tag{41}$$

(This definition is independent from the measure under which dW is a Brownian motion.) Because of $\overline{B}_k(t) = B_k(t) D_k(t)$, the $\overline{B}_k$ volatility in (39) can now be written as follows:

$$\overline{\alpha}_k(t) = \alpha_k(t) + \alpha_k^D(t). \tag{42}$$

There is again a recursion formula for the $\alpha_k^D(t)$

$$\alpha_{k+1}^D(t) = \alpha_k^D(t) + \frac{\delta_k H_k(t) \sigma_k^H(t)}{1 + \delta_k H_k(t)}. \tag{43}$$

The following formula is similar to equation (32), it describes the change from $\overline{P}_k$ to $\overline{P}_{k+1}$:

$$\mathbf{E}^{\overline{P}_k}\left[\, X \,\right] = \frac{1}{1 + \delta_k \overline{F}_k(0)} \mathbf{E}^{\overline{P}_{k+1}}\left[\, (1 + \delta_k \overline{F}_k(T_k)) X \,\right]. \tag{44}$$

4.7.3. *Change of Measure from Forward- to Survival Measure.* By (30) and (39) the Brownian motions under the T_k forward measure P_k and the T_k survival measure $\overline{P}_k$ differ by

$$d\overline{W}_k(t) = dW_k(t) + \alpha_k^D(t)dt. \tag{45}$$

Thus we can change between P_k and $\overline{P}_k$ directly, without having to go through the spot martingale measure Q. The density for this change of measure is

$$\mathbf{E}^{\overline{P}_k}\left[\, X \,\right] = \frac{1}{D_k(0)} \mathbf{E}^{P_k}\left[\, I(t) D_k(t) X \,\right] = \frac{B_k(0)}{\overline{B}_k(0)} \mathbf{E}^{P_k}\left[I(t) \frac{\overline{B}_k(t)}{B_k(t)} X \right]. \tag{46}$$

The drift change (45) to the dynamics of the Brownian motions can also be achieved using a different probability measure $\overline{P}'_k$ with the following density $L^D_k(t)$

$$\frac{d\overline{P}'_k}{dP_k}(t) = L^D_k(t) = e^{-\int_0^t \lambda(s)ds} \frac{D_k(t)}{D_k(0)} \tag{47}$$

Using equations (16) and (17) it is easily shown that $L^D_k(t)$ is a P_k-martingale with initial value one and that $L^D_k(t)$ satisfies the stochastic differential equation

$$dL^D_k(t) = -\alpha^D_k(t)dW_k(t).$$

This new measure $\overline{P}'_k$ has the same effect on the Brownian motions as $\overline{P}_k$. For random variables X that are measurable with respect to the filtration generated by the Brownian motions[6], the expected values under both of these measures will coincide, and $\overline{P}'_k$ may be used instead of $\overline{P}_k$.

4.8. **Further Probability Measures.** Following the introduction of the survival measure, the definition and analysis of further, survival-based probability measures is straightforward. To save space we refrain from introducing a defaultable analog to the *spot Libor measure* Q', which is useful in many numerical implementation algorithms. Later on we will need a defaultable *swap measure* to price options on default swaps. The default-free versions of these probability measures were introduced by Jamshidian (1997).

5. Drift Restrictions for the Discrete Tenor Case

Using the results of the previous section we can now derive the dynamics of the defaultable and default-free forward rates under the new probability measures. We only give the dynamics of each process under one of the measures, the dynamics under the other measures follow from the respective change of drift formulae.

5.1. **Default-Free Forward Rates:** B_k/B_{k+1} is a Martingale under the T_{k+1}-*forward measure*. Hence

$$F_k = \frac{1}{\delta_k}(\frac{B_k}{B_{k+1}} - 1) \tag{48}$$

is also a martingale under the T_{k+1}-*forward measure* and its dynamics are (according to the lognormal assumption)

$$dF_k(t) = F_k(t)\sigma^F_k dW_{k+1}(t). \tag{49}$$

Under the T_{k+1} survival measure, the dynamics of F_k are

$$dF_k(t) = F_k(t)\sigma^F_k(d\overline{W}_{k+1}(t) - \alpha^D_{k+1}(t)dt). \tag{50}$$

5.2. **Defaultable Forward Rates:** For the defaultable forward rates we use that $\overline{B}_k/\overline{B}_{k+1}$ is a martingale[7] under the T_{k+1}-*survival measure*, therefore

$$\overline{F}_k = \frac{1}{\delta_k}(\frac{\overline{B}_k}{\overline{B}_{k+1}} - 1) \tag{51}$$

is a martingale under the T_{k+1}-*survival measure*. Again, its dynamics are

$$d\overline{F}_k(t) = \overline{F}_k(t)\sigma^{\overline{F}}_k d\overline{W}_{k+1}(t). \tag{52}$$

[6]Intuitively speaking, if X does not contain any direct reference to default and survival events $I(t)$ it satisfies this condition. I.e. $\tau, N(t)$ or $I(t)$ do not occur in X, but S or $\overline{B}$ may occur in X.
[7]Strictly speaking $\overline{B}_k/\overline{B}_{k+1}$ is only defined up to default. After default we consider the process stopped.

Calculating the dynamics of $\overline{F}_k$ under default-free forward measures does not make much sense, as the defaultable forward rates are only meaningful in survival events.

5.3. **Forward Spreads:** The dynamics of the forward spreads under the T_{k+1} survival measure are

$$dS_k = F_k \sigma_k^F \alpha_{k+1}^D dt + S_k \sigma_k^S d\overline{W}_{k+1}. \tag{53}$$

5.4. **Forward Intensities:** The forward discrete default intensities H_k have the following dynamics under $\overline{P}_{k+1}$:

$$dH_k = \frac{F_k \sigma_k^F}{1+\delta_k F_k} \left((1+\delta_k H_k)\alpha_{k+1}^D - \delta_k H_k \sigma_k^H \right) dt + H_k \sigma_k^H d\overline{W}_{k+1}. \tag{54}$$

5.5. **Independence:** If the default-free bond prices $B(t)$ and the time of default τ are independent under Q, then it is easily seen that also the default-free forward rates F and the discrete default intensities H are independent under Q and under all other pricing measures. In this case, many of the relationships above simplify significantly. Independence in this sense means zero covariation between forward rates and default intensities:

$$\sigma_k^F \sigma_l^H = 0 \qquad \forall\, k, l \leq K \tag{55}$$

In particular independence is given when the default-free forward rates F_k and the discrete-time default intensities H_k are driven by different components of the vector Brownian motion W_Q[8]. Note that by equation (11) credit spreads S_k and default-free forward rates F_k have a nonzero covariation even if H_k and F_k are independent.

Under independence $\alpha^D \sigma^F = 0$ holds, hence by equations (49), (53) and (54), the default-free forward rate F_k, the credit spreads S_k and the discrete default intensities H_k are martingales under the T_{k+1} survival measure

$$\frac{dF_k}{F_k} = \sigma_k^F d\overline{W}_{k+1} \qquad \frac{dH_k}{H_k} = \sigma_k^H d\overline{W}_{k+1} \tag{56}$$

$$\frac{d\overline{F}_k}{\overline{F}_k} = \sigma_k^{\overline{F}} d\overline{W}_{k+1} \qquad \frac{dS_k}{S_k} = \sigma_k^S d\overline{W}_{k+1}. \tag{57}$$

Even if independence does not hold, the drift of of the default intensities H_k and of the credit spreads S_k is of a small order of magnitude: risk-free interest rates times the covariation between credit spreads and the k-th risk-free forward rate. A good strategy for model calibration under correlation is to first calibrate the model to the closed-form solutions that are reached under the assumption of independence, and then to iteratively adjust the parameters to the case of correlation, which should be not too far away. For pricing purposes, closed-form solutions under independence can be used as control variates to increase the accuracy of simulations.

6. Positive Recovery of Par

6.1. **The Recovery Model:** Most recovery mechanisms in intensity-based models of default risk prescribe the recovery on defaultable zero coupon bonds. Then all defaultable claims (in particular all defaultable coupon bonds) are decomposed into defaultable zero coupon bonds, and their recovery payoff is determined by summing up the recovery values of the individual zero coupon bonds.

[8]By an orthogonal transformation of W this structure can always be achieved if (55) holds.

Representative of this approach is the *equivalent recovery*[9] (or *recovery of treasury*) model, where one defaultable bond $\overline{B}(t,T)$ has a recovery of c equivalent *default-free* bonds $B(t,T)$ at the time of default. Another popular zero-bond recovery model is the *fractional recovery* model (or *recovery of market value* model) used in Duffie / Singleton (1997; 1999) and the multiple-defaults model by Schönbucher (1998). Here a defaulted bond pays off a fraction q of its *pre-default value.*

Unfortunately, by decomposing coupons and principal of defaultable coupon bonds into the same asset class, these modelling approaches ignore the fundamental difference between principal and coupon claims in real-world default proceedings. The claim of a creditor on the defaulted debtor's assets is only determined by the outstanding principal and accrued interest payments of the defaulted loan or bond, any future coupon payments do *not* enter the consideration. The recovery rate gives the fraction of this claim that is paid off after a default, and this payoff is measured in cash and not in terms of default-free bonds or pre-default market value.

This distinction becomes important when it comes to the calibration of the model to the prices of traded coupon bonds or default swaps. Here, using a structurally incorrect recovery model can yield misleading results. This can happen for defaultable bonds with high coupons, for defaultable bonds that trade far from their par value[10], or when defaultable bonds of similar maturity but different coupon size are used.

Instead of the zero-bond approach we propose to view the recovery value of a defaultable security as a *default-contingent payoff*, a (possibly random) payoff at default. In particular, defaultable coupon bonds and loans should be decomposed in two distinct classes of elementary claims: zero-recovery claims $\overline{B}(t,T)$, and positive recovery claims $\overline{B}^p(t,T)$ which have a recovery of π times their face value in cash at default.

Formally, the recovery of par model in the discrete-tenor setup is as follows:

Assumption 4 (Recovery of Par)**.** *If a defaultable coupon bond defaults in the time interval* $]T_k, T_{k+1}]$ *then its recovery is composed of the recovery rate* π *times the sum of the notional of the bond (here normalised to* 1*) and the accrued interest over* $]T_k, T_{k+1}]$*. The accrued interest can be*

(a) c*, a constant in the case of a fixed-coupon bond with coupon* c*,*
recovery is $\pi(1+c)$

(b) F_k *in the case of a floating rate bond*[11]*,*
recovery is $\pi(1+\delta_k F_k(T_k))$

The recovery payoffs occur in cash at $T_{\kappa(\tau)}$ *i.e. at the next tenor date* T_{k+1} *if a default was in* $]T_k, T_{k+1}]$*.*
We denote with $e_k(t)$ *the time-*t *value of receiving* 1 *at* T_{k+1} *if and only if a default has occurred in the preceding time interval* $]T_k, T_{k+1}]$*.*

Modelling the recovery of a defaultable bond as a fraction of its par value was first suggested by Duffie (1998), who used this model in an affine term-structure setup but did not model recovery of accrued interest. For a comparison of traditional recovery models see e.g. Schönbucher (1999).

[9] This model is used in Jarrow / Turnbull (1995), Jarrow / Lando / Turnbull (1997), Lando (1998) and many others.

[10] In particular bonds of issuers that are close to default tend to trade around their expected recovery value irrespective of their maturity or coupon amount.

[11] Defaultable floating rate notes usually pay Libor F plus a constant spread x. In this case recovery is $\pi(1+x+\delta_k F_k(T_k))$

We assume that all claims of the same seniority have the same recovery rate π at the time of default. The recovery rate π can be stochastic in $[0, 1]$ but its distribution is assumed to be independent of the default-free interest rates, and time-invariant. For pricing purposes it is then sufficient to work with the expected recovery rate which we will do from now on.

Using the notation of assumption 4 the price of a claim with maturity T_N, no coupons, notional 1 and positive recovery can be represented as follows:

$$\overline{B}^p(0) = \overline{B}_N(0) + \pi \sum_{i=1}^{N} e_{i-1}(0). \tag{58}$$

The price of a defaultable fixed coupon bond with N fixed coupons of c at $T_i,\ i = 1, \dots N$ and a notional of 1 is

$$\overline{B}_N(0) + \sum_{i=1}^{N} \left(c\overline{B}_i(0) + (1 + c)\pi e_{i-1}(0) \right). \tag{59}$$

At a default there is positive recovery π on the notional of 1 and on the next outstanding coupon c. Positive recovery has the effect of enhancing the coupon. For *floating coupon* debt the decomposition is slightly different but the fundamental idea remains the same: The coupons have zero recovery, and the recovery depends only on the notional and the coupon that was outstanding at the time of default. By adjusting the numbers of the recovery claims e_i other, non-standard exposure profiles like amortising debt or can be represented in this framework.

6.2. **Discrete-Tenor Defaults.** We restricted recovery payments to the next tenor date $T_{\kappa(\tau)}$ following default. This is not a strong restriction for a number of reasons: First, most defaults do indeed occur on payment dates — at least, they become publicly *apparent* when a payment has to be made and cannot be made. Even if strictly speaking the default had happened between two payment dates, many debtors tend to hang on and hope for resurrection until the next payment is due. Some credit derivatives even define a default event as the event of a missed payment on one or several defaultable bonds. A missed payment can obviously only occur on a payment date. Second, if the tenor dates are spaced reasonably closely (i.e. quarterly or closer) the error will be very small. Third, given the large uncertainty that prevails about recovery rates, the error committed by restricting defaults to the tenor dates is of second order importance.

Finally, the effect of this assumption is a *postponement* of the default from somewhere in $]T_k, T_{k+1}]$ to T_{k+1}. There is an approximate correction to this error by adjusting the recovery rate upwards as follows: We assume that continuously compounded short rate r and default intensity λ are constant over $[T_k, T_{k+1}]$. Then, given $H = H_K(T_k)$, $F := F_k(T_k)$ and $\delta := T_{k+1} - T_k$, the default-intensity is $\lambda := \frac{1}{\delta}\ln(1 + \delta H)$ and the continuously compounded short rate is $r := \frac{1}{\delta}\ln(1 + \delta F)$. Given a default happens in $]T_k, T_{k+1}]$, the T_{k+1}-value of π received at default and invested at r until T_{k+1} is

$$\pi' := \frac{\lambda}{\lambda + r} \frac{F(1 + \delta H)}{H(1 + \delta F)} \pi \geq \pi. \tag{60}$$

Thus, as a correction we can use π' instead of π and work with recovery payoffs at the next tenor date T_{k+1}. Typically, this adjustment amounts to a factor π'/π of 1.005 to 1.02, it increases with high interest rates and long time steps δ_k, and is rather insensitive to changes in the default intensity λ. A similar adjustment can be constructed for the alternative case when only accrued interest until τ is taken into consideration for the recovery, i.e. for a default at $\tau \in]T_k, T_{k+1}]$ the recovery is $\pi(1 + (\tau - T_k)c)$ where c is the coupon. In this case the adjustment will be even smaller.

6.3. **Valuation of the Recovery Payoffs under Independence.** The disadvantage of this 'recovery of par' modelling approach is that we now have to do some work to reach even the price of a simple defaultable coupon bond. Compared to this, the derivation of the prices of fixed-coupon bonds is much easier in the equivalent recovery model or the fractional recovery model. Nevertheless, this is not lost labour for two reasons: This analysis is also a necessary ingredient to price credit default swaps, and for defaultable *floating* coupon debt there are no simple formulae in the alternative recovery models either.

The following two propositions give the prices of the recovery payoffs for the two most important cases: fixed payoff and fixed plus floating payoff. The time argument T_0 was suppressed to simplify the notation. The proofs can be found in the appendix.

Proposition 2 (Recovery Payoffs under Independence)**.**
Under independence of H_k and F_l, $\forall\, k, l$ we have:

(i) Fixed Payment at Default:
The value of a payment of 1 at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is

$$e_k := \overline{B}_{k+1}\delta_k H_k \tag{61}$$

(ii) Floating Payment at Default:
The value of a payment of $1+\delta_k F_k(T_k)$ at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is

$$\overline{B}_{k+1}\delta_k S_k \tag{62}$$

(iii) Floating Coupon in Survival:
The value of a payment of $F_k(T_k)$ at T_{k+1} if no default occurs until T_{k+1} is

$$\overline{B}_{k+1}F_k. \tag{63}$$

If independence between defaults and default-free interest rates does not hold we need to resort to approximative solutions for the pricing equations. The error of these approximations should be very low for reasonable parameter values. It will certainly be an order of magnitude lower than the approximative correlation correction itself which in turn is of the order of a few basis points. The following proposition gives these approximations.

Proposition 3 (Recovery Payoffs under Correlation)**.**
If H_k and F_l are not independent then:

(i) The approximate volatility of L_k^D over $[T_0, T_m]$ is

$$A_{k,m}^D := \sum_{l=0}^{k-1} \frac{\delta_l H_l \sigma_l^H}{1+\delta_l H_l} T_{l\wedge m}. \tag{64}$$

(ii) Fixed Payment at Default:
The value of a payment of 1 at T_{k+1} if a default happens in $]T_k, T_{k+1}]$ is

$$\begin{aligned} e_k &= \overline{B}_{k+1}\delta_k\,\mathbf{E}^{\overline{P}_{k+1}}\left[\, H_k \,\right] \\ &= \delta_k H_k \overline{B}_{k+1} + \overline{B}_k\, \text{cov}^{P_k}\left(L_k^D(T_k)\,,\ \frac{1}{1+\delta_k F_k(T_k)}\right) \\ &\approx \delta_k H_k \overline{B}_{k+1} - \overline{B}_k \frac{\delta_k F_k}{1+\delta_k F_k}\left(\exp\left\{ \frac{\sigma_k^F A_{k,k}^D}{1-\delta_k F_k}\right\} - 1\right), \end{aligned} \tag{65}$$

(iii) Floating Payment at Default:
The value of $1+\delta_k F_k(T_k)$ at T_{k+1} if a default occurs in $]T_k, T_{k+1}]$ is

$$\begin{aligned} \overline{B}_{k+1}\delta_k\,\mathbf{E}^{\overline{P}_{k+1}}\left[\, S_k \,\right] &= \overline{B}_{k+1}\delta_k S_k - \delta_k \overline{B}_{k+1}\, \text{cov}^{P_{k+1}}\left(L_{k+1}^D(T_k)\,,\ F_k(T_k)\right) \\ &\approx \overline{B}_{k+1}\delta_k S_k - \overline{B}_{k+1}\delta_k F_k \left(\exp\left\{ A_{k+1,k}^D \sigma_k^F \right\} - 1\right). \end{aligned}$$

(iv) Floating Coupon in Survival:
The value of $F_k(T_k)$ at T_{k+1} if no default occurs until T_{k+1} is

$$\overline{B}_{k+1}\,\mathbf{E}^{\overline{P}_{k+1}}\,[\;F_k\;] \approx \overline{B}_{k+1}F_k e^{A^D_{k+1,k}\sigma^F_k} \tag{66}$$

7. Basic Credit Derivatives

In naming the counterparties for credit derivatives we will use the convention that counterparty **A** will be the insured counterparty (i.e. the counterparty that receives a payoff if a default happens or the party that is long the credit derivative), and counterparty **B** will be the insurer (who has to pay in default). Party **C** will be the reference credit.

7.1. Default Swap.

7.1.1. *Description:* In a *default swap* (also known as *credit swap*) **B** agrees to pay the default payment to **A** *if a default has happened.* If there is no default of the reference security until the maturity of the default swap, counterparty **B** pays nothing.

A pays a fee for the default protection. The fee can be either a lump-sum fee up front (default put) or – more commonly – a regular fee at intervals until default or maturity (default swap).

Different types of default swaps usually only differ in the specification of the default payment. Here we only consider the standard default swap without going into the problems of the fine print of the specification of the default payment.

- (fee stream) **A** pays s at T_i until T_N or default.
- (default payment) **B** pays the difference between the post-default price of the reference asset (usually a bond issued by **C**) and its par value at default.

7.1.2. *The Fee.* The value of the fee stream can be directly determined as

$$s\sum_{k=1}^{N}\overline{B}_k(0) \tag{67}$$

This valuation is valid for all fee streams of credit derivatives that pay fees until default.

7.1.3. *The Default Payment.* The typical reference asset is a defaultable coupon bond with fixed coupon c. In this case the value of the reference asset in default is $\pi(1+c)$, so the default payment is $1-\pi(1+c)$ at default. The value of this contingent payoff is

$$D^{\text{Def Put}} = (1-\pi(1+c))\sum_{k=0}^{N-1} e_k. \tag{68}$$

7.1.4. *The Default Swap Rate.* The default swap rate is the level s of the fee payment that makes the default swap fairly priced.

$$\overline{s} = (1 - \pi(1+c))\frac{\sum_{k=0}^{N-1} e_k}{\sum_{k=0}^{N-1} \overline{B}_{k+1}} \tag{69}$$

$$\overline{s} = (1 - \pi(1+c))\sum_{k=0}^{N-1} \overline{w}_k \delta_k H_k \qquad \text{(for independence), and} \tag{70}$$

$$\overline{s} \approx (1 - \pi(1+c))\sum_{k=0}^{N-1} \overline{w}_k \left(\delta_k H_k + (1+\delta_k H_k)(e^{(1-F_k(0))A^D_{k,k}\sigma^F_k} - 1)\right) \tag{71}$$

under correlation, where $\overline{w}_k := \overline{B}_{k+1}/\sum_{j=0}^{N-1}\overline{B}_{j+1}$. Thus, under independence the default swap rate $\overline{s}$ is a weighted average of the H_k with weights $\overline{w}_k$. There is an equivalent representation of a plain vanilla fixed-for-floating interest rate swap rate as a weighted average of default-free forward rates

$$s = \sum_{k=0}^{N-1} w_k \delta_k F_k$$

with weights $w_k := B_{k+1}/\sum_{j=0}^{N-1} B_{j+1}$. This property will be useful later on in the pricing of options on default swaps.

7.2. **Asset Swap Packages.**

7.2.1. *Description:* An *asset swap package* is a combination of a defaultable fixed coupon bond (the asset) with a fixed-for-floating interest rate swap whose fixed leg is chosen such that the value of the whole package is the par value of the defaultable bond.

The payoffs of the asset swap package are:
B sells to **A** for 1 (the nominal value of the **C**-bond):

- a fixed coupon bond issued by **C** with coupon c payable at coupon dates $t_i,\ i = 1, \ldots, N$,
- a fixed for floating swap (as below).

The payments of the swap: At each coupon date $t_i,\ i \leq N$ of the bond

- **A** pays to **B**: c, the amount of the fixed coupon of the bond,
- **B** pays to **A**: Libor + a.

a is called the *asset swap spread* and is adjusted to ensure that the asset swap package has initially the value of 1.

The asset swap is not a credit derivative in the strict sense, because the swap is unaffected by any credit events. Its main purpose is to transform the payoff streams of different defaultable bonds into the same form: *Libor + asset swap spread* (given that no default occurs). **A** still bears the full default risk and if a default should happen, the swap would still have to be serviced.

7.2.2. *Pricing:* To ensure that the value of the asset swap package (asset swap plus bond) to **A** is at par at time $t = 0$ we require:

$$C + (s + a - c)A = 1 \tag{72}$$

where C is the initial price of the bond, s is the fixed-for-floating swap rate for the same maturity and payment dates T_i, and A is the value of an annuity paying 1 at all times $T_i,\ i = 1, \ldots, N$.

All these quantities can be readily observed in the market at time $t = 0$. To ensure that the value of the asset-swap package is one, the asset swap rate must be chosen as

$$a = \frac{1}{A}(1 - C) + c - s.$$

Note that the asset swap rate would explode at a default of **C**, because then $(1 - C(t))$ would change from being very small to a large number. Using the definition of the fixed-for-floating swap rate: $sA = 1 - B_N$ this can be rearranged to yield:

$$Aa = \underbrace{B_N + cA}_{\text{def. free bond}} - \underbrace{C}_{\text{defaultable bond}}, \tag{73}$$

the asset swap rate a is the price difference between the defaultable bond C and an equivalent default free coupon bond (with the same coupon c, it has the price $B_N + cA$) in the numeraire asset A.

Asset swap packages are very popular and liquid instruments in the defaultable bonds market, sometimes their market is even more liquid than the market for the underlying defaultable bond alone. They also serve frequently as underlying assets for options on asset swaps, so called *asset swaptions*. An asset swaption gives **A** the right to enter an asset-swap package at some future date T at a pre-determined asset swap spread a.

8. Options on Default Swaps

Options are frequently embedded in defaultable securities and credit derivatives. Many loans and bonds feature options for the obligor: prepayment options (which amount to a call option on the bond at face value) or extension options (which are equivalent to a put of the bond to the creditors). Credit default swaps also often have extension- or callability options which are basically call or put options on default swaps. Many of these options can be reduced to options on credit default swaps, for which closed-form and semi-closed form solutions are given in this section.

The semi-closed form approximation is based upon the weighted-average representation of the default-swap rate in equation (70) and (71). Similar approximations for prices of options on interest-rate swaps in default-free Libor market models were given by Brace / Gatarek / Musiela (1997), Andersen and Andreasen (1998) and Zühlsdorff (1999). Here, we only consider the case of independence between H and F, and to remain in the Libor-modelling framework we need to make some approximations regarding the dynamics of the forward default swap rates. These simplifications are *not* central to the derivation of the pricing formulae (85) and (87).

We will also show how these formulae can be derived without needing approximations if the volatility of the default-swap rate is known.

8.1. **Description and Payoffs:** A call on a default swap (*default swaption*) gives the buyer **A** the right to enter a default swap at at pre-determined spread $\overline{s}^*$ at time T_K.

There are two alternatives for the treatment of an early default before the exercise time T_K of the option. Either the option is knocked out and its value drops to zero, or the option remains valid. The former case will be treated below, the pricing problem in the latter case can be reduced to the valuation of an option that is knocked out at default as follows:

If the default swaption is still alife at T_K even though a default has happened before that, **A** will certainly exercise the default swaption at T_K, enter the default swap and immediately receive the payoff $(1 - \pi)$. The value of this default-protection component of the default swaption is

$$(1 - \pi)(B_K - \overline{B}_K). \tag{74}$$

The full value of the default swaption consists thus of the sum of value of the value of the default payment given above, and the value of the right to enter the default swap if no default has happened before T_K, i.e. the value of the default payment and a default swaption which is knocked out at default. Therefore we will concentrate on the pricing of this option.

As mentioned before, options on default swaps frequently appear as components of more complicated credit derivatives. A typical case is a standard default swap to which an *option to extend* is added. If the underlying default swap runs from T_0 to T_K and **A** can choose at T_K to extended its maturity until T_N, **A** effectively holds a plain default swap from T_0 to T_K and a call option with exercise time T_K on a default swap from T_K to T_N. If a default has already triggered the default swap before T_K then it obviously cannot be extended any more, so the option is knocked out at a default of the reference credit. In many cases the default swap rate at which the default swap can be extended is higher than the rate for the first protection period T_0 to T_K, in this case the structure is known as a *callable stepup default swap.* Similarly, early *cancellation rights* constitute put options on default swaps.

To price the default swaption we first have to derive the value of its payoff at maturity of the option. At time $t \leq T_K < T_N$, a default swap with maturity T_N that is entered at time T_K at a default swap rate of $\overline{s}^*$ and that is knocked out at defaults before T_K has the value

(75) $$(\overline{s}(t, T_K, T_N) - \overline{s}^*) \sum_{j=K}^{N-1} \overline{B}_j(t).$$

Here $\overline{s}(t, T_K, T_N)$ is the forward default swap rate. The forward default swap rate is the market rate at time t of a default swap for the future protection period $[T_K, T_N]$. According to equation (70) the forward default swap rate is given by

$$\overline{s}(t, T_K, T_N) = \sum_{k=K}^{N-1} \overline{w}_k \delta_k H_k,$$

where now $\overline{w}_k = \overline{B}_{k+1} / \sum_{j=K}^{N-1} \overline{B}_{j+1}$.[12]

If no default has occurred before T_K the default swaption will only be exercised if it is in-the-money at T_K, i.e. if $\overline{s}^* < \overline{s}(T_K, T_K, T_N)$. Then the payoff function of the default swaption is

(76) $$(\overline{s}(T_K) - \overline{s}^*)^+ \sum_{k=K}^{N-1} \overline{B}_k.$$

8.2. **Dynamics of the Forward Default Swap Rate.** To price an option on the default swap we need to know the dynamics of the default swap rate, and most importantly its volatility (the drift will follow from a no-arbitrage argument). Let $H^T := (H_K, H_{K+1}, \ldots, H_{N-1})$ denote the vector of forward spreads, and $\overline{w}^T := (\overline{w}_K, \overline{w}_{K+1}, \ldots, \overline{w}_{N-1})$ the vector of the weights of these rates in the forward default swap rate. Without loss of generality we set the tenor distances $\delta_k = 1$ equal to one (for general distances the following orthogonality argument would become only slightly more complicated), and we ignore the constant introduced by the positive recovery and the coupon. We also write $\overline{s}$ for the *forward* default swap rate. Note, that

(77) $$\overline{s} = H^T \overline{w} \qquad \text{and} \qquad \sum_{k=K}^{N-1} \overline{w}_k = 1 = \mathbf{1}^T \overline{w},$$

[12] This holds under independence, in general the forward default swap rate will be defined similar to (69) and (71).

where $\mathbf{1}^T = (1, 1, \ldots, 1)$ is a vector composed of ones. Then (given survival) the dynamics of $\overline{s}$ are given by

$$d\overline{s} = \overline{w}^T dH + H^T d\overline{w} + d < \overline{w}, H >, \tag{78}$$

and the dynamics of H are

$$dH_i = \ldots dt + H_i \sigma_i^H dW, \tag{79}$$

where σ_i^H are d-dimensional row vectors. For now we are only concerned with the volatility of $\overline{s}$, so we do not yet specify the measure under which dW is a Brownian motion.

In the next step we make two approximations[13]:

Assumption 5. *(i)* "The effect of the changes in the weights are negligible."

$$H^T d\overline{w} \approx 0$$

(ii) "The H_i are only driven by parallel shifts."

$$H_i \sigma_i^H dW = H_i \sum_{j=1}^{d} \sigma_{ij}^H dW_j \approx H_i \sigma_0^H dW_0$$

Where W_0 is a Brownian motion that is reached by a suitable rotation of the other $W_1, \ldots, W_d$, such that the first column of the volatility matrix σ_{ij}^H equals $\sigma_0^H \mathbf{1}$.

Note that by equation (77), we have $\mathbf{1}^T \overline{w} = 0$, so the first approximation is exact when the term structure of intensities H is *flat.* The quality of this approximation therefore depends on

(a) the deviation of H from a flat structure (should be small)
(b) the volatility of $\overline{w}$ (should be small).

Both conditions are usually satisfied in practice.

In the second approximation, the other components of the rotated variance-covariance matrix are ignored. The error of this approximation depends on the weight that higher order components have in the dynamics of the term structure of default intensities H. Principal component decompositions of the variance/covariance matrix of interest-rates typically exhibit a strongly dominating first component which is almost flat. The larger such a component is for credit default swaps, the better the approximation will work.

These approximations have been tested for interest-rate swap rates and have proven to be highly precise. This gives us reason to expect similarly good performance in the default-risk setting. After these approximations the resulting volatility of the forward default swap rate is *constant*. From (78) follows

$$d\overline{s} = \ldots dt + \sum_{i=K}^{N-1} H_i \overline{w}_i \sigma_0^H dW_0 = \ldots dt + \overline{s} \sigma_0^H dW_0, \tag{80}$$

where the drift of the default swap rate is left unspecified.

Instead of going through the approximations above, one could also *directly* specify the dynamics (80), i.e. a *constant* volatility σ_0^H for the forward default swap rate. This amounts to changing from a Libor-based market-model framework to a swap-based market model framework, a common technique introduced by Jamshidian (1997).

[13]The approximation argument in this subsection is based upon Zühlsdorff (1999) and also Andersen and Andreasen (1998).

8.3. **Pricing the Option, the Default Swap Measure.** The key point to note for pricing is that $\overline{s}(t)\sum_{k=K}^{N-1} I(t)\overline{B}_k(t)$ is a traded asset in the market: It is the value of the default-protection component of the forward default swap over $[T_K, T_N]$. Thus, in analogy to the swap-market measure introduced by Jamshidian (1997) and to the introduction of the survival measure before, we can take

$$X(t) := I(t)\sum_{k=K}^{N-1} \overline{B}_k(t) \tag{81}$$

as numeraire asset for a new probability measure $\overline{P}^s$. We call this measure the *default swap measure.*

We do not go through the derivation of the Radon-Nikodym density of this measure with respect to the other martingale measures which is exactly analogous to the derivations in the previous sections. The measure $\overline{P}^s$ is associated with the Brownian motion $\overline{W}^s$, and under this measure prices of (defaultable) traded assets divided by the new numeraire $X(t)$ are martingales. The measure is again a *survival-based* measure, i.e. the probability of a default until T_K is zero under the default swap measure.

Starting from the dynamics in equation (80) we now know that $\overline{s}$ is a martingale under $\overline{P}^s$ because $\overline{s}$ is a price in terms of the associated numeraire asset X. Therefore its dynamics under $\overline{P}^s$ exhibit no drift:

$$d\overline{s} = \overline{s}\sigma_0^H d\overline{W}^s. \tag{82}$$

As mentioned before, we could take the direct specification of the dynamics of $\overline{s}$ under $\overline{P}^s$ in equation (82) as starting point without having to go through the approximations in the previous subsection, and also without having to use independence of H and F.

Using the measure $\overline{P}^s$, we can now directly price the option. Starting from

$$C(0) = \mathbf{E}^Q\left[\beta(T_K)I(T_K)\sum_{k=K}^{N-1}\overline{B}_k(T_K)(\overline{s}(T_K)-\overline{s}^*)^+\right] \tag{83}$$

the change of measure to $\overline{P}^s$ yields

$$= \left(\sum_{k=K}^{N-1}\overline{B}_k(0)\right)\ \mathbf{E}^{\overline{P}^s}\left[\ (\overline{s}(T_K)-\overline{s}^*)^+\ \right]. \tag{84}$$

Evaluating the expectation yields the following proposition:

Proposition 4. *The value of a European Call option to enter at time T_K a default swap with maturity T_N and strike default swap rate $\overline{s}^*$, which is knocked out at defaults before $\overline{s}$ is*

$$C = \left(\sum_{k=K}^{N-1}\overline{B}_k(0)\right)\ \{\overline{s}(0)N(d_1) - \overline{s}^*N(d_2)\}, \tag{85}$$

where d_1 and d_2 are given by

$$d_{1;2} = \frac{\ln(\overline{s}/\overline{s}^*) \pm \frac{1}{2}(\sigma^H)^2 T_K}{\sigma^H\sqrt{T_K}}. \tag{86}$$

An European Put option to enter as protection seller *the same default swap at time T_K has the price*

$$P = \left(\sum_{k=K}^{N-1}\overline{B}_k(0)\right)\ \{\overline{s}^*N(-d_2) - \overline{s}(0)N(-d_1)\}. \tag{87}$$

The recovery rate does not enter this pricing formula explicitly. The reason is, that the value of the payoff of the default swap does not directly involve the recovery rate: It is the value of the swap at the strike $\overline{s}^*$ minus the value of an offsetting default swap at the market rate $\overline{s}$. Thus, only the difference between the fee streams is paid out, until a default happens. Before maturity, the default swap can be knocked out at default, but the recovery rate does not enter directly either. Of course, the recovery rate is still present, but in indirect form: Depending on the assumed recovery rate, the calibrated values of the zero-recovery bonds $\overline{B}_k$ can vary much.

9. Numerical Implementation

Because of their great importance in practice there is a quickly growing literature on the implementation and calibration of Libor and Swap market models, and we cannot mention all contributions in this area. The question of calibration is addressed by Rebonato (1998; 1999a; 1999b), advanced techniques for Monte-Carlo simulation can be found e.g. in Glasserman and Zhao (2000) and the survey article by Broadie and Glasserman (1998). On the background of this large literature we restrict ourselves to the details of the implementation that are specific to the case of credit risk modelling.

9.1. **Setup.** First, a choice has to be made whether to model the discrete-time default intensities H_k or the credit spreads S_k as lognormal. Given the scarcity of available data it is unlikely that a statistical test would be able to decide between the two specifications because their effects on the defaultable forward rates are very similar.

Spreads are more intuitive to work with and their drift modification under the survival measure is simpler, but the H_k and their volatilities appear more frequently in the pricing formulae and they are more closely associated with the numeraire of the survival measure and the change between survival and forward measure. It seems that the advantages of having a lognormal H_k outweigh the advantages of lognormal S_k particularly for the simplification in calibration, but this judgement depends on the security to price.

Next, the tenor structure has to be chosen such that all payoff relevant dates are covered and the distances between the dates are not too large. Then the dimension of the driving Brownian motion for the combined model has to be determined. Usually, given the scarcity of data, only one Brownian motion is needed in addition to the Brownian motions that drive the default-free term structure of interest-rates.

9.2. **Calibration:** For details to the calibration of the default-free part of the model the reader is referred to Rebonato (1998; 1999a; 1999b). Second, the volatility vectors σ_k^H for the H_k have to be specified. Typically, these will involve correlation with the first principal component ('level') of the default-free interest rates and the idiosyncratic movements of the credit spreads / intensities.

Given this information, the defaultable bond prices B_k in the model can be calibrated to observed defaultable bond prices, default swap rates and asset swap rates using the closed-form or approximate solutions given in the paper. If independence between H and F is assumed, this fitting can be achieved without the need to refer to volatility input. In all cases the expected recovery rate π is needed as an input, too.

10. Conclusion

In this paper we showed how default risk can be incorporated in the modelling framework of the so called market models for interest rates. The change of measure technique which already was important for default-free market models, now becomes the most important tool for analyzing the relationships between forward rates, default intensities and credit spreads and for the derivation of prices and implementation of the model. In particular, a new class of probability measures, the *survival measures*, provides the appropriate tools for the pricing of default-dependent payoffs. These survival measures can be viewed as the probability measures that are reached, when the default-free forward measure is conditioned on survival.

In the modelling of the recovery of defaultable bonds we chose to use the 'recovery of par' modelling approach. This recovery model has the advantage of being able to accurately represent real-world recovery rules. We showed how to price a number of basic defaultable securities in this setup, including defaultable fixed- and floating coupon bonds, asset swaps and credit default swaps. For a fully general specification of the volatilities of credit spreads and interest rates these prices were given by using approximate solutions, under independence of defaults and interest-rate dynamics closed-form solutions are given.

The modification of this model to other specifications of the recovery at default is straightforward: For recovery in equivalent default-free bonds all pricing problems can be reduced to the pricing of zero-recovery bonds (which is already solved here), and the extension to fractional recovery (Duffie / Singleton (1999) and Schönbucher (1998)) should not present any problems either.

We then addressed the pricing of some popular credit derivatives. Most of the work for the pricing of default swaps had already been done in the analysis of the par recovery model, and the pricing formula for asset swap packages is entirely model-independent. To be able to price options on default swaps we again had to transfer and extend notions from the default-free market model world: The introduction of the *default swap measure* — the defaultable analogy of Jamshidian's (1997) swap market measure — enabled us to derive closed-form solutions for these second-generation instruments. As default swaps are becoming more and more liquid and standardised, a modelling approach based on the default swap measure making default swap rates to martingales has much potential for the future.

APPENDIX

PROOF OF PROPOSITION 2 AND 3

To lighten notation, the time index is dropped for $t = 0$, i.e. $\overline{B}_k$ stands for $\overline{B}_k(0)$ etc.

Valuation of the Recovery Payoffs under Independence. We assume independence of defaults and default-free interest rates. The indicator function of default in $]T_k, T_{k+1}]$ is $I(T_k) - I(T_{k+1})$. We call $e'_k(X)$ the value of receiving X at T_{k+1} if a default happens in this interval:

$$\begin{aligned} e'_k(X) &:= \mathbf{E}^Q\left[\ \beta(T_{k+1})(I(T_k) - I(T_{k+1}))X\ \right] \\ &= \mathbf{E}^Q\left[\ \beta(T_{k+1})X\ \right](D_k - D_{k+1}) = \delta_k H_k \overline{B}_{k+1}\,\mathbf{E}^{P_{k+1}}\left[\ X\ \right]. \end{aligned} \tag{88}$$

We consider two cases:
X can either be (a) a fixed payment or (b) principal plus floating rate.

In **case (a)** $X = 1$. Equation (61) follows directly.

In **case (b)** we can use that F_k is a martingale under P_{k+1}. Therefore using (6) yields

$$e'_k(1 + \delta_k F_k(T_k)) = \delta_k S_k \overline{B}_{k+1}. \tag{89}$$

Equation (63) follows from $\mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_{k+1})F_k\ \right] = \mathbf{E}^{P_{k+1}}\left[\ F_k\ \right] D_{k+1} = F_k \overline{B}_{k+1}$.

Valuation under Correlation of Defaults and Interest Rates. We start again from the representation for $e'_k(X)$

$$\begin{aligned} e'_k(X) &:= \mathbf{E}^Q\left[\ \beta(T_{k+1})(I(T_k) - I(T_{k+1}))X\ \right] \\ &= \mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_k)X\ \right] - \mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_{k+1})X\ \right] \end{aligned} \tag{90}$$

The first term in (90)**:**

$$\begin{aligned} \mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_k)X\ \right] &= B_{k+1}\,\mathbf{E}^{P_{k+1}}\left[\ I(T_k)X\ \right] \\ &= B_k\,\mathbf{E}^{P_k}\left[\frac{1}{1+\delta_k F_k(T_k)} X I(T_k)\right] = \overline{B}_k\,\mathbf{E}^{\overline{P}_k}\left[\frac{X}{1+\delta_k F_k(T_k)}\right]. \end{aligned} \tag{91}$$

Case (a): $X = 1$. Changing to $\overline{P}_{k+1}$ yields

$$\mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_k)\ \right] = \overline{B}_{k+1}\left(1 + \delta_k\,\mathbf{E}^{\overline{P}_{k+1}}\left[\ H_k(T_k)\ \right]\right). \tag{92}$$

Case (b): $X = 1 + \delta_k F_k(T_k)$. The solution is found directly

$$\mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_k)X\ \right] = \overline{B}_k.$$

The Second Term of (90)**:**

$$\mathbf{E}^Q\left[\ \beta(T_{k+1})I(T_{k+1})X\ \right] = \overline{B}_{k+1}\,\mathbf{E}^{\overline{P}_{k+1}}\left[\ X\ \right]. \tag{93}$$

Case (a) yields the result $\overline{B}_{k+1}$.

In **case (b)** the value is

$$\overline{B}_{k+1}(1 + \delta_k\,\mathbf{E}^{\overline{P}_{k+1}}\left[\ F_k(T_k)\ \right]). \tag{94}$$

Combining these results, the values of the payoffs are:
in case (a): (fixed payment)

$$e'_k(1) = \overline{B}_{k+1}\delta_k\,\mathbf{E}^{\overline{P}_{k+1}}\left[\ H_k(T_k)\ \right] \tag{95}$$

in case (b): (floating coupon plus principal payment)

$$e_k(1+\delta_k F_k(T_k)) = \overline{B}_{k+1}\delta_k \, \mathbf{E}^{\overline{P}_{k+1}} [\; S_k(T_k) \;] \,. \tag{96}$$

The value of a floating coupon payment of $\delta_k F_k$ paid at T_{k+1} in survival is

$$\mathbf{E}^{Q} [\; \beta(T_{k+1}) I(T_{k+1}) \delta_k F_k \;] = \overline{B}_{k+1}\delta_k \, \mathbf{E}^{\overline{P}_{k+1}} [\; F_k(T_k) \;] \,. \tag{97}$$

Thus, because of correlation the values of forward default intensity or credit spread have been replaced with their expectation under the T_{k+1}-survival measure.

Approximative Solutions under Correlation. In equations (95) and (96) we have to evaluate the expectation of certain forward rates under the respective survival measures. This is done in two steps: As all random variables are measurable w.r.t. the realisations of the Brownian motions, we can consider the expectations under the survival measure $\overline{P}'_{k+1}$ as expectations under $\overline{P}'_{k+1}$. In a first step we transform the pricing problem to the problem of the calculation of the covariance of a rate with the Radon-Nikodym density L^D of the change from the forward measure to the measure $\overline{P}'$. In a second step we give approximations to the values of these covariances.

For equation (92) we change to the T_k-forward measure using equation (47) to reach

$$\mathbf{E}^{\overline{P}_k}\left[\; \frac{1}{1+\delta_k F_k(T_k)} \;\right] = \mathbf{E}^{\overline{P}'_k}\left[\; \frac{1}{1+\delta_k F_k(T_k)} \;\right] = \mathbf{E}^{P_k}\left[\; L^D_k(T_k)\frac{1}{1+\delta_k F_k(T_k)} \;\right].$$

Both $\frac{1}{1+\delta_k F_k}$ and L^D_k are martingales under the T_k-forward measure, thus

$$\mathbf{E}^{\overline{P}_k}\left[\; \frac{1}{1+\delta_k F_k(T_k)} \;\right] = \frac{1}{1+\delta_k F_k} + \mathrm{cov}^{P_k}\Big(L^D_k(T_k)\,,\; \frac{1}{1+\delta_k F_k(T_k)} \Big). \tag{98}$$

Similarly, for (94) we change to the T_{k+1}-forward measure:

$$\mathbf{E}^{\overline{P}_{k+1}} [\; F_k(T_k) \;] = \mathbf{E}^{\overline{P}'_{k+1}} [\; F_k(T_k) \;] = \mathbf{E}^{P_{k+1}} \left[\; L^D_{k+1}(T_k) F_k(T_k) \;\right].$$

Again, both expressions F_k and L^D_{k+1} are martingales under the T_{k+1}-forward measure

$$\mathbf{E}^{\overline{P}_{k+1}} [\; F_k(T_k) \;] = F_k + \mathrm{cov}^{P_{k+1}}\Big(L^D_{k+1}(T_k)\,,\; F_k(T_k) \Big). \tag{99}$$

There are no closed-form expressions for the covariances in the previous expressions. We are going to use the following common approximation: We approximate both processes with log-normal processes by setting the stochastic components in the diffusion parameters equal to their values at time $t=0$ and evaluate the covariance of these processes[14].

The volatility of L^D_k is

$$\begin{aligned} \alpha^D_k(t) &= \int_t^{T_k} \sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\, ds \\ &= \sum_{l=\kappa(t)}^{k-1} \frac{\delta_l H_l(t) \sigma^H_l(t)}{1+\delta_l H_l(t)} + \int_t^{\kappa(t)} \sigma^{\overline{f}}(t,s) - \sigma^f(t,s)\, ds \end{aligned}$$

and approximated

$$\approx \sum_{l=\kappa(t)}^{k-1} \frac{\delta_l H_l(0) \sigma^H_l(0)}{1+\delta_l H_l(0)}.$$

[14]For the default-free market models, Brace / Gatarek / Musiela (1997) interpret this approximation as a first-order chaos expansion. Rebonato (1998) reports very good results for similar approximations.

Integrating the approximated volatility yields the aggregate volatility of L_k^D over $[0, T_m]$

$$\int_0^{T_m} \alpha_k^D(t)dt \approx \sum_{l=0}^{k-1} \frac{\delta_l H_l(0)\sigma_l^H(0)}{1+\delta_l H_l(0)} T_{l\wedge m} =: A_{k,m}^D.$$

In the approximation we set $H_l(t)$ and $\sigma_l^H(t)$ to their values at $t = 0$. If H and not S is taken as fundamental process, then σ^H is constant anyway. Furthermore we set the volatilities at the short end (before the next tenor time) to zero. [15].

The dynamics of $X(t) := \frac{1}{1+\delta_k F_k(t)}$ are easily found by Itô's lemma as $dX = -X(1-X)\sigma_k^F dW_k$. We choose to use $Y(t) := 1 - X(t)$ instead, which follows

$$dY(t) = Y(t)(1-Y(t))\sigma_k^F dW_k \approx Y(t)(1-Y(0))\sigma_k^F dW_k \tag{100}$$

The values of $Y(t)$ are typically close to zero, thus the lognormal approximation should be very accurate and better than approximating $X(t)$ as lognormal.

This yields the following approximative value for the covariance in equation (98)

$$\begin{aligned} \text{cov}^{P_k}\left(L_k^D(T_k)\,,\,\frac{1}{1+\delta_k F_k(T_k)}\right) &= -\text{cov}^{P_k}\left(L_k^D(T_k)\,,\,Y(T_k)\right) \\ &\approx -\frac{\delta_k F_k(0)}{1+\delta_k F_k(0)}\left(\exp\left\{\frac{1}{1-\delta_k F_k(0)} A_{k,k}^D \sigma_k^F\right\} - 1\right) \end{aligned} \tag{101}$$

and for the covariance in equation (99)

$$\text{cov}^{P_{k+1}}\left(L_{k+1}^D(T_k)\,,\,F_k(T_k)\right) \approx F_k(0)\left(e^{A_{k+1,k}^D \sigma_k^F} - 1\right). \tag{102}$$

The error of these approximation should be very low for reasonable parameter values. It will certainly be an order of magnitude lower than the approximative correlation correction itself which in turn is of the order of a few basis points. The sign of the correction depends on the sign of the correlation between the default intensities H and the default-free interest rates F.

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[15]In fact we only need to set the *covariation* with the default-free interest rates to zero.

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