The LIBOR Market Model: A Recombining Binomial Tree Methodology

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Abstract

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We propose an implementation of the *LIBOR* market model, adapting the recombining node methodology of Ho, Stapleton and Subrahmanyam (1995). Initial tests suggest that the method provides a fast and accurate approach for the valuation of path dependent interest rate derivatives such as Bermudan-style swaptions. The lattice based approach illustrated here provides an efficient alternative to Monte-Carlo simulation implementation of the LMM.

1 Introduction and Motivation

The *LIBOR* Market Model (LMM) is the most common implementation in practice of the general Heath, Jarrow and Morton (1990) forward rate approach to the valuation of interest-rate derivatives. First proposed by Miltersen, Sandmann and Sondermann (1997) (MSS) and Brace, Gatarek and Musiella (1997) (BGM), the model assumes that the London Interbank Offer Rate *LIBOR* has a conditional probability distribution which is lognormal. This paper addresses two problems that arise with the LMM. First, the proof of the continuous-time LMM is somewhat obscure. Hull (2002), for example simply states the drift of the forward rates, under the risk-neutral measure, without proof. However, without a knowledge of why the drift is as stated in the model, it is hard to use the model with confidence. We show here that the drift of the forward rate can be derived very simply from the pricing of Forward Rate Agreements (FRAs) in a no-arbitrage setting.

Second, multi-factor versions of the LMM are difficult to implement, especially for the pricing of Bermudan-style swaptions. We employ the recombining binomial-tree methodology of Ho, Stapleton and Subrahmanyam (1995) (HSS) to construct a swaption pricing model, which does not have to rely on Monte-Carlo simulation or the lower-bound approximations commonly employed. As an illustration, we implement a two-factor example of the LMM and price various swaptions and other interest-rate sensitive contracts.

2 Main Features of *LIBOR* Market Models

The LMM is a term-structure model which recovers caplet and floorlet values that are consistent with the market practice of applying the Black model to price options on interest rates, defined on a *LIBOR* basis. Many market participants build LMMs and use them to price path-dependent interest-rate derivatives such as Bermudan-style swaptions. The main assumption of the model is that the forward rate is conditional lognormal under the risk-neutral measure. Since in the model the drift of the forward rate is stochastic, the *LIBOR* is not unconditional log-normal. If it was, then as in the Black and Karasinski (1991) spot-rate model, the Black model would not hold for caplets and floorlets, due to the effects of stochastic discounting[1].

The Black model for a caplet is given by:

[1] Pricing a caplet (floorlet) using the BK model leads to caplet (floorlet) prices which are less than (greater than) those obtained from the Black model.

Definition 1 [The Black Model: Interest-Rate Caplet]

$$caplet_t = \frac{A}{1+f_{t,t+T}\delta}\delta[f_{t,t+T}N(d_1) - kN(d_2)]B_{t,t+T}$$

where

$$d_1 = \frac{ln(\frac{f_{t,t+T}}{k}) + \sigma^2(T)/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where

A is the principal value of the caplet.

$B_{t,t+T}$ is the value at t of a zero-coupon bond paying 1 unit of currency at $t+T$.

δ is the interest-rate reset interval (ex. 3 months) as a proportion of a year.

k is the strike rate.

$f_{t,t+T}$ is T-period forward *LIBOR* at time t.

σ is the volatility of T-period *LIBOR*.

The *LIBOR* Market Model is a model of the stochastic evolution of interest rates that is consistent with the above formula holding for all caplets, with maturities $T = 1, 2, ..., N$. The model allows other interest-rate dependent contingent claims, such as European-style and Bermudan-style swaptions, to be priced in a way that is consistent with the pricing of the caplets.

The LMM is rooted in financial theory. In particular it uses the ideas of the risk-neutral measure, forward parity, and no-arbitrage asset pricing relationships. These ideas are well documented in texts such as Pliska (1997). For convenience, we re-state the most important results here. Since we will be concerned with the pricing of zero-coupon bonds, the relevant ideas concern zero-dividend paying assets. We have the following results.

For a zero-dividend paying asset:

1. The no-arbitrage spot asset price is given by[2]

$$S_t = B_{t,t+1} E_t[B_{t+1,t+2} E_{t+1}[B_{t+2,t+3}[....E_{t+T-1}[B_{t+T-1,t+T} S_T]]]],$$

   where the expectation is taken under the period-by-period risk-neutral measure.

2. The T-period forward price of the asset is given by

$$F_{t,t+T} = S_t / B_{t,t+T}$$

3. The expected one-period-ahead forward price of the asset is[3]

$$E_t(F_{t+1,t+T}) = F_{t,t+T} - cov_t(F_{t+1,t+T}, B_{t+1,t+T}) \frac{B_{t,t+1}}{B_{t,t+T}}.$$

Since forward rates are closely related to forward prices of zero-coupon bonds, and since we will be interested in the drift of forward rates, we now apply these results to price forward contracts on zero-coupon bonds.

Lemma 1 (Zero-Coupon Bond Forward Prices) *When expectations are taken under the risk-neutral measure:*

1. *The T-period forward price of a one-period maturity zero-coupon bond is*

$$B_{t,t+T,t+T+1} = E_t(B_{t+1,t+T,t+T+1}) + \frac{B_{t,t+1}}{B_{t,t+T}} cov_t(B_{t+1,t+T,t+T+1}, B_{t+1,t+T})$$

2. *The one-period ahead forward price of a long maturity bond is:*

$$B_{t,t+1,T} = E_t(B_{t+1,t+2} B_{t+1,t+2,T})$$

Lemma 1 is directly useful if we are building a stochastic process of forward prices. It shows how the drift of the forward price depends upon the covarariance of the forward prices. A similar effect is found in the drift of forward rates. In the following section we use a corollary of the lemma in the analysis of the forward rate drift.

[2] See, for example Pliska (1997), chapter 2

[3] This follows from taking expectations and using the definition of covariance.

3 The *LIBOR* Market Model

The *LIBOR* Market Model is constructed by forming a process for the evolution of forward rates of various maturities. We have to be careful with the notation. We adopt the following:

The LMM: Notation

- $B_{t,t+\delta}$ = Value at t of a zero-coupon bond paying 1 unit of currency at $t+\delta$.
- δ = Interest-rate reset interval (ex. 3 months) as a proportion of a year
- $B_{t,t+T}$ = Value at t of a zero-coupon bond paying 1 unit of currency at $t+T$.
- x_t = Cash flow paid at time t
- S_0 = Value at $t=0$ of a cash flow paid at time t
- $f_{t,t}$ = Spot *LIBOR* at time t:

$$B_{t,t+\delta} = \frac{1}{1 + f_{t,t}\,\delta}$$

- $f_{t,t+T}$ = T-period forward *LIBOR* at time t
- $B_{t,t+T,t+T+\tau}$ = Forward price at t for delivery of a zero-coupon bond (with maturity τ) at T.

We begin the derivation of the LMM by defining the following standard contract: A *Forward Rate Agreement* (FRA) on *LIBOR*, with maturity T, has a payoff

$$\frac{(f_{t+T,t+T} - k)\delta}{1 + f_{t+T,t+T}\delta}$$

at date $t+T$. Note that the definition assumes that the contract is settled at time $t+T$ on a discounted basis at time t. Also, for simplicity we assume that δ is a constant. In practice the precise payoff depends on the day count. Hence the above contract can be thought of as a theoretical, or idealised FRA. We now have the following result:

FRA Pricing and the Drift of the Forward rate

We begin by stating the following corollary of Lemma 1:

Corollary 1 *One and Two-Period Forward Rates*

From Forward Parity:

$$B_{t,t+1,t+3} = B_{t,t+1,t+2} B_{t,t+2,t+3}$$

Using

$$B_{t,t+1,t+3} = E_t(B_{t+1,t+2} B_{t+1,t+2,t+3})$$

It follows that

$$E_t \left(\frac{1}{1+f_{t+1,t+1}} \frac{1}{1+f_{t+1,t+2}} \right) = \frac{1}{1+f_{t,t+1}} \frac{1}{1+f_{t,t+2}}.$$

We first price a one-period FRA. Since the FRA must have a zero value:

$$E_t \left[\frac{(f_{t+1,t+1} - f_{t,t+1})\,\delta}{1+f_{t+1,t+1}\delta} \right] = 0,$$

where expectations are taken under the risk-neutral measure. It follows that

$$E_t \left(\frac{f_{t+1,t+1}}{1+f_{t+1,t+1}\delta} \right) = \frac{f_{t,t+1}}{1+f_{t,t+1}\delta}.$$

Also

$$E_t(f_{t+1,t+1}) - f_{t,t+1} = -cov \left(f_{t+1,t+1}, \frac{1}{1+f_{t+1,t+1}\delta} \right) (1+f_{t,t+1}\delta) \geq 0$$

The difference between the expected forward rate at $t+1$ and the forward rate at t, i.e. the drift of the forward rate over the first period, depends on the covariance of the rate with the bond price. In the case of the two period forward rate we have a similar, but more complicated result, which we state in the following proposition:

Proposition 1 *Drift of Two-Period Forward*

Since a two-period FRA has a zero value:

$$E_t \left[\left(\frac{f_{t+1,t+2} - f_{t,t+2}}{1+f_{t+2,t+2}\delta} \right) \frac{1}{1+f_{t+1,t+1}\delta} \right] = 0.$$

It follows that

$$E_t(f_{t+1,t+2}) - f_{t,t+2} = -cov\left[f_{t+1,t+2}, \frac{1}{1+f_{t+1,t+2}\delta}\cdot\frac{1}{1+f_{t+1,t+1}\delta}\right](1+f_{t,t+1}\delta)(1+f_{t,t+2}\delta)$$

For small changes, $\frac{dx}{x} = d\ln x + k$, *and*

$$cov(\cdots) = cov\left[\ln f_{t+1,t+2}, \ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\frac{1}{1+f_{t+1,t+1}\delta}\right)\right]\frac{f_{t,t+2}}{(1+f_{t,t+1}\delta)(1+f_{t,t+2}\delta)}$$

Hence

$$\begin{aligned}
& E_t\left[f_{t+1,t+2}\right] - f_{t,t+2} \\
= \quad & -cov\left[\ln f_{t+1,t+2}, \ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\frac{1}{1+f_{t+1,t+1}\delta}\right)\right] f_{t,t+2} \\
= \quad & -cov\left[\ln f_{t+1,t+2}, \ln\left(\frac{1}{1+f_{t+1,t+2}\delta}\right)\right] f_{t,t+2} - cov\left[\ln f_{t+1,t+2}, \ln\left(\frac{1}{1+f_{t+1,t+1}\delta}\right)\right] f_{t,t+2}
\end{aligned}$$

Proposition 1 shows that the drift of the two period ahead forward rate depends on the two covariance terms. We now evaluate these covariance terms using a well known property of normally distributed variables. First, we state the following modification of Stein's Lemma:

Lemma 2 *Stein's Lemma for Lognormal Variables*

Assume that x and y are lognormal variables. Then

$$cov\left[\ln x, \ln\left(\frac{1}{1+y}\right)\right] = E\left[\frac{-y}{1+y}\right] cov(\ln x, \ln y)$$

Hence,

$$\begin{aligned}
cov\left[\ln f_{t+T,t+T}, \ln\left(\frac{1}{1+f_{t+T,t+T}\delta}\right)\right] &= E\left(\frac{-f_{t+T,t+T}\delta}{1+f_{t+T,t+T}\delta}\right) var(\ln f_{t+T,t+T}) \\
&= \left(\frac{-f_{t,t+T}}{1+f_{t,t+T}\delta}\right) var(\ln f_{t+T,t+T})
\end{aligned}$$

Stein's lemma, applied above to the case of lognormal variables, provides the key to evaluating the covariance terms that determine the drift of the forward rates. It turns covariances in to logarithmic covariances. We now state the drift terms, in the general case, assuming first for simplicity that $\delta = 1$. We have, given Stein's Lemma, for small changes in forward rates

$$f_{t,t+2} = E_t\left[f_{t+1,t+2}\right] - f_{t,t+2}\frac{f_{t,t+2}}{1+f_{t,t+2}\delta}\sigma_{2,2} - f_{t,t+2}\frac{f_{t,t+1}}{1+f_{t,t+1}\delta}\sigma_{1,2}$$

which states that the drift is dependent, as in HJM, on a series of discounted covariances. We are now in a position to state the drift in the BGM version of the LMM. First we make an additional assumption. We assume that the covariances (of the logarithms of the forward rates) depend only on the maturity of the forward rates, i.e. they are not time dependent. We then can establish:

Proposition 2 *The BGM Model*

Given intertemporal stability of the covariances, if $f_{t,t}$ is the m-month LIBOR rate then

$$B_{t,t+m} = \frac{1}{1+\delta f_{t,t}}, \qquad \delta \approx \frac{m}{12}$$

If the period length t to $t+1$ is also m months, then

$$E\left[f_{t+1,t+T}\right] - f_{t,t+T} = \frac{\delta f_{t,t+T} f_{t,t+T}}{1+\delta f_{t,t+T}}\sigma_{T,T} + \frac{\delta f_{t,t+T-1} f_{t,t+T}}{1+\delta f_{t,t+T-1}}\sigma_{T-1,T} + \cdots$$

In the LMM, the drift of the forward rate at a point in time depends upon the level of the rate. It also depends on the sum of a series of discounted covariances. Since, the drift depends on the forward rate, it is stochastic. This is the property that causes the problem in implementing the model. The stochastic drift can be handled by a Monte-Carlo simulation. But, to avoid this brute force approach we show in the following section that we can model the stochastic drift using adjusted conditional probabilities.

4 The HSS Recombining Node Methodology and the *LIBOR* Market Model

Ho, Stapleton and Subrahmanyam (1995) [HSS] suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process. An adaptation of this methodology has been used by Peterson, Stapleton and Subrahmanyam (2003) [PSS] to build a two-factor spot rate model of the term-structure. In this section we show that a similar application can be made in the case of the LMM. There are some differences in this case however. First, we will assume in this version of the LMM that the stochastic factors driving the term structure are independent log-Brownian motions. Hence, there is no mean reversion or correlation in the factors.

We assume that the proportionate up and down movements in the log-binomial process due to factor $i = 1, 2$ are $u(T)(i)$ and $d(T)(i)$ respectively, for the forward rate with maturity T. This imples that the up and down moves, depend on the maturity of the forward, but not on the time t.

In the HSS method, the T-period forward rate at time t, in state r, s, [after r down-moves in factor 1 and s down-moves in factor 2] is given by

$$f_{t,t+T,r,s} = f_{t-1,t+T}[u_T(1)]^{t-r}[d_T(1)]^r[u_T(2)]^{t-s}[d_T(2)]^s \tag{1}$$

where

$$d_T(i) = \frac{2}{1 + e^{2\sigma_T(i)\sqrt{\delta}}}$$
$$u_T(i) = 2 - d_T(i),$$

for

$$t = 1, 2, ..., N$$
$$T = 0, 1, ..., N - t.$$

Here we have assumed that volatilities are time independent (i.e. they are dependent only on maturity T)

4.1 Forward Rate Drifts and HSS Probabilities

In this section we describe the computation of the drifts of the forward rate process and the HSS conditional probabilities.

Let $m_{t,t+T}(i)$ denote the drift per period of the T-period forward rate at time t due to factor i. In general, the drift of the forward rate at time t is

$$m_{t,t+T,r,s}(i) \quad = \quad \delta \left[\frac{\delta f_{t,t+1,r,s}}{1+\delta f_{t,t+1,r,s}} \sigma_{0,T}(i) + \frac{\delta f_{t,t+2,r,s}}{1+\delta f_{t,t+2,r,s}} \sigma_{1,T}(i) + ... + \frac{\delta f_{t,t+T,r,s}}{1+\delta f_{t,t+T,r,s}} \sigma_{T,T}(i) - \frac{[\sigma_{T,T}(i)]^2}{2} \right]$$

and the probability of an up move is

$$\begin{aligned} q_{t,t+T,r,s}(i) &= [m_{t,t+T,r,s}(i) + (t-r)\ln u_{T+1}(i) + r \ln d_{T+1}(i) - (t-r)\ln u_T(i) - r \ln d_T(i) \\ &- \ln d_T(i)]/[\ln u_T(i) - \ln d_T(i)]. \end{aligned}$$

4.2 Bond Prices

First compute the forward, one-period bond prices $B_{t,t+T,t+T+1,r,s}$. These are given by

$$B_{t,t+T,t+T+1,r,s} = \frac{1}{1+\delta f_{t,t+T,r,s}},$$

for $T = 0, 1, 2, ..., N-1$ [when $T = 0$, these are spot prices] Then the long bond prices are given by:

$$B_{t,t,t+T+1,r,s} = B_{t,t,t+T,r,s} B_{t,t+T,t+T+1,r,s}$$

for $t = 0, 1, 2, ..., N-1$ and $T = 0, 1, 2, ..., N-1$. Define the spot prices by

$$B_{t,t,t+T,r,s} \equiv B_{t,t+T,r,s}$$

4.3 Caplet Prices

The European-style Caplet is priced using the equations:

$$\begin{aligned} cap(\tau, t, r, s) &= \{q_{t,\tau,r,s}(1) q_{t,\tau,r,s}(2) cap(\tau, t+1, r, s) \\ &+ q_{t,\tau,r,s}(1)[1 - q_{t,\tau,r,s}(2)] cap(\tau, t+1, r, s+1) \\ &+ [1 - q_{t,\tau,r,s}(1)] q_{t,\tau,r,s}(2) cap(\tau, t+1, r+1, s) \\ &+ [1 - q_{t,\tau,r,s}(1)][1 - q_{t,\tau,r,s}(2)] cap(\tau, t+1, r+1, s+1)\} \left[\frac{1}{1+f_{t,t,r,s}\delta} \right] \end{aligned}$$

where

$$cap(\tau,\tau,r,s) = max[f_{\tau,\tau} - k, 0]\delta\frac{1}{1+f_{t,t,r,s}\delta}$$

4.4 Swaption Prices

Lemma 3 *The expected n-period swap rate is given by*

$$E(s_{n,t}) = \sum_{T=1}^{n} \alpha_T E(f_{t,t+T}) \tag{2}$$

where

$$\alpha_T = \frac{1 - B_{t,t+T}}{\sum_{T=1}^{n} B_{t,t+n}}$$

The swap rate at time t, for a swap with maturity n, is given by

$$s_{t,n,r,s} = \frac{1 - B_{t,t+n,r,s}}{\delta \sum_{\tau=1}^{n} B_{t,t+\tau}} \tag{3}$$

The payoff, at time t on a (payer) swaption with strike rate k, is

$$swn_{t,n,r,s} = max(s_{t,n,r,s} - k, 0)\delta[\sum_{\tau=1}^{n} B_{t,t+\tau,r,s}] \tag{4}$$

The value of the swaption at time t is

$$max(swn_{t,n,r,s}, Swn_{t,n,r,s}) \tag{5}$$

where

$$Swn_{t,n,r,s} = E_t(Swn_{t+1,n-1,r,s})B_{t,t+1,r,s} \tag{6}$$

5 Calibration to Swaption Volatilities

First compute the swap weights: Using the bond prices $B_{0,t,0,0} \equiv B_{0,t}$, we have

$$\omega_{1,n} = \frac{B_{0,1}}{\sum_{\tau=1}^{n} B_{0,\tau}}$$

The volatility of the swap rate is then given by

$$swapvol_{t,n} = \left(\sum_{\tau=1}^{n} \sum_{T=1}^{n} \omega_\tau \omega_T \rho_{\tau,T} \sigma_{t,t+\tau} \sigma_{t,t+T} \right)^{0.5}$$

The model can be calibrated to selected swaption vbolatilities by minimising the distance, measured by the sum of squared differences, between the computed swaption vols and the market vols. For example, if equal weight is given to each of the swaption vols, the model is calibrated by selecting the caplet vols so as to solve:

$$\min_{\{\sigma_{1,1+T}\}} \left(swapvol_{t,n} - swapvol_{t,n}^{m} \right)^2$$

where $swapvol_{t,n}^{m}$ refer to the (Black model) market quotes for swaptions with option maturity t and underlying swap maturity n.

Swaption Volatility Quotes

The forward swap rate at time 0, for delivery at τ, for a swap with final maturity T, is given by

$$s_{0,\tau,T} \quad = \quad \frac{1 - B_{0,\tau,T}}{m[B_{0,\tau,\tau+1} + B_{0,\tau,\tau+2} + ... + B_{0,\tau,T}]} \tag{7}$$

The Black model for swaption quotes is

$$A_{0,\tau,T} B(s_{0,\tau,T}, \tau, \sigma_{\tau,T-\tau}, k) = swn_{0,\tau,T} \tag{8}$$

where $A_{0,\tau,T}$ is the forward swap annuity:

$$A_{0,\tau,T} = B_{0,\tau,\tau+1} + B_{0,\tau,\tau+2} + ... + B_{0,\tau,T} \tag{9}$$

and where $\sigma_{\tau,T-\tau}$ is the implied volatility of the swaption with maturity τ and swap maturity $T-\tau$.

$$\begin{aligned} swn_{t,\tau,T,i,j} &= B_{t,t+1,i,j}\{q_x(t+1,i,j)[q_y(t+1,j)swn_{t+1,\tau,T,i,j} \\ &+ (1-q_y(t+1,j))swn_{t+1,\tau,T,i,j+1}] \\ &+ (1-q_x(t+1,i,j))[q_y(t+1,j)swn_{t+1,\tau,T,i+1,j} \\ &+ (1-q_y(t+1,j))swn_{t+1,\tau,T,i+1,j+1}]\} \end{aligned} \tag{10}$$

where

$$swn_{\tau,\tau,T,i,j} = swn_{\tau,T,i,j} \tag{11}$$

6 Some Preliminary Tests of the Model

In this section we present results based on small scale (16 period and 24 period) versions of the model. We first show that the model is capable of accurately re-pricing caplets, when compared to the Black model. We then proceed to price a variety of swaption contracts: European-style, Fixed-tail Bermudan-style and American-style swaptions. Where possible, our prices are compared with those recorded in the literature for similar contracts and data input.

6.1 Caplet Pricing

In Table 1, we show the results of pricing caplets, given a volatility structure and the forward rates at time $t = 0$. The first and fourth columns show the Black model prices at strike prices of 5% and 6% respectively. The third and sixth columns show our one-factor model prices at strike prices of 5% and 6% respectively. The second and fifth columns show our two-factor model prices at strike prices of 5% and 6% respectively. The volatility structure is a constant 20% in each case. In the case of the two-factor model, the allocation constants are $a_{1,0} = 0.8$ and $b = 0.7$. It is clear from the table that in most cases (excluding the very short term caplets in the one-factor case) the pricing of the caplets is consistent with the Black model. This confirms that the drift factors and probabilities are correct.

6.2 Swaption Pricing

In Table 2 we show results from using a one-factor version of the model to price European-style and 'fixed-tail' Bermudan-style swaptions. The results are directly comparable with those recorded from the LMM and from Carr and Yang's Markov model. As in Andersen (2000), the Bermudan-style swaptions have a 'lockout period' shown in column four of the table. In most cases our model overprices both European-style and Bermudan-style swaptions by a small number of basis points. This is probably due to using the probability of the longest forward in the underlying swap. Other tests show that using the probability of the shortest forward in the swap underprices the swaptions.

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8 Appendix: Computational Procedure: Forward Volatilities and Covariances

8.1 Inputs

1. Input time 0 structure of forward *LIBOR* rates

$$f_{0,T},\ T = 0, 1, ..., N$$

2. Input time 0 structure of caplet volatilities, denote these

$$\sigma_{t,t},\ \ t = 1, 2, .., N$$

8.2 Compute Forward Volatilities

The one-period forward volatilities solve the following 'bootstrap' equations:

$$\begin{aligned}
\sigma^2_{1,1} &= \sigma^2_{1,1} \\
\sigma^2_{1,2} &= 2\sigma^2_{2,2} - \sigma^2_{1,1} \\
\sigma^2_{1,3} &= 3\sigma^2_{3,3} - \sigma^2_{1,1} - \sigma^2_{1,2} \\
... &= ... \\
\sigma^2_{1,N} &= N\sigma^2_{N,N} - \sigma^2_{1,1} - \sigma^2_{1,2} - ... - \sigma^2_{1,N-1}
\end{aligned}$$

The two-period forward volatilities solve the following 'bootstrap' equations:

$$\begin{aligned}
\sigma^2_{2,2} &= \sigma^2_{2,2} \\
\sigma^2_{2,3} &= [3\sigma^2_{3,3} - \sigma^2_{1,1}]/2 \\
\sigma^2_{2,4} &= [4\sigma^2_{4,4} - \sigma^2_{1,1} - \sigma^2_{1,2}]/2 \\
... &= ... \\
\sigma^2_{2,N} &= [N\sigma^2_{N,N} - \sigma^2_{1,1} - \sigma^2_{1,2} - ... - \sigma^2_{1,N-1}]/2
\end{aligned}$$

Also, in general we need for $t = 1, ..., N-1$:

$$\begin{aligned}
\sigma^2_{t,t} &= \sigma^2_{t,t} \\
\sigma^2_{t,t+1} &= [(t+1)\sigma^2_{t+1,t+1} - \sigma^2_{1,1}]/t \\
\sigma^2_{t,t+2} &= [(t+2)\sigma^2_{t+2,t+2} - \sigma^2_{1,1} - \sigma^2_{1,2}]/t \\
... &= ... \\
\sigma^2_{t,N} &= [N\sigma^2_{N,N} - \sigma^2_{1,1} - \sigma^2_{1,2} - ... - \sigma^2_{1,N-1}]/t
\end{aligned}$$

8.3 Computing Factor Loadings

1. Input constants $a_{1,0}, ..., a_{1,N-1}$ [for convenience, assume $a_{1,T} = (a_{1,0})^{b(T+1)}$, then only input $a_{1,0}$ and b.]

2. Compute the relative factor loadings for factor 2 using:

$$a_{2,T} = (1 - a^2_{1,T})^{0.5} \; T = 0, 1, ..., N-1 \tag{12}$$

3. Compute the absolute factor loadings for factor 1 and 2 using:

$$\begin{aligned}
\beta_{1,T} &= a_{1,T}\sigma_{1,T+1}, \; T = 0, 1, ..., N-1 &(13)\\
\beta_{2,T} &= a_{2,T}\sigma_{1,T+1}, \; T = 0, 1, ..., N-1 &(14)
\end{aligned}$$

8.4 Compute Covariances

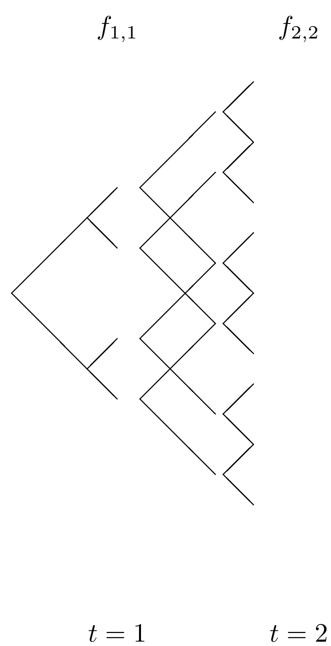
Compute array of $cov_{i,\tau,T}$, for factors $i = 1, 2$ and for $\tau = 0, 1, ..., N-1$ and $T = 0, 1, ..., N-1$, using

$$cov_{i,\tau,T} = \beta_{i,\tau}\beta_{i,T} \tag{15}$$

Also compute the covariance:

$$cov_{\tau,T} = \beta_{1,\tau}\beta_{1,T} + \beta_{2,\tau}\beta_{2,T} \tag{16}$$

Figure 1
Recombining Tree: Two Factors



Note: The model produces 4 nodes at $t = 1$, 9 nodes at $t = 2$, and so on. The lattice recombines in two dimensions.

Table 1: Caplet Prices: Black, One-Factor and Two-Factor Models

5% Strike			6% Strike		
Black	2-factor	1-factor	Black	2-factor	1-factor
0.00049	0.00049	0.00061	0.00002	0.00000	0.00000
0.00069	0.00067	0.00063	0.00009	0.00009	0.00003
0.00083	0.00082	0.00089	0.00017	0.00017	0.00019
0.00095	0.00093	0.00092	0.00025	0.00025	0.00019
0.00104	0.00103	0.00108	0.00033	0.00032	0.00036
0.00113	0.00111	0.00112	0.00041	0.00039	0.00037
0.00120	0.00119	0.00122	0.00048	0.00046	0.00050
0.00127	0.00125	0.00125	0.00055	0.00052	0.00052
0.00133	0.00131	0.00134	0.00061	0.00058	0.00062

1. Caplets assume $\delta = 0.25$.
2. Assumes volatility is 20%, the forward curve is flat 10%.

Table 2: Swaption Prices (basis points): One-Factor Model

European			Bermudan		
option maturity	swap end	price	lockout	swap	price
1	4	118 (122)	1	4	164 (158)
2	4	113 (111)	2	5	196 (188)
3	4	67 (66)			
2	5	167 (162)			
3	5	133 (128)			
4	5	74 (72)			

1. All options are on $\delta = 0.25$ reset swaps. Strike price is 10%, volatility is 20%, the forward curve is flat 10%.
2. The prices can be compared with those for Carr and Yang shown in Andersen (2000), Table 1. These prices are shown in brackets.