Libor Market Model

-Building the One Factor Model-

André Delgado Maciel

Faculdade de Economia e Gestão

Universidade Católica Portuguesa

Email: amaciel@porto.ucp.pt

Report done in the scope of MeF 202

"International Financial Markets"

Lecturer: Professor Richard Stapleton

Porto, March 2002

Abstract

The goal of this paper is to present the steps needed to build the one factor LIBOR Market Model and to show how this model can be applied to value a Bermudan-style caplet. The main conclusion of this paper is that the model though intuitively appealing by the use of market rates, can be quite cumbersome to implement because of its non-Markov nature. In what concerns the drift of the forward rate in the LIBOR Market Model, it is stochastic – as it is a function of the stochastic forward rates - and depends on the rate and the position on the tree. The paper starts with procedures to build the one-factor LIBOR Market Model in Section 2, and then proceeds to its application to price a Bermudan-style caplet in Section 3. Section 4 is concerned with the comparison between the LIBOR Market Model and the Black-Karasinski model and Section 5 discusses the drift of the forward rate in the LIBOR Market Model. In Section 6, some concluding remarks are presented.

1. Introduction

The goal of this paper is to present the steps needed to build the one factor LIBOR Market Model[1] and to show how this model can be applied to value a Bermudan-style caplet.

The main conclusion of this paper is that the LMM though intuitively appealing by the use of market rates, can be quite cumbersome to implement because of its non-Markov nature. In what concerns the drift of the forward rate in the LMM, it is stochastic – as it is a function of the stochastic forward rates - and depends on the rate and the position on the tree.

The paper starts with procedures to build the one-factor LMM in Section 2, and then proceeds to its application to price a Bermudan-style caplet in Section 3. Section 4 is concerned with the comparison between the LMM and the Black-Karasinski[2] model and Section 5 discusses the drift of the forward rate in the LMM. In Section 6, some concluding remarks are presented.

2. Building the model

The LMM is a stochastic interest rate model rooted on forward rate models (Heath-Jarrow-Morton[3]) and forward price models (Ho-Lee[4]). It exists a variety of LMMs from single factor to multi-factor models, but in this paper I will focus on the one-factor LMM. A detailed presentation of the model can be found in Section 4, where the LMM is compared with the BK model.

The first step to build the forward interest rate term-structure tree is to compute the forward volatilities, through the use of bootstrapping, and to determine the covariances for a given set of inputs formed by the structure of forward LIBOR rates at time zero and the structure of caplet volatilities at the same moment in time.

Next, one must calculate the factor that will introduce the volatility in the tree. At the same time one determines the drift for each maturity of the first period (or step of the tree). Having these two elements one can determine the LIBOR forward rates of the subsequent moment for each maturity, which will be equal to the previous moment

[1] LMM hereafter
[2] BK hereafter
[3] HJM hereafter
[4] HL hereafter

forward rate plus the drift plus a volatility term. After this, one repeats this process - determine drift of the second period and forward rates at the subsequent moment – until one reaches the final nodes of the tree.

A numerical example can be found in Appendix 1.

Before one proceeds, it is worth to make some brief observations concerning this model. First, notice that the tree does not recombine. This implies that the tree will increase exponentially in size for each additional step that it is added. To solve for this problem one can force the tree to recombine. Second, the (estimated) forward rates eventually turn into (estimated) spot rates as time proceeds. This also results in smaller term structures at each subsequent moment of time in the tree.

3. Pricing a Bermudan-style caplet

In this section, I will apply the LMM developed in the preceding section to price a Bermudan-style caplet.

A Bermudan-style option, also known as Mid-Atlantic option, is an option that can be exercised at specific moments during the option's life. It differs from the European option because the latter can only be exercised at maturity and it also differs from the American-style option because the Bermudan-style option can only be exercised at specific moments.

The characteristics of the caplet to price and its valuation are presented in Appendix 2. In order to price the caplet I assume that in the case of the option being exercised before maturity, the payoff will be received at the end of the period the option is exercised.

Notice that at maturity the value of the caplet is similar to an European-style option. The difference arises at the nodes before maturity, in which one has to see whether it is optimal to exercise or not the option for a Bermudan-style option, whereas one just discounts the value at maturity back to moment zero for an European-style option.

The discounting is done through the one-period estimated spot zero-coupon bonds (discount factors) derived from the interest rate tree constructed in Section 2.

4. Comparison with the Black-Karasinski model

4.1 LIBOR Market Model

The LMM is a no-arbitrage model stochastic interest rate model. Historically it is posterior to the BK model, because the latter belongs to the first no-arbitrage interest rate models family – spot rate models -, while the former belongs to a more recent family of models: market models. Between these two families the forward rate models, pioneered by HJM, were developed.

Inside the market models, Brace, Gatarek and Musiela (BGM model), and Miltersen, Sandmann and Sondermann (MSS model), developed market models based on money market rates. Jamshidian developed an equivalent market model based on swap rates. In this paper the LMM refers to the one factor BGM model.

As previously mentioned the LMM roots are the HJM model and the HL model. The connection with HJM has to do with the fact that both are forward rate models, but LMM overcomes the HJM problem of HJM's factor (instantaneous forward rate) not being observable. LMM substitutes the instantaneous forward rate by the LIBOR rates, which are observable in the market and more familiar to traders.

In what concerns the relation to the HL model, notice that LMM is nothing more than the HL model expressed in terms of rates.

LMM is also consistent with the Black Market Model of 1976 in pricing caplets. This is in fact one of the weaknesses of the model, for one knows that the volatility smile proves that the Black Market Model of 1976 is not correct.

Another characteristic is that the LMM starts with the term structure of forward rates at time zero and assumes the forward rates are conditionally lognormal distributed over each period. Additionally, the tree does not recombine because the volatility is a stochastic function, although some extensions of the LMM force the tree to recombine. If the tree is not forced to recombine, its construction can be quite cumbersome for long periods or for short time-steps.

The main uses of the LMM are to value complex derivatives: American and Bermudan-style options.

One final note to mention that the LMM can have several factors, and for this reason one cannot talk about *the* LMM, but to a specific representation of LMMs. This is one of the strengths of the LMM, as it is very easy to extend this model to accommodate additional factors.

4.2 Black-Karasinski

The BK model is also a no-arbitrage model stochastic interest rate model. However it is a spot rate model, while LMM is a market model. Furthermore its roots differ from LMM, since BK is related to the Black-Derman-Toy[5] model and to the Hull-White[6] one factor model.

The BK model is an extension of the BDT model with only one change introduced to the latter: it breaks the link between mean reversion and the volatility structure that existed in the BDT model. BK can be also viewed as a special case of the HW one factor model.

BK starts with the bond prices at time zero, matching the initial yield curve. The initial term structure of volatilities is also matched under the BK model. The volatility in BK is time varying but deterministic, i.e., is constant within a period of time allowing the tree to recombine.

In BK the only factor is the short rate, and is this factor that is lognormal distributed (unconditionally), avoiding negative interest rates. BK also incorporates the mean reversion effect.

The world in which BK operates is assumed to have no taxes, no transaction costs, no default risk, no extra costs for borrowing bonds and it is also assumes that all securities are perfectly correlated in continuous time.

The main uses of the BK are pricing European and American-style bond options.

4.3 Comparison

The LMM and the BK are both no-arbitrage stochastic interest rate models, but belong to different families of models: the first is a market model (the factor being a forward market model), while the latter is a spot rate model.

The models' roots differ. While LMM is related to HJM, HL and Black (1976), the BK model is associated with the BDT and the HW one-factor models.

The number of factors can also be a difference between the two models because the LMM model can incorporate more than one factor.

---

[5] BDT hereafter

[6] HW hereafter

If one considers the LMM with one factor, this can be any forward LIBOR rate, but nevertheless an observable market rate. In the BK model the driving factor is the spot short rate, which is unobservable. The above models assume different distributions for their factors: the spot short rate on the BK model is assumed (unconditionally) lognormal distributed and the forward rate on the LMM is conditionally lognormal distributed.

The BK incorporates the mean reversion effect, and the LMM does not because the forward rates are already expectations.

Additionally, the LMM gives higher caplet and floorlet values for out-of-the money contracts. Finally, the LMM distribution of the rates is more in the tails than the BK's distribution.

5. Drift of the forward rate in the LMM

In what respects the drift of the forward rate in the LMM, one encounters several drifts in the model. To start with, the drift differs from maturity to maturity. To exemplify, notice on Appendix 1, that the drift for the first period is different for each of the forward rates depending on their maturity: 0,00002 for the f_{01}, 0,00005 for the f_{02} and 0,00010 for the f_{03}.

Additionally, if one only considers one of those maturities the drift also varies depending on where the rate is on the tree. Let us consider for example the f_{03} rate. The first drift is 0,00010. In the second period we have two different drifts depending on which state of the nature one considers: 0,00007 if the rate goes up or 0,00005 if the rate goes down. In the next period we have four drifts, which are also all different as it can be observed on Appendix 1. As this example demonstrates, the drift varies according to which point in time the rate is and according to which state of the nature the rate is.

In short, the drift is stochastic and depends on the rate and the position on the tree. The reason why it is stochastic has to do with the fact that the drift depends on the forward rates, which are also stochastic.

One final note concerning the drift to mention that the computation of the drift is assuming inter-temporal stability of the covariances.

6. Conclusion

As a conclusion one can say that the LMM though intuitively appealing by the use of market rates, can be quite cumbersome to implement because of its non-Markov nature of all HJM based models. This non-recombining feature can be overcome.

In what concerns the drift of the forward rate in the LMM, it is stochastic – as it is a function of the stochastic forward rates - and depends on the rate and the position on the tree.

References

Black, Fisher and Piotr Karasinski. 1991. Bond and Option Pricing when Short Rates are Lognormal. *Financial Analysts Journal* July-August: 52-59.

Yan, Hong. 2001. Dynamic Models of the Term Structure. *Financial Analysts Journal* July-August: 60-76.