Stable implied calibration of a multi-factor LIBOR model via a semi-parametric correlation structure *

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Abstract

We will study the thorny issues around simultaneous calibration of LIBOR models to cap(let) and swaption prices in the markets. We will show in general that low factor market models calibrated to these prices tend to imply unrealistic instantaneous correlations between different forward LIBOR rates. Many-factor models, however, have in general a large parameter dimension and therefore tend to be unstable. In this paper we handle this problem by using a semi-parametric full rank correlation structure in a Brace-Gatarek-Musiela/Jamshidian framework, [1, 5] subject to certain natural constraints which enforce realistic behaviour of forward correlations. A LIBOR market model equipped with this correlation structure has essentially the same parameter dimension as a general two-factor model and we show that calibration of such a model to market swaption and cap(let) volatilities is very stable. Moreover, the implied instantaneous forward LIBOR correlation matrix is consistent with estimations from historical data. Further, application of principal component analysis to the thus obtained multi-factor model yields stably calibrated low-factor models.

1 Introduction

In the last years, several models for LIBOR rates and valuation methods for LIBOR rate related derivatives have appeared, see e.g. Brace, Gatarek and Musiela (1997), [1], Jamshidian (1997), [5]. The advantage of these approaches is that they model the LIBOR rate process directly as the primary object in an arbitrage free way instead of deriving it from the term structure of instantaneous rates modelled in a HJM framework by Heath, Jarrow and Morton (1992), [3]. In particular, by choosing a deterministic volatility structure in the general LIBOR dynamics we get the so called LIBOR market model in which it is possible to price cap(lets) by Black-Scholes formulas and swaptions by analytical approximations, [1]. In this paper we consider the LIBOR model within the framework of Jamshidian and study the implied calibration of LIBOR market models to market prices of cap(let)s and swaptions. In section (2) we recall briefly the LIBOR market model in Jamshidians setting, [5] and illustrate the intrinsic shortcomings of low factor models. In section (3) we introduce a multi-factor market model with a special correlation structure which is *semi-parametric* in the sense that the parameter dimension of the, say $n \times n$, correlation matrix is $\mathcal{O}(n)$ rather than $\mathcal{O}(n^2)$ in which case we would speak of a *non*-parametric structure. Although for typical n, e.g. $n = 40, 80$ a parameter dimension of $\mathcal{O}(n)$ is still relatively large, extra regularity constraints on the correlation parameters ensure *both* stability *and* realistic implied model correlations. In this paper we will motivate and study this particular correlation structure, which was previously suggested in Schoenmakers, Coffey (1999), [9], in detail. In section (4) we outline the calibration of the presented LIBOR model to the cap and swap market by using approximate relationships between caplet and swaption volatilities, e.g. see Rebonato (1996) and Schoenmakers, Coffey (1999), [9] where it is analysed in some more detail which approximations are really made. Empirical results based on market quotes of caplet and swaption volatilities are presented and it is shown that it is possible to achieve a very good fit to caplet/swaption volatilities while the forward correlations are matched realistically. We note here that the proposed way of, in a sense, indirect calibration to Black swaption volatilities by using an approximate relationship rather than direct calibration to swaption prices by Monte Carlo simulation is not strictly correct. However, the latter calibration method is extremely slow whereas least square fitting of the swaption volatility matrix can be done in a few seconds and, as it turns out, the swaption prices obtained by Monte Carlo simulation of the thus calibrated model are well in accordance with Black prices computed from the market swaption volatilities, hence the market prices.

2 The LIBOR market model

We consider a Jamshidian LIBOR market model [5] for the forward LIBOR processes L_i with respect to a given tenor structure $0 < T_1 < T_2 < \ldots < T_n$, in the terminal bond numeraire $\mathbb{P}_n$,

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^{(n)}, \tag{1}$$

where, for $i = 1, \ldots, n-1$, the L_i are defined in the intervals $[t_0, T_i]$, $\delta_i = T_{i+1} - T_i$ are the day count fractions and $\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,d})$ are given deterministic functions, called factor loadings, defined in $[t_0, T_i]$, respectively. In (1), $(W^{(n)}(t) \mid t_0 \le t \le T_{n-1})$ is a standard d-dimensional Wiener process under $\mathbb{P}_n$, where d, $d \le n-1$, is the number of driving factors.

2.1 Problems with low factor models

It is known that low factor models have intrinsic problems to match (instantaneous) correlations between forward LIBORS realistically, see, e.g. [6, 9]. To illustrate this fact we consider a two factor version of the LIBOR model (1), where

$$\gamma_i(t) = g_i(t) e_i, \quad e_i \in \mathbb{R}^2, \quad i = 1, \ldots, n-1 \tag{2}$$

and e_i are unit vectors specifying the instantaneous correlations which we assume to be time independent. In general, the volatility norms $g_i = |\gamma_i|$ will be taken to be time dependent. For instance, a plausible assumption is

$$g_i(t) = c_i g(T_i - t), \tag{3}$$

where the $i-$independent function g takes care of the typical "hump shaped" volatility behaviour as function of time to LIBOR maturity. However, as instantaneous correlations are completely determined by the choice of e_i, the choice of g_i does not effect the problem we sketch below. Indeed, we have

$$\frac{\gamma_i(t) \cdot \gamma_j(t)}{|\gamma_i(t)||\gamma_j(t)|} = e_i \cdot e_j = \cos(\phi_i - \phi_j),$$

for a set of angles $\phi_1, \ldots, \phi_{n-1}$ with $\phi_1 := 0$ and it is clear that any two factor market model with constant instantaneous correlations can be represented in this way.

Now suppose, for instance, that $n = 20$ and that the market tells us the correlations $\rho_{1,j}$ behave like $\rho_{1j} = 18/(17+j)$, thus falling down from 1 to 0.5. Then, if we calibrate this two-factor model, i.e. the ϕ_i, to these correlations it is easily seen that, as an immediate consequence, the correlations $\rho_{j,19}$ have to be $\rho_{j,19} = \frac{9}{17+j} + \frac{\sqrt{3}}{2}\sqrt{1-\left(\frac{17}{18}+\frac{j}{18}\right)^{-2}}$, see figure (1). However, the behaviour of the correlations $\rho_{j,19}$ in figure (1) is clearly *not* consistent

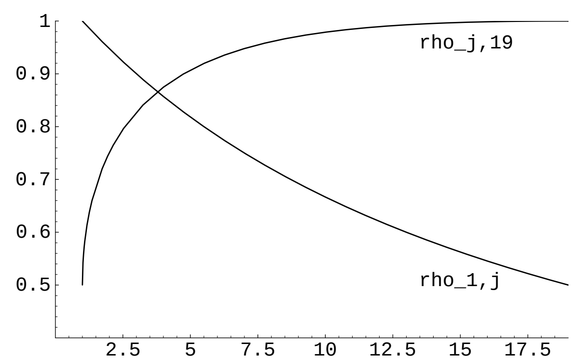

Figure 1:

with their real behaviour in the market which should look more or less the same as $\rho_{1,j}$, mirrored at $j = 10$. Of course the situation will be better when we increase the number of factors, but the two factor example reveals the problem most clearly.

As a solution for this intrinsic low factor calibration problem we propose an alternative market model by the identification of a natural correlation structure which matches correlation behaviour observed in practice with a relatively small number of essential parameters, in fact, the same number as in a two factor market model.

3 Market model with semi-parametric correlation structure

Assumption 3.0.1 (semi-parametric correlation structure) *We propose a LIBOR market model (1) with* $d = n - 1$ *and a deterministic volatility of the following structure*

$$\gamma_i(t) = g_i(t) e_i, \quad e_i \in \mathbb{R}^{n-1}, \tag{4}$$

where $g_i = |\gamma_i|$ *are in general suitable humped shaped real functions and* e_i *are time independent unit vectors determining an instantaneous correlation structure of the form*

$$\rho_{ij} := e_i \cdot e_j = \frac{\min(b_i, b_j)}{\max(b_i, b_j)}, \tag{5}$$

where the sequence $b = (b_1, \ldots, b_{n-1})$*, is required to be positive, strictly increasing, and such that*

$$i \longrightarrow b_i/b_{i+1} \text{ is strictly increasing.} \tag{6}$$

Without further restriction we may assume in (5) that $b_1 = 1$.

As the calibration of correlation structure (5) involves the identification of $\mathcal{O}(n)$ parameters whereas the calibration of a general or *non-parametric* correlation matrix, would require the identification $\mathcal{O}(n^2)$ entries, we call (5) a *semi-parametric* correlation structure. In the next section we will motivate (5) and show in particular that (5) defines a correlation structure indeed.

3.1 Motivation of the semi-parametric correlation structure (5)

Let b_i, $i = 1, \ldots, m$, be an arbitrary positive increasing sequence with $b_1 = 1$ and let $a_1 = b_1 = 1$, $a_i := \sqrt{b_i^2 - b_{i-1}^2}$, for $i = 2, \ldots, m$. Let further Z_i, $i = 1, \ldots, m$, be standard normally distributed independent real random variables and consider the random variables

$$Y_i := \sum_{k=1}^{i} a_k Z_k. \tag{7}$$

Then, for $i \leq j$ the covariance between Y_i and Y_j is given by

$$Cov(Y_i, Y_j) = \sum_{k=1}^{i} a_k^2 = b_i^2$$

and so the correlation is given by $\rho_{Y_i,Y_j} = b_i/b_j$. Hence, it follows that the correlation structure of Y is given by (5), where $m = n - 1$ and, in particular, that (5) defines a correlation structure indeed.

Remark 3.1.1 We note that (7) is not the only Gaussian vector with correlation structure (5): It is not difficult to prove that for an $m \times m$ matrix B and $Z := (Z_1, \ldots, Z_m)^\top$, the centered Gaussian vector $U := BZ$ has correlation structure (5) if and only if there exists a positive diagonal matrix Λ and orthogonal matrix Q such that $B = \Lambda A Q$, where $A_{ik} := a_k 1_{k \leq i}$.

We may assume without restriction in (4) that the time independent $(n-1) \times (n-1)$ matrix E defined by $E_{ik} = e_{i,k}$ is lower triangular, otherwise, we may apply an orthogonal transformation to the Brownian vector $W^{(n)}$. So, we have

$$\begin{aligned} \Delta L_i &= \ldots \Delta t + L_i \gamma_i \cdot \Delta W^{(n)} \\ &= \ldots \Delta t + L_i g_i \Delta t \sum_{k=1}^{i} e_{i,k} Z_k. \end{aligned}$$

By the imposed correlation structure (5) and remark (3.1.1) it then follows that

$$\begin{aligned} e_{i,k} &= \frac{1}{b_i} a_k 1_{k \le i}, \quad \text{and so} \\ \Delta L_i &= \ldots \Delta t + L_i g_i \sum_{k=1}^{i} \frac{a_k}{\sqrt{\sum_{l=1}^{i} a_l^2}} \Delta W_k^{(n)} = \ldots \Delta t + L_i g_i \sum_{k=1}^{i} \frac{\sqrt{b_k^2 - b_{k-1}^2}}{b_i} \Delta W_k^{(n)} \\ &= \ldots \Delta t + L_i g_i \sqrt{1 - \frac{b_{i-1}^2}{b_i^2}} \Delta W_i^{(n)} + \frac{L_i g_i}{L_{i-1} g_{i-1}} \frac{b_{i-1}}{b_i} \Delta L_{i-1} \end{aligned} \tag{8}$$

where $b_0 := 0$. The interpretation of (8) is clear: The risky part of a forward LIBOR increment ΔL_i at time t, $t < T_{i-1}$ is a linear combination of the forward increment ΔL_{i-1} and an independent random shock $\Delta W_i^{(n)}$, with coefficients determined by $L_{i-1}(t)$, $L_i(t)$, $g_{i-1}(t)$, $g_i(t)$, b_{i-1} and b_i, respectively. We emphasize that decomposition (8) is due to the special structure of $e_{i,k}$ hence the correlation structure (5) and does not hold for a general correlation structure. Finally, the additional assumption (6) forces that for fixed p

$$i \longrightarrow \rho_{i,i+p} = Cor(\Delta L_i, \Delta L_{i+p}) \quad \textit{is increasing.} \tag{9}$$

This is a very important realistic feature of the model which states, for instance, that the correlation between a seven and a nine year forward is higher than the correlation between a three and a five year forward. Moreover, it turns out that the extra constraint (6) makes the calibration of the model very stable. For more interesting features around the correlation structure (5), we refer to Curnow & Dunnett (1962), [2].

3.2 Alternative characterization of the correlation structure (5) and the generation of (5)-consistent low parametric structures

By the next theorem it is possible to transform the non-linear constraints on the sequence $(b_1, \ldots, b_{n-1})$ in (5) to the region $\mathbb{R}_+^{n-2}$.

Theorem 3.2.1 *Every correlation structure of type (5) can be represented by*

$$b_i = \exp\left[\sum_{l=2}^{m} \min(l-1, i-1)\Delta_l\right], \tag{10}$$

$$\tag{11}$$

for a sequence of nonnegative numbers $\Delta_i, \Delta_i \ge 0$, $i = 2, \ldots, m := n-1$.

Conversely, (10) satisfies (5) for any sequence Δ_l, $\Delta_l \ge 0$ $l = 2, \ldots, m$.

Proof. For a sequence (b_i) satisfying (5) let us define $\xi_i := \ln b_i$, $1 \le i \le m := n-1$. Then, $\xi_1 = 0$ since $b_1 = 1$ and for ξ_i we have the following constraints:

$$\begin{aligned} \xi_i &\le \xi_{i+1} \quad 1 \le i \le m-1 \\ \xi_{i-1} + \xi_{i+1} &\le 2\xi_i \quad 2 \le i \le m-1. \end{aligned} \tag{12}$$

We introduce the new variables,

$$\begin{aligned}
\Delta_i &:= 2\xi_i - \xi_{i-1} - \xi_{i+1} = \xi_i - \xi_{i-1} - (\xi_{i+1} - \xi_i) \geq 0, \quad 2 \leq i \leq m-1, \\
\Delta_m &:= \xi_2 - \sum_{l=2}^{m-1} \Delta_l
\end{aligned} \tag{13}$$

and so we have for $2 \leq i \leq m$,

$$\begin{aligned}
\xi_i = \xi_i - \xi_1 &= \sum_{k=2}^{i} \xi_k - \xi_{k-1} \\
&= \sum_{k=2}^{i} \{\xi_k - \xi_{k-1} - (\xi_2 - \xi_1) + \xi_2 - \xi_1\} \\
&= (i-1)\xi_2 + \sum_{k=3}^{i} \sum_{l=2}^{k-1} \{\xi_{l+1} - \xi_l - (\xi_l - \xi_{l-1})\} \\
&= (i-1)\xi_2 - \sum_{k=3}^{i} \sum_{l=2}^{k-1} \Delta_l \\
&= (i-1)\xi_2 - \sum_{l=2}^{i-1} \sum_{k=l+1}^{i} \Delta_l \\
&= (i-1)\xi_2 - \sum_{l=2}^{i-1} (i-l)\Delta_l
\end{aligned} \tag{14}$$

where an empty sum is defined to be zero. It follows that

$$\xi_{i+1} - \xi_i = \xi_2 - \sum_{l=2}^{i} \Delta_l$$

and the constraints (12) transform into

$$\Delta_i \geq 0, \qquad 2 \leq i \leq m. \tag{15}$$

Then by (13) and (14) we may express ξ_i, resp. b_i in the new coordinates Δ_i via

$$\begin{aligned}
\xi_i &= (i-1) \sum_{l=2}^{m} \Delta_l - \sum_{l=2}^{i-1} (i-l)\Delta_l \\
&= (i-1) \sum_{l=i}^{m} \Delta_l + \sum_{l=2}^{i-1} (l-1)\Delta_l \\
&= \sum_{l=2}^{m} \min(l-1, i-1)\Delta_l, \\
b_i &= \exp(\xi_i).
\end{aligned} \tag{16}$$

The converse follows straightforwardly by checking (5) for the sequence (b_i) defined by (10).

For a correlation structure (5) representation by (10) it now follows that

$$\rho_{ij} = \exp[-\sum_{l=i+1}^{m} \min(l-i, j-i)\Delta_l] \qquad i < j. \tag{17}$$

From representation (10) in theorem (3.2.1) or, equivalently, from (17) we may derive conveniently various low parametric structures consistent with (5). Below we give some examples.

Example 3.2.2 Let us take $\Delta_2 = \ldots = \Delta_{m-1} =: \alpha \geq 0$ and $\Delta_m =: \beta \geq 0$. Then, (17) yields the correlation structure

$$\rho_{ij} = e^{-|i-j|(\beta+\alpha(m-\frac{i+j+1}{2}))}, \quad i,j = 1,\ldots,m. \tag{18}$$

Note that for $\alpha = 0$ we get $\rho_{ij} = e^{-\beta|i-j|}$, a simple correlation structure frequently used in practice in spite of the, in fact, unrealistic consequence that $i \to Cor(\Delta L_i, \Delta L_{i+p})$ is *constant* rather than increasing for fixed p. Let us introduce new parameters $\rho_\infty := \rho_{1m}$ and $\eta := \alpha(m-1)(m-2)/2$, hence

$$\beta = -\frac{\alpha}{2}(m-2) - \frac{\ln \rho_\infty}{m-1} = \frac{-\eta - \ln \rho_\infty}{m-1}$$

and then (18) becomes

$$\boxed{\rho_{ij} = e^{-\frac{|i-j|}{m-1}(-\ln \rho_\infty + \eta \frac{m-i-j+1}{m-2})}, \quad 0 \leq \eta \leq -\ln \rho_\infty, \quad i,j = 1,\ldots,m.} \tag{19}$$

While the structures (18) and (19) are essentially the same, by the reparametrization of (18) into (19) the parameter stability is improved: Relatively small movements in the $b-$sequence connected with (19), thus the (market) correlations, causes relatively small movements in the parameters ρ_∞ and η. In fact, this can be seen also by analytical comparison of the parameter sensitivities (derivatives) in (18) and (19).

The following three parametric structure is refinement of (18):

Example 3.2.3 Suppose $m > 2$ and let Δ_i be linear dependent of i, for $2 \leq i \leq m-1$, with

$\Delta_2 = \alpha_1 \geq 0$, $\Delta_{m-1} = \alpha_2 \geq 0$, and $\Delta_m = \beta \geq 0$. Hence for $i = 2, \ldots, m-1$,

$$\Delta_i = \alpha_1 \frac{m-i-1}{m-3} + \alpha_2 \frac{i-2}{m-3}.$$

Then, from (17) we get by rather tedious but elementary algebra the correlation structure

$$\begin{aligned}\rho_{ij} = \exp\Bigg[-|j-i|\Bigg(\beta &- \frac{\alpha_2}{6m-18}(i^2+j^2+ij-6i-6j-3m^2+15m-7) \\ &+ \frac{\alpha_1}{6m-18}(i^2+j^2+ij-3mi-3mj+3i+3j+3m^2-6m+2)\Bigg)\Bigg].\end{aligned} \tag{20}$$

Note that (20) collapses too (18) for $\alpha_1 = \alpha_2 = \alpha$. As in (18) we now re-parameterize (20) by $\rho_\infty = \rho_{1m}$ which yields

$$\beta = \frac{-\ln \rho_\infty}{m-1} - \frac{\alpha_1}{6}(m-2) - \frac{\alpha_2}{3}(m-2).$$

In order to gain parameter stability we set, as in (19),

$$\alpha_1 = \frac{6\eta_1 - 2\eta_2}{(m-1)(m-2)}, \quad \alpha_2 = \frac{4\eta_2}{(m-1)(m-2)}$$

and then (20) becomes

$$\begin{aligned} \rho_{ij} \;&=\; \exp\left[-\frac{|j-i|}{m-1}\left(-\ln\rho_\infty + \eta_1 \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)} + \right.\right. \\ &\left.\left. -\eta_2 \frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(m-2)(m-3)}\right)\right], \\ &i,j = 1,\ldots,m, \quad \eta_1 \geq 0,\ \eta_2 \geq 0,\ 0 \leq \eta_1 + \eta_2 \leq -\ln\rho_\infty. \end{aligned} \tag{21}$$

Obviously, for $\eta_1 = \eta_2 = \eta/2$, (21) yields (19) again.

Remark 3.2.4 (An optimal two-parametric correlation structure) *In section (4) we will see that correlation structure (21) suits very well in practice. However, calibrating a three parametric structure takes longer than a two-parametric one, of course. Furthermore, the calibration experiments discussed in (4) reveal that $\eta_2 \approx 0$, hence $\alpha_2 \approx 0$ in (20), which implies that the magnitude of concavity of sequence $\ln b_i$ in (5) is decreasing to zero rather than being constant like in (18). We thus advocate the structure*

$$\begin{aligned} \rho_{ij} \;&=\; \exp\left[-\frac{|j-i|}{m-1}\left(-\ln\rho_\infty + \eta \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)}\right)\right], \\ &i,j = 1,\ldots,m, \quad \eta > 0,\ 0 < \eta < -\ln\rho_\infty, \end{aligned} \tag{22}$$

which follows from (21) by setting $\eta_2 := 0$, $\eta := \eta_1$, as being, in a sense, an optimal two-parametric structure in practice.

Example 3.2.5 It is easily checked that the sequence b defined by

$$b_i = e^{\beta(i-1)^\alpha}, \quad 1 \leq i \leq m = n-1,\ \beta > 0,\ 0 < \alpha < 1, \tag{23}$$

satisfies the requirements in assumption (3.0.1) and thus defines a correlation structure by (5), which was earlier proposed in Schoenmakers, Coffey [9] and used for LIBOR simulations in Kurbanmuradov, Sabelfeld, Schoenmakers, [4]. By theorem (3.2.1) the structure (23) has a representation (10) with

$$\begin{aligned} \Delta_i &= 2\beta(i-1)^\alpha - \beta(i-2)^\alpha - \beta i^\alpha, \quad 2 \le i \le m-1, \\ \Delta_m &= \beta - \sum_{l=2}^{m-1} \Delta_l = \beta(m-1)^\alpha - \beta(m-2)^\alpha, \end{aligned} \tag{24}$$

where indeed $\Delta_i > 0$ for $1 \le i \le m$ and, in fact, Δ_i is decreasing for $2 \le i \le m-1$. By introducing $\rho_\infty := 1/b_m$ we get from (23),

$$\rho_{ij} = e^{\ln \rho_\infty \left| \left(\frac{i-1}{m-1}\right)^\alpha - \left(\frac{j-1}{m-1}\right)^\alpha \right|}, \quad i,j = 1, \ldots, m, \ 0 < \alpha < 1, \ \beta > 0. \tag{25}$$

Here we note that structure (25) has, in principal, similar properties as (22), but, calibration experiments outlined in (4) show that (22) performs a little better, which implies in fact that, in practice, the *linear* decreasing form of Δ_i, $2 \le i < m$ connected with (22) suits better than the (decreasing) form in (24).

Remark 3.2.6 Let us consider the parametrization of market correlations used by Rebonato (1999), [8], which has for a an equidistant tenor structure the following form

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty) \exp\left[-|j-i|(\beta - \alpha \max(i,j))\right]. \tag{26}$$

The structure (26) can be seen as a perturbation of the correlation structure

$$\hat{\rho}_{ij} = \rho_\infty + (1-\rho_\infty) \exp\left[-|j-i|\beta\right] \tag{27}$$

and has the desirable property that $i \to \rho_{i,i+p} = \rho_\infty + (1-\rho_\infty) \exp\left[-p(\beta - \alpha(i+p))\right]$ is *increasing* for $\alpha > 0$ and thus may produce for a given tenor structure realistic market correlations for properly chosen ρ_∞, $\beta > 0$ and (small) $\alpha > 0$, see [8]. However, while (26) may produce realistic correlations, it should be noted that its $(\alpha, \beta, \rho_\infty)$ domain of positivity is not explicitly specified. Hence, for a particular choice of parameters it is not directly guaranteed that (26) defines a correlation structure indeed. In particular. it can be verified easily that (26) does not fit in the framework of (5). It is clear that all the correlation structures consistent with (5), in particular (18), (20), (23) and their equivalent representations do not suffer for this problem as, from the way they are constructed, they are endogenously positive definite and incorporate the economically realistic properties $j \to \rho_{ij}$ is *decreasing* for $j > i$ *and* $i \to \rho_{i,i+p}$ is *increasing* for fixed p, for any choice of parameters *in their well specified domain.* This property makes the correlation structures consistent with (5), presented in this section, particularly suitable for calibration purposes as search routines are thus prevented for searching in parameter regions where the matrix ρ_{ij} fails to be a correlation structure in fact.

4 Calibration to caps and swaptions; empirical results

Rather than calibrating the market model (1) directly to market prices of swaptions, for instance by Monte Carlo simulation, we will take advantage of the following well known approximate relationship between (local) swap volatilities, LIBOR volatilities and LIBOR correlations:

$$S_{p,q}^2\sigma_{p,q}^2 \approx \sum_{i,j=p}^{q-1} w_i^{p,q} w_j^{p,q} L_i L_j |\gamma_i||\gamma_j|\rho_{ij}, \tag{28}$$

where $w_i^{p,q} := \delta_i B_{i+1} / \sum_{k=p}^{q-1} \delta_k B_{k+1}$. See e.g. eq. 56, [9], where it is shown that this approximation is good as long as, loosely speaking, small increments $d\ln w_i^{p,q}$ can be neglected to increments $d\ln L_i$. In practice, this appears to be the case indeed.

By writing the swap rate as a weighted sum of forward LIBORS in (28) and the assumptions in (3), we may write (28) in the following form,

$$\frac{1}{T_p - t}\int_t^{T_p} \sigma_{p,q}^2(s)ds \approx \sum_{i,j=p}^{q-1} \frac{\rho_{ij}}{T_p - t}\int_t^{T_p} \frac{w_i^{p,q} w_j^{p,q} L_i L_j}{\sum_{k,l=p}^{q-1} w_k^{p,q} w_l^{p,q} L_k L_l} g_i(s) g_j(s) ds. \tag{29}$$

We now assume in addition, although inconsistently, that the swap volatilities can be considered as deterministic as well, so that the l.h.s. of (29) is given by the market as implied Black swap-volatilities $\sigma_{p,q}^B$. Next, we note that the fractions in the integrand of the r.h.s. of (29) can be regarded as weights, which tend to vary relatively slowly and we thus approximate them by their initial values (several works confirm that this is a good approximation). We then have,

$$(\sigma_{p,q}^B)^2 \approx \sum_{i,j=p}^{q-1} \frac{w_i^{p,q} w_j^{p,q} L_i L_j}{\sum_{k,l=p}^{q-1} w_k^{p,q} w_l^{p,q} L_k L_l}(t)\, \frac{\rho_{ij}}{T_p - t}\int_t^{T_p} g_i(s) g_j(s) ds.$$

Next, we assume volatility norms g_i of the form (3) and choose a specific functional form for the "hump function" g. As g has to act, in principle, on $[0,\infty[$, it is plausible to take a constant plus a linear combination of the first two Laguerre functions $e^{-s/2}$ and $(s-1)e^{-s/2}$, properly scaled. Without restriction we require $g(0) = 1$ in (3) and by chosing $g_\infty := \lim_{s\to\infty} g(s)$ as parameter, we thus set

$$\begin{aligned} g_i(t) &=: c_i\, g(T_i - t); \\ g(s) &:= g_{a,b,g_\infty}(s) := g_\infty + (1 - g_\infty + as)e^{-bs}, \quad a, b, g_\infty > 0, \end{aligned} \tag{30}$$

where it has to be noted that, in fact, (30) is consistent with a functional form proposed by Rebonato, (1999) [7]. Then, in general, the parameters a, b, g_∞ and c_i are to be determined consistent with the Black caplet volatilities γ_i^B, via

$$\begin{aligned} (\gamma_i^B)^2 &= \frac{1}{T_i - t}\int_t^{T_i} g_i^2(s)ds = \frac{c_i^2}{T_i - t}\int_t^{T_i} g^2(T_i - s)ds \\ &= \frac{c_i^2}{T_i - t}\int_0^{T_i - t} g_{a,b,g_\infty}^2(s)ds. \end{aligned} \tag{31}$$

We next introduce for $p \le \min(i,j)$ the quantities

$$\begin{aligned}\alpha^{a,b,g_\infty}_{i,j,p} &:= \frac{1}{T_p - t}\int_t^{T_p} \frac{g_i(s)g_j(s)}{\gamma_i^B\gamma_j^B} ds = \frac{1}{T_p - t}\frac{c_ic_j}{\gamma_i^B\gamma_j^B}\int_t^{T_p} g(T_i - s)\, g(T_j - s) ds \\ &= \frac{\sqrt{T_i - t}\sqrt{T_j - t}}{T_p - t}\frac{\int_t^{T_p} g_{a,b,g_\infty}(T_i - s)\, g_{a,b,g_\infty}(T_j - s) ds}{\sqrt{\int_0^{T_i - t} g^2_{a,b,g_\infty}(s) ds}\sqrt{\int_0^{T_j - t} g^2_{a,b,g_\infty}(s) ds}}. \end{aligned} \tag{32}$$

We note that (32) is easily evaluated analytically.[1] Hence, the coefficients c_i are eliminated and the basis for the calibration of the market model will be the equation

$$(\sigma^B_{p,q})^2 = \sum_{i,j=p}^{q-1} \frac{w_i^{p,q}(t) w_j^{p,q}(t) L_i(t) L_j(t)}{S^2_{p,q}(t)} \gamma_i^B \gamma_j^B \alpha^{a,b,g_\infty}_{i,j,p}\, \rho_{ij}. \tag{33}$$

It is clear that equation (33) is in general over determined for a correlation structure of type (5) and therefore we are going to "solve" (33) in least square sense. I.e. for a particular (5)-consistent parametrisation of the correlation structure, say $\rho_{ij}(\eta)$, where $\eta \in \mathcal{N} \subset \mathbb{R}^k$ for some $k-$dimensional parameter set $\mathcal{N}$, $k \le n-1$, we minimize the 'root mean square' distance,

$$RMS(a,b,g_\infty;\eta) := \sqrt{\frac{2}{(n-1)(n-2)}\sum_{1\le p\le q-2,\ q\le n}\left(\frac{\sigma^B_{p,q} - \sigma_{p,q}(a,b,g_\infty;\eta)}{\sigma^B_{p,q}}\right)^2} \longrightarrow \min_{a,b,g_\infty;\eta\in\mathcal{N}}, \tag{34}$$

where

$$\sigma_{p,q}(a,b,g_\infty;\eta) := \sum_{i,j=p}^{q-1} \frac{w_i^{p,q}(t) w_j^{p,q}(t) L_i(t) L_j(t)}{S^2_{p,q}(t)} \gamma_i^B \gamma_j^B \alpha^{a,b,g_\infty}_{i,j,p}\, \rho_{ij}(\eta). \tag{35}$$

In summary, if we choose a certain parametrisation of the correlation structure, for example, the three parametric structure (21), we have to carry out a least squares search for the six parameters a, b, g_∞, η_1, η_2, ρ_∞. Then the c_i are determined by (31) and the calibration of the multi factor LIBOR model is done.

As a matter of fact, the above sketched method of calibration to caps and (at the money) swaptions via the approximative relationship (33) is rather indirect in the sense that we do not calibrate directly to swaption prices and therefore the question arises whether for a thus calibrated LIBOR model with volatility norms g_i and correlation structure $\rho_{ij}(\eta_0)$, say, simulation prices of (at the money) swaptions are well in accordance with market prices. This question can be equivalently formulated as follows. Given a correlation structure $\rho_{ij}(\eta_0)$ and volatility norms g_i, are then the implied Black volatilities of simulated (ATM) swaption prices in accordance with (35) for $\eta = \eta_0$? A back test comparison of (ATM) market swaption prices with Monte Carlo simulated model prices shows indeed that for typical initial term structures, volatilities and correlations these prices well agree within spreads. Hence calibration to caps and swaptions via (33) is, in a certain sense, legitimate.

[1] We omit the rather long expressions, to prevent errors in the tedious calculations one might produce the results easily with a program like, Mathematica or Maple, e.g..

Remark 4.0.7 In a rougher approximation one might choose the volatility norms to be time independent, hence $a = b = 0$ in (30) and then $\alpha_{i,j,p}^{0,0,\cdot} \equiv 1$ in (32). However, generally, $\alpha_{i,j,p}^{a,b,g_\infty}$ may be less or greater than 1, depending on a, b and g_∞. See figure (2) below for a typical choice of a, b and g_∞, yielding a function g as plotted in figure (3).

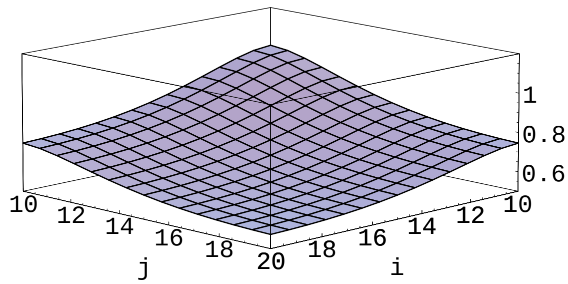

Figure 2: $\alpha_{i,j,p}^{0.5,0.4,0.6}$; $p = 10 \leq i, j \leq 20$

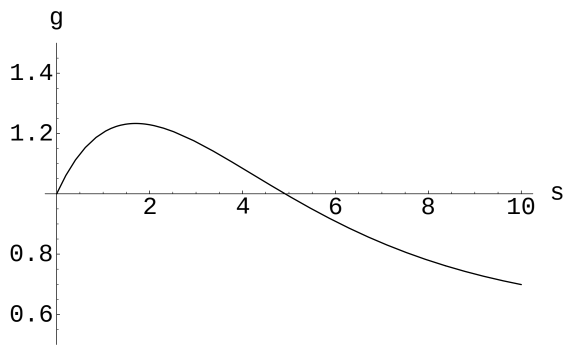

Figure 3: $s \to g_{0.5,0.4,0.6}(s)$

4.1 Empirical results

We have carried out several tests with Euro-market data. As an example we give results for a quarterly tenor structure starting at April, 14, 1998. The initial yield curve $j \to R_j$, which is define here as

$$\frac{1}{B_j(0)} =: (1 + R_j)^j,$$

is plotted in figure (4). The Black caplet and swaption volatilities are given in figure (5) and table (1), respectively. For the calibration we have chosen correlation structure (21), hence the calibration parameters

$$\eta_1, \ \eta_2, \ \rho_\infty \text{ and } a, \ b, \ g_\infty.$$

The calibration procedure outlined in this section has then given the following result:

$$\begin{aligned}\eta_1 \approx 0.25,\ \eta_2 \approx, 0.00004,\ \rho_\infty \approx 0.77,\\ a \approx 3.34,\ b \approx 0.99,\ g_\infty \approx 1.96,\end{aligned} \tag{36}$$

with an RMS error of 2.1%. We note that all our tests resulted in $\eta_2 \approx 0.0$, which advocates that we actually may use the computationally faster structure (22). With this structure the calibration procedure takes typically only 30 sec. to 1 minute.

The by parameters (36) implied g " hump"-function and $\alpha-$surface are plotted in figures (6) and (7), respectively. The implied correlation structure may be computed by substituting (36) in (21). The with (21) connected $b-$sequence is plotted in figure (8).

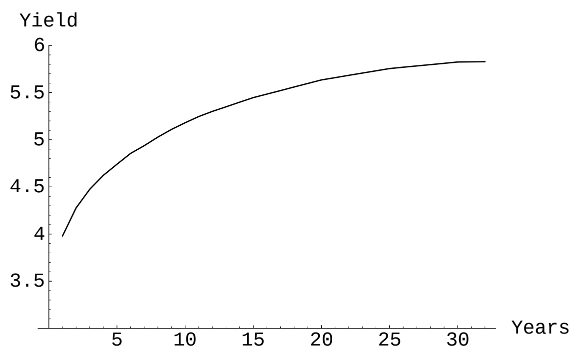

Figure 4: $j \to R_j$

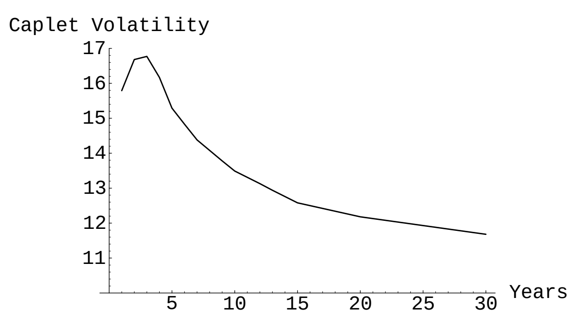

Figure 5: $j \to \gamma_j^B$

$p\backslash\backslash q$	1	2	3	4	5	6	7	8	9	10	20	29
1	16.3	15.5	14.6	14.3	13.4	12.8	12.3	12	11.7	11.4	10.2	10.2
2	15.8	15.3	14.5	13.7	12.9	12.4	11.9	11.5	11.1	9.84	9.24	
3	15	14.3	13.5	12.9	12.1	11.8	11.5	11.1	9.68	9.21		
4	14.2	13.3	12.7	12.1	11.7	11.2	10.8	9.71	9.03			
5	13.5	12.5	12	11.4	11.1	10.8	9.43	8.99				
6	12.6	11.9	11.4	10.9	10.6	9.22	8.84					
7	11.7	11.3	10.8	10.3	8.97	8.71						
8	11.4	11	10.6	8.92	8.64							
9	11.2	10.8	8.85	8.57								
10	10.9	8.76	8.52									
20	8.8											

Table 1: *Swaption volatilities*

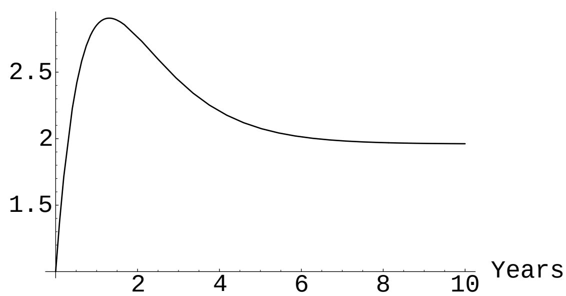

Figure 6: $s \to g_{3.34,0.99,1.96}(s)$

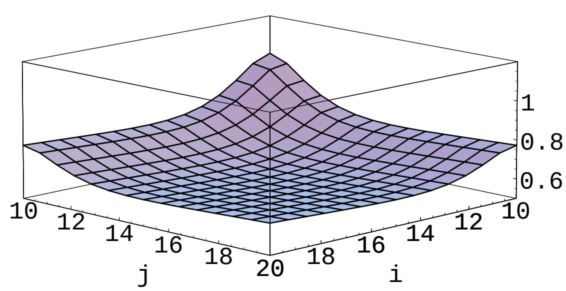

Figure 7: $\alpha_{i,j,p}^{3.34,0.99,1.96}$; $p = 10 \leq i, j \leq 20$

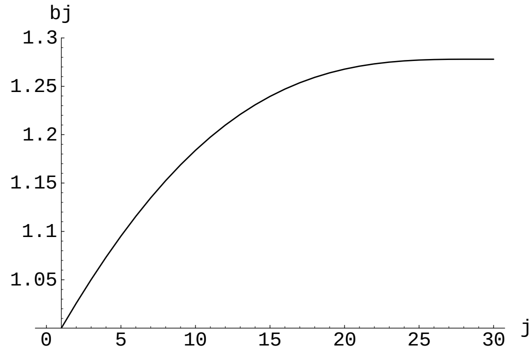

Figure 8: $j \to b_j$

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