On the joint calibration of the Libor market model to caps and swaptions market volatilities *

Damiano Brigo Cristina Capitani Fabio Mercurio

Product and Business Development Group
Banca IMI, San Paolo–IMI Group
Corso Matteotti 6
20121 Milano, Italy
Fax: + 39 02 76019324
E-mail: {brigo,fmercurio}@bancaimi.it cristina.capitani@t-online.de

First Version: January 5, 1999. This Version: July 17, 2001

Abstract

In this paper we consider several parametric assumptions for the instantaneous covariance structure of the Libor market model. We examine the impact of each different parameterization on the evolution of the term structure of volatilities in time, on terminal correlations and on the joint calibration to the caps and swaptions markets. We present a number of cases of calibration in the Euro market. In particular, we consider calibration via a parameterization establishing a controllable one to one correspondence between instantaneous covariance parameters and swaptions volatilities, and assess the benefits of smoothing the input swaption matrix before calibrating.

1 Introduction

In this paper we consider the calibration to market option data of one of the most popular and promising family of interest rate models: The Libor market model. The lognormal forward–Libor model (LFM), or Libor market model (known sometimes also as BGM model) owes its popularity to its compatibility with Black's formula for caps. Although theoretically incompatible with Black's swaption formula, the discrepancy between the swaption prices implied by the two models is usually low, so that the LFM is "practically compatible" with the swaption market formula. Before market models were introduced, there was no interest–rate dynamics that was compatible with Black's formula for both caps and swaptions.

We begin by recalling the LFM dynamics under different forward measures. We recall how caps are priced in agreement with Black's cap formula, and explain how to one can compute terminal

*The paper has been presented by the first author at the Annual Research Conference in Financial Risk, 12th - 14th July 2001, Budapest, Hungary.

correlations analytically. An approximation leading to an analytical swaption pricing formula is also recalled. We suggest several parametric forms for the instantaneous covariance structure in the LFM model, and examine their impact on the evolution of the term structure of volatilities. A part of the parameters in this structure can be obtained directly from market quoted cap volatilities, whereas other parameters can be obtained by calibrating the model to swaptions prices. The calibration to swaption prices can be made computationally efficient through the analytical approximation above. We also present a particular parametric structure inducing a one to one correspondence between covariance parameters of the LFM and market swaptions volatilities. We present some numerical cases from the market concerning the goodness of fit of the LFM to both the caps and swaptions markets, based on EURO data. We finally consider the implications of the different covariance structures in the market calibrations presented, and point out the benefits of smoothing the input swaption matrix before proceeding with the calibration.

2 The lognormal forward Libor model (LFM)

Consider a set $\mathcal{E} = \{T_0, \ldots, T_M\}$ of expiry-maturity pairs of dates for a family of spanning forward rates. We shall denote by $\{\tau_0, \ldots, \tau_M\}$ the corresponding year fractions, meaning that τ_i is the year fraction associated with the expiry-maturity pair T_{i-1}, T_i for $i > 0$, and τ_0 is the year fraction from settlement to T_0. Times T_i will be usually expressed in years from the current time. We set $T_{-1} := 0$. Consider the generic simply compounded forward rate $F_k(t) := F(t; T_{k-1}, T_k)$, $k = 1, \ldots, M$, resetting at its expiry T_{k-1} and with maturity T_k, which is "alive" up to its expiry. Consider now the probability measure Q^k associated to the zero coupon bond numeraire with maturity T_k, denoted by $P(\cdot, T_k)$. The Libor market model assumes the following (driftless) lognormal dynamics for F_k under the T_k forward adjusted measure Q^k:

$$dF_k(t) = \sigma_k(t)\ F_k(t) dZ_k^k(t), \quad t \leq T_{k-1}, \tag{1}$$

where $Z_k^k(t)$ is the k-th component of an M-dimensional Brownian motion $Z^k(t)$(under Q^k) with instantaneous covariance $\rho = (\rho_{i,j})_{i,j=1,\ldots M}$,

$$dZ^k(t)\ dZ^k(t)' = \rho\ dt\ .$$

Notice that the upper index in the Brownian motion denotes the measure, while the lower index denotes the vector component. We will often omit the upper index. The time function $\sigma_k(t)$ bears the usual interpretation of instantaneous volatility at time t for the forward Libor rate F_k. We will often consider piecewise constant instantaneous volatilities, $\sigma_k(t) = \sigma_{k,\beta(t)}$, (with $\sigma_k(0) = \sigma_{k,1}$) where in general $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$, so that $t \in (T_{\beta(t)-2}, T_{\beta(t)-1}]$. At times we will use the notation $Z_t = Z(t)$. Concerning correlations, notice that for example historical one–factor short–rate models imply forward rate dynamics that are perfectly instantaneously correlated, i.e. with $\rho_{i,j} = 1$ for all i, j. Forward rates in these models are then too correlated. One needs to lower the correlation of the forward rates implied by the model. Some authors refer to this objective as to achieving *decorrelation*. This is pursued not only by "lowering" instantaneous correlations, but also by carefully redistributing integrated variances of forward rates (obtained from market caplets) over time. See Rebonato (1998) or Brigo and Mercurio (2001) for some numerical examples.

2.1 Instantaneous volatility structures

Under the general "piecewise constant" assumption, it is possible to organize instantaneous volatilities in a matrix as follows:

TABLE 1

Instant. Vols	Time: $t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\sigma_{1,1}$	Dead	Dead	...	Dead
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	...	$\sigma_{M,M}$

Several assumptions can be made on the entries of Table 1 so as to reduce the number of volatility parameters. For an extensive description of possibilities and related implications see Brigo and Mercurio (2001). Here we only consider the following separable piecewise constant parameterization:

$$\textbf{S-P-C PARAMETERIZATION :} \qquad \sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)} \tag{2}$$

for all t. This is the product of a structure which only depends on the time–to maturity (the ψ's) by a structure which only depends on the maturity (the Φ's). Benefits of this choice will be clear later on. Now let us leave piecewise constant structures and consider now the linear-exponential formulation

$$\textbf{L-E FORMUL. :} \qquad \sigma_i(t) = \Phi_i \; \psi(T_{i-1} - t; a, b, c, d) := \Phi_i \; \left([a(T_{i-1} - t) + d] e^{-b(T_{i-1}-t)} + c \right) \; . \tag{3}$$

This form too can be seen as having a parametric core ψ, depending only on time to maturity, which is locally altered for each maturity T_i by the Φ's.

Under any of the volatility formulations, and under the driftless lognormal dynamics for F_k under Q^k, the dynamics of F_k under the forward adjusted measure Q^i in the three cases $i < k$, $i = k$ and $i > k$ are, respectively,

$$\begin{aligned} i < k, \quad & t \le T_i : \; dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j} \; \tau_j \; \sigma_j(t) \; F_j(t)}{1 + \tau_j F_j(t)} \, dt + \sigma_k(t) F_k(t) dZ_k(t), \\ i = k, \quad & t \le T_{k-1} : \; dF_k(t) = \sigma_k(t) F_k(t) dZ_k(t), \\ i > k, \quad & t \le T_{k-1} : \; dF_k(t) = -\sigma_k(t) F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j} \; \tau_j \; \sigma_j(t) \; F_j(t)}{1 + \tau_j F_j(t)} \, dt + \sigma_k(t) F_k(t) dZ_k(t), \end{aligned} \tag{4}$$

where $Z = Z^i$ is Brownian motion under Q^i. It is also easy, by the change of numeraire, to obtain the dynamics under the discrete bank-account numeraire (see for example Brigo and Mercurio (2001) for details).

The above dynamics describe the lognormal forward Libor model (LFM), and do not feature known marginal or transition densities. As a consequence, no analytical formula or simple numeric integration can be used in order to price contingent claims depending on the joint dynamics.

2.2 Calibration of the LFM to caps and floors prices

The calibration to caps and floors prices for the LFM model is almost automatic, since one can simply input in the model volatilities σ given by the market in form of Black–like implied volatilities for cap prices. Caps are a series of caplets. If the strike rate is K, each caplet indexed by i pays out $\tau_i(F_i(T_{i-1}) - K)^+$ at T_i. This looks like a call option on F_i, which has a lognormal distribution under Q^i. Indeed, in the market caplets are priced according to the Black formula, consisting of the Black and Scholes price for a stock call option whose underlying "stock" is F_i, struck at K, with maturity T_{i-1}, with 0 constant "risk-free rate" and instantaneous percentage volatility $\sigma_i(t)$:

$$\mathrm{Cpl}^{\mathrm{Black}}(0,T_{i-1},T_i,K) = P(0,T_i)\tau_i \ \mathrm{Bl}(K, F_i(0), \sqrt{T_{i-1}}\ v_{T_{i-1}-\mathrm{caplet}}) \ ,$$

$$\mathrm{Bl}(K,F,v) := F\Phi(d_1(K,F,v)) - K\Phi(d_2(K,F,v)), \quad d_{1,2}(K,F,v) := \frac{\ln(F/K) \pm v^2/2}{v},$$

where

$$v^2_{T_{i-1}-\mathrm{caplet}} := \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 dt \ . \tag{5}$$

The quantity $v_{T_{i-1}-\mathrm{caplet}}$ is termed T_{i-1}-*caplet volatility* and has thus been defined as the square root of the average percentage variance of the forward rate $F_i(t)$ for $t \in [0, T_{i-1})$. The market quotes caps, not caplets. One then may strip caplet volatilities back from cap quotes, and then work with caplet volatilities as market inputs. See Brigo and Mercurio (2001) for more details. Notice that since a cap is split additively in caplets, each depending on a single forward rate, cap prices do not depend on the instantaneous correlation ρ. Thus, only the σ's have impact on cap prices.

We now review some implication of the different volatility structures introduced in Section 2 as far as calibration is concerned. In general, for the structure of Table 1, we have

$$v^2_{T_{i-1}-\mathrm{caplet}} = \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma^2_{i,\beta(t)} dt = \frac{1}{T_{i-1}} \sum_{j=1}^{i} (T_{j-1} - T_{j-2})\ \sigma^2_{i,j} \tag{6}$$

In particular, if we assume the S-P-C parameterization, we obtain

$$T_{i-1}\ v^2_{T_{i-1}-\mathrm{caplet}} = \Phi_i^2 \sum_{j=1}^{i} (T_{j-1} - T_{j-2})\ \psi^2_{i-j+1} \ . \tag{7}$$

If the caplet volatilities are read from the market, the parameters Φ can be given in terms of the parameters ψ as

$$\Phi_i^2 = \frac{T_{i-1}\ (v^{\mathrm{MKT}}_{T_{i-1}-\mathrm{caplet}})^2}{\sum_{j=1}^{i} (T_{j-1} - T_{j-2})\ \psi^2_{i-j+1}} \ . \tag{8}$$

Therefore the caplet prices are incorporated in the model by determining the Φ's in terms of the ψ's. The parameters ψ, together with the instantaneous correlation of forward rates, can be used in the calibration to swaption prices.

Finally, if we assume the L-E Formulation for instantaneous volatilities, we obtain :

$$T_{i-1} v^2_{T_{i-1}-\mathrm{caplet}} =: \Phi_i^2 \int_0^{T_{i-1}} \left([a(T_{i-1} - t) + d] e^{-b(T_{i-1}-t)} + c \right)^2 \ dt \ = \Phi_i^2\ I^2(T_{i-1}; a, b, c, d) \ . \tag{9}$$

Now the Φ's parameters can be used to calibrate automatically caplet volatilities. Indeed, if the v_i are inferred from market data, we may set

$$\Phi_i^2 = \frac{T_{i-1} (v^{\mathrm{MKT}}_{T_{i-1}-\mathrm{caplet}})^2}{I^2(T_{i-1}; a, b, c, d)} \ . \tag{10}$$

Thus caplet volatilities are incorporated by expressing the parameters Φ's as functions of the parameters a, b, c, d, which are still free and can be used, together with instantaneous correlations ρ, to fit swaption prices.

2.3 The term structure of volatility

The term structure of volatility at time T_j is a graph of expiry times T_{h-1} against average volatilities $V(T_j, T_{h-1})$ of the forward rates $F_h(t)$ up to that expiry time itself, i.e. for $t \in (T_j, T_{h-1})$. In other terms, at time $t = T_j$, the volatility term structure is the graph of points

$$\{(T_{j+1}, V(T_j, T_{j+1})), (T_{j+2}, V(T_j, T_{j+2})), \ldots, (T_{M-1}, V(T_j, T_{M-1}))\}$$

where

$$V^2(T_j, T_{h-1}) = \frac{1}{T_{h-1} - T_j} \int_{T_j}^{T_{h-1}} \frac{dF_h(t)\ dF_h(t)}{F_h(t)F_h(t)} = \frac{1}{T_{h-1} - T_j} \int_{T_j}^{T_{h-1}} \sigma_h^2(t)dt$$

for $h > j + 1$. The term structure of volatilities at time 0 is given simply by caplets volatilities plotted against their expires.

Different assumptions on the behaviour of instantaneous volatilities imply different evolutions for the term structure of volatilities in time as $t = T_0$, $t = T_1$, etc. We now examine the impact of two different formulations of instantaneous volatilities on the evolution of the term structure. Under the S-P-C parameterization, we have two extreme situations. If all Φ's are equal, the term structure remains the same in time: As we plot it at later instants, it loses its tail. On the contrary, if all ψ are equal, the structure changes: As we plot it at later instants, it loses its head, thus losing the humped shape that is usually considered a good qualitative property. The general S-P-C case lies in between, and by constraining all Φ's to be close to one, we can obtain a "hump-maintaining" evolution. Details, examples, and figures in Brigo and Mercurio (2001). Finally, consider the L-E formulation. Given the same separable structure, the same qualitative behaviour can be expected: The term structure of volatilities can maintain its humped shape if initially humped and if all Φ's are close to one. This form has been suggested for example by Rebonato (1999).

2.4 An approximated formula for terminal correlations

In general, if one is interested in terminal correlations of forward rates at a future time instant, as implied by the LFM model, then the computation has to be based on a Monte Carlo simulation technique. Indeed, assume we are interested in computing the terminal correlation between forward rates F_i and F_j at time T_α, $\alpha \leq i - 1 < j$, say under the measure Q^γ, $\gamma \geq \alpha$. Then we need to compute

$$\text{Corr}^\gamma(F_i(T_\alpha), F_j(T_\alpha)) = \frac{E^\gamma\left[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))\right]}{\sqrt{E^\gamma[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))^2]}\ \sqrt{E^\gamma[(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))^2]}} \qquad (11)$$

Recalling the dynamics of F_i and F_j under Q^γ, the expected values appearing in the above expression can be obtained by simulating the above dynamics for $k = i$ and $k = j$ respectively, thus simulating F_i and F_j up to time T_α. The simulation can be based for example on a discretized Milstein dynamics. However, at times traders may need to quickly check reliability of the model's terminal correlations, so that there could be no time to run a Monte Carlo simulation. Fortunately, there does exist approximated formulas that allow us to compute terminal correlations algebraically from the LFM parameters ρ and $\sigma(\cdot)$. The first approximation we introduce is a partial freezing of the drift in the dynamics, and a collapse of all forward measures. Following Brigo and Mercurio (2001), we obtain easily

$$\text{Corr}(F_i(T_\alpha), F_j(T_\alpha)) = \frac{\exp\left(\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\rho_{i,j}dt\right) - 1}{\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_i^2(t)dt\right) - 1}\ \sqrt{\exp\left(\int_0^{T_\alpha} \sigma_j^2(t)dt\right) - 1}}\ . \qquad (12)$$

Notice that a first order expansion of the exponentials appearing in this last formula yields a second formula (Rebonato's (1999) terminal correlation formula)

$$\mathrm{Corr}^{\mathrm{REB}}(F_i(T_\alpha), F_j(T_\alpha)) = \rho_{i,j} \frac{\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\ dt}{\sqrt{\int_0^{T_\alpha} \sigma_i^2(t)dt}\ \sqrt{\int_0^{T_\alpha} \sigma_j^2(t)dt}} \ . \tag{13}$$

An immediate application of Schwartz's inequality shows that *terminal correlations are always smaller, in absolute value, than instantaneous correlations* when computed via Rebonato's formula. In agreement with this general observation, recall that through a careful repartition of integrated volatilities (caplets) in instantaneous volatilities $\sigma_i(t)$ and $\sigma_j(t)$ we can make the terminal correlation $\mathrm{Corr}^{\mathrm{REB}}(F_i(T_\alpha), F_j(T_\alpha))$ arbitrarily close to zero, even when the instantaneous correlation $\rho_{i,j}$ is one. See Brigo and Mercurio (2001) for more details, examples, and for numerical tests against Monte Carlo showing that the above approximations are both good in non-pathological situations.

2.5 Swaptions and the lognormal forward–swap model (LSM)

Denote by $S_{\alpha,\beta}(t)$ a forward swap rate at time t for a swap first resetting at T_α and exchanging payments at $T_{\alpha+1}, \ldots, T_\beta$. Consider now the related swaption with strike K. If we assume unit notional amount, the swaption payoff at maturity T_α can be written as

$$(S_{\alpha,\beta}(T_\alpha) - K)^+\ C_{\alpha,\beta}(T_\alpha), \quad C_{\alpha,\beta}(t) := \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i). \tag{14}$$

The process C is the so called present value for basis point and is a numeraire under which the forward swap rate $S_{\alpha,\beta}$ follows a martingale. Let $Q^{\alpha,\beta}$ denote the related (swap) measure. By assuming a lognormal dynamics,

$$d\ S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\ dW_t^{\alpha,\beta}, \tag{15}$$

where $W^{\alpha,\beta}$ is a standard Brownian motion under $Q^{\alpha,\beta}$, we obtain the price as Black's formula for swaptions, where the volatility parameter is given by the square root of the average percentage variance $v^2_{\alpha,\beta}(T_\alpha)$ of the forward swap rate,

$$v^2_{\alpha,\beta}(T) = \frac{1}{T} \int_0^T (\sigma^{(\alpha,\beta)}(t))^2\ dt. \tag{16}$$

This model for the evolution of forward swap rates is known as lognormal forward–swap model (LSM), since each swap rate $S_{\alpha,\beta}$ has a lognormal distribution under its swap measure $Q^{\alpha,\beta}$. However, this model is not compatible with the LFM, since the distributions of S under a given numeraire are different under the two models (see for example Brigo and Mercurio (2001)).

However, we have chosen the LFM as central model, so that we need to price swaptions under the LFM. Recall that the forward swap rate can be expressed as a quasi-average of spanning forward rates:

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)\ F_i(t), \quad w_i(F_{\alpha+1}(t), \ldots, F_\beta(t)) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}} \ . \tag{17}$$

Then we can simulate forward rates through a discretization of the LFM dynamics (4), so as to obtain zero coupon bonds P and the forward swap rate (17). It is then possible to price the swaption with a Monte Carlo simulation. Usually analytical approximations are available for swaptions in the LFM, as we will see. Finally, notice that, contrary to caps, swaption payoffs cannot be decomposed additively in payoffs depending only on single forward rates, so that swaption prices will depend on the correlation ρ, differently from caps.

In general the full instantaneous correlation matrix ρ needed to price a swaption features $M\ (M-1)/2$ parameters, which can be a lot (where now $M = \beta - \alpha$ is the number of forward rates embedded in the relevant swap rate). Therefore, a parsimonious parametric form has to be found for ρ, based on a reduced number of parameters. We will take $dZ(t) = B\, dW(t)$, with W an n-dimensional standard Brownian motion, $n \leq M$. $B = (b_{i,j})$ is a suitable n-rank $M \times n$ matrix such that $\rho^B = BB'$ is an n-rank correlation matrix. If $n << M$ this reduces drastically the noise factors. We will take $n = 2$. It was observed both in Brace, Dun and Barton (1998) and in De Jong, Driessen and Pelsser (1999) that two factors are usually enough, provided one chooses a flexible volatility structure. We follow Rebonato (1999) by taking M angles θ to parameterize B:

$$b_{i,1} = \cos\theta_i, \quad b_{i,2} = \sin\theta_i, \quad \rho^B_{i,j} = b_{i,1}b_{j,1} + b_{i,2}b_{j,2} = \cos(\theta_i - \theta_j)\ , \quad i = 1, \ldots, M. \tag{18}$$

2.6 An approximated formula for Black swaptions volatilities

Recall the forward swap rate dynamics (15) underlying the LSM model, leading to Black's formula for swaptions. A crucial role in the LSM model is played by the Black swap volatility $v_{\alpha,\beta}(T_\alpha)$ entering Black's formula for swaptions, expressed by (16). One can compute, under a number of approximations, an analogous quantity $v^{\text{LFM}}_{\alpha,\beta}$ in the LFM model. We have presented a derivation of two such approximations in Brigo and Mercurio (2001). These are based again on "partially freezing the drift" and on "collapsing all measures" in the LFM dynamics. These formulas appeared earlier in Rebonato (1998) and Hull and White (1999) respectively. In Brigo and Mercurio (2001) we have also tested both formulas against Monte Carlo simulations, and found that the differences are negligible in non-pathological situations, so that we present here only the simplest formula:

$$(v^{\text{LFM}}_{\alpha,\beta})^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\ dt\ . \tag{19}$$

The quantity $v^{\text{LFM}}_{\alpha,\beta}$ can be used as a proxy for the Black volatility $v_{\alpha,\beta}(T_\alpha)$ of the swap rate $S_{\alpha,\beta}$. Putting this quantity in Black's formula for swaptions allows one to compute approximated swaptions prices with a closed form formula under the LFM model.

2.7 Calibration to swaptions prices

Since traders already know standard swaptions prices from the market, they wish a chosen model to incorporate as many such prices as possible. In case we are adopting the LFM model, we need to find the instantaneous volatility and correlation parameters σ and ρ in the LFM dynamics that reflect the swaptions prices observed in the market. First let us quickly see how the market organizes swaption prices in a table. Traders consider a matrix of at-the-money-swaptions prices or Black's swaptions volatilities organized as follows. To simplify ideas, assume we are interested only in swaptions with maturity and underlying swap length given by multiples of one year. Typically one then organizes data in a matrix, where each row is indexed by the swaption maturity T_α, whereas each column is indexed in terms of the underlying swap length, $T_\beta - T_\alpha$. The $x \times y$-swaption is then the swaption in the table whose maturity is x-years and whose underlying swap is y years long. Here we consider maturities of $1, 2, 3, 4, 5, 7, 10$ years and underlying swap lengths of $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ years. A typical example of table of swaption volatilities is shown below:

Black implied volatilities of at the money swaptions from INTERCAPITAL on may 16, 2000.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4%	15.8%	14.6%	13.8%	13.3%	12.9%	12.6%	12.3%	12.0%	11.7%
2y	17.7%	15.6%	14.1%	13.1%	12.7%	12.4%	12.2%	11.9%	11.7%	11.4%
3y	17.6%	15.5%	13.9%	12.7%	12.3%	12.1%	11.9%	11.7%	11.5%	11.3%
4y	16.9%	14.6%	12.9%	11.9%	11.6%	11.4%	11.3%	11.1%	11.0%	10.8%
5y	15.8%	13.9%	12.4%	11.5%	11.1%	10.9%	10.8%	10.7%	10.5%	10.4%
7y	14.5%	12.9%	11.6%	10.8%	10.4%	10.3%	10.1%	9.9%	9.8%	9.6%
10y	13.5%	11.5%	10.4%	9.8%	9.4%	9.3%	9.1%	8.8%	8.6%	8.4%

Its entries are the implied volatilities obtained by inverting the related at-the-money swaption prices through Black's formula for swaptions. However, this matrix is not necessarily updated uniformly: While the most liquid swaptions are updated regularly, some entries refer to older market situations. This "temporal misalignment" can cause troubles, since when we try a calibration the model parameters might reflect this misalignment by assuming weird, almost "singular" values. If one trusts the model, this can indeed be used in turn to detect these misalignments: The model is calibrated to the liquid swaptions, and then one prices the remaining swaptions, looking at the values that most differ from the corresponding market values. Instead, in case we can trust the matrix, our purpose is incorporating as much information as possible from such a table in the LFM model parameters. To focus ideas, consider the LFM model with volatility Formulation L-E (given by Formula (3)) and rank-2 correlations expressed by the angles θ. We search for those values of the parameters a, b, c, d and θ (Φ's being determined by (10)) such that the LFM swaption prices approach as much as possible the market prices given in the table. We can for instance minimize the sum of the squares of the differences of the corresponding swaption prices. We approach this problem by using Rebonato's formula to price swaptions under the LFM model.

We need also to point out that of late attention has moved on a pre-selected instantaneous correlation matrix. Typically, one estimates the instantaneous correlation ρ historically from time series of zero rates at a given set of maturities. With this approach, the parameters that are left for the swaptions calibration are the free parameters in the volatility structure: with the L-E Formulation, the Φ's are determined by the cap market as functions of a, b, c and d, which are in turn parameters to be used in the calibration to swaptions prices. However, if the number of swaptions is large, four parameters are not sufficient for practical purposes. One then needs to consider richer parametric forms for the instantaneous volatility, and we will see that in many respects the easiest calibration is obtained with the general piecewise constant structure of Table 1 when instantaneous correlations are left out of the calibration.

3 Calibration to caps and/or swaptions: Market cases

We now consider some examples of joint calibration of the LFM model to caps/floors and swaptions. We will investigate the difficulties of such a task when using different parameterizations of the instantaneous covariance structure of the model. However, before proceeding, we would like to make ourselves clear by pointing out that the examples presented here are a first attempt at underlying the relevant choices as far as the LFM model parameterization is concerned. We do not pretend to be exhaustive or even particularly systematic in these examples, and we do not employ statistical testing or econometric techniques in our analysis. We will base our considerations only on cross sectional calibration to the market quoted volatilities, *although we will check implications of the obtained calibrations as far as the future time-evolution of key structures of the market are concerned*. Within such cross-sectional approach we are not being totally systematic either. We are aware there are several other issues concerning the number of factors, different possible parameterizations, modeling correlations implicitly as inner products of vector-instantaneous volatilities, and on and on. Yet, most of the available literature on interest rate models does not deal with the questions and examples we raise here on the market model. We thought about presenting the examples below in order to let the reader appreciate what are the current problems with the LFM model, especially as far as practitioners and traders are concerned.

We will try and calibrate the following data: "Annualized" initial curve of forward rates, "annualized" caplet volatilities, and swaptions volatilities.

We actually take as input the vector of initial semi-annual forward rates as of may 16, 2000, and the semi-annual caplet volatilities, $v0 = [v_{1y-\text{Caplet}}, v_{1.5y-\text{Caplet}}, \dots, v_{19.5y-\text{Caplet}}]$ the first for the semi-annual caplet resetting in one year and paying at 1.5y, the last for the semi-annual caplet resetting in 19.5y and paying at 20y, all other reset dates being six-months spaced.

These volatilities have been provided by our interest rate traders, based on a stripping algorithm combined with personal adjustments applied to cap volatilities.

We transform semi-annual data in annual data and work with the annual forward rates $F0 = [F(\cdot; 1y, 2y), F(\cdot; 2y, 3y), \dots, F(\cdot; 19y, 20y)]$ and their associated yearly caplet volatilities. The "annualizing procedure" is outlined in Brigo and Mercurio (2001). In the transformation formula, infra-correlations are set to one. Notice that infra-correlations might be kept as further parameters to ease the calibration. In our data from may 16, 2000, the initial spot rate is $F(0; 0, 1y) = 0.0452$, and the other initial forward rates and caplet volatilities are showed in the first part of the table in the next subsection. Finally, the values of swaptions volatilities are the same as in the table given in Section 2.7.

3.1 Joint calibration with the S-P-C structure

In order to satisfactorily calibrate the above data with the LFM model, we first try the S-P-C volatility structure, with a local algorithm of minimization for finding the fitted parameters $\psi_1, \dots, \psi_{19}$ and $\theta_1, \dots, \theta_{19}$ starting from the initial guesses $\psi_i = 1$ and $\theta_i = \pi/2$. We thus adopt Formula (2) for instantaneous volatilities and obtain the Φ's directly as functions of the parameters ψ by using the (annualized) caplet volatilities and Formula (8). We compute swaptions prices as functions of the ψ's and θ's by using Rebonato's Formula (19). We also impose the constraints $-\pi/2 < \theta_i - \theta_{i-1} < \pi/2$ to the correlation angles. This implies that $\rho_{i,i-1} > 0$. We thus require that adjacent rates have positive correlations. As we shall see, this requirement is obviously too weak to guarantee the instantaneous correlation matrix coming from the calibration to be "reasonable". The inputs and the parameters we obtain are showed in the following table:

Index	initial $F0$	v_{caplet}
1	0.050114	0.180253
2	0.055973	0.191478
3	0.058387	0.186154
4	0.060027	0.177294
5	0.061315	0.167887
6	0.062779	0.158123
7	0.062747	0.152688
8	0.062926	0.148709
9	0.062286	0.144703
10	0.063009	0.141259
11	0.063554	0.137982
12	0.064257	0.134708
13	0.064784	0.131428
14	0.065312	0.128148
15	0.063976	0.127100
16	0.062997	0.126822
17	0.061840	0.126539
18	0.060682	0.126257
19	0.059360	0.125970

Index	ψ	Φ	θ
1	1.3392	0.1346	2.3649
2	1.3597	0.1419	1.2453
3	1.1771	0.1438	0.7676
4	1.3563	0.1353	0.3877
5	1.2807	0.1287	0.2870
6	1.2696	0.1218	1.8578
7	1.1593	0.1193	1.9923
8	1.7358	0.1105	2.2133
9	0.8477	0.1114	2.1382
10	1.9028	0.1030	2.4456
11	0.6717	0.1043	1.5708
12	2.8610	0.0891	1.5708
13	0.6620	0.0897	1.5708
14	1.2649	0.0883	1.5708
15	0.1105	0.0906	1.5708
16	0.0000	0.0934	1.5708
17	0.0000	0.0961	1.5708
18	0.0000	0.0986	1.5708
19	0.6551	0.1004	1.5708

where the Φ's have been computed through (8). The fitting quality is as follows. The caplets are fitted exactly, whereas we calibrated the whole swaptions volatility matrix except for the first column of $S \times 1$ swaptions. This is left aside because of possible misalignments with the annualized caplet volatilities, since we are basically quoting twice the same volatilities. A more complete approach can be obtained by keeping semi-annual volatilities and by introducing semi-annual infra-correlations as new fitting parameters. The matrix of percentage errors in the swaptions calibration,

100*(Market swaptions vol - LFM swaption vol)/Market swaptions vol

is reported below.

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.24%	1.29%	0.61%	-0.59%	-0.28%	0.97%	-0.64%	1.02%	-0.87%
2y	-1.65%	-1.29%	-1.09%	-0.33%	0.03%	-0.61%	-0.37%	-1.04%	-0.89%
3y	1.03%	1.11%	0.51%	1.45%	0.79%	1.08%	1.03%	1.30%	0.79%
4y	0.13%	-1.05%	-0.80%	-0.29%	-0.33%	-0.16%	0.49%	0.02%	0.23%
5y	0.89%	0.07%	-0.09%	-0.16%	-1.27%	-0.50%	0.00%	-0.80%	0.37%
7y	1.15%	0.53%	0.59%	0.12%	-0.33%	-0.66%	-0.58%	0.10%	0.93%
10y	-1.23%	-0.61%	0.64%	0.07%	0.46%	0.45%	-0.37%	-0.64%	-0.20%

Errors are actually small, and from the point of view of the fitting error, this calibration would seem to be satisfactory, considering that we are trying to fit 19 caplets and 63 swaption volatilities!

The first observation is that the calibrated θ's above imply quite erratic instantaneous correlations. Consider the first ten columns of ρ^B:

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	1.000	0.436	- 0.026	- 0.395	- 0.486	0.874	0.931	0.989	0.974	0.997
2y	0.436	1.000	0.888	0.654	0.575	0.818	0.734	0.567	0.627	0.362
3y	- 0.026	0.888	1.000	0.929	0.887	0.462	0.339	0.125	0.199	- 0.107
4y	- 0.395	0.654	0.929	1.000	0.995	0.101	- 0.034	- 0.252	- 0.179	- 0.468
5y	- 0.486	0.575	0.887	0.995	1.000	0.000	- 0.134	- 0.348	- 0.277	- 0.555
6y	0.874	0.818	0.462	0.101	0.000	1.000	0.991	0.937	0.961	0.832
7y	0.931	0.734	0.339	- 0.034	- 0.134	0.991	1.000	0.976	0.989	0.899
8y	0.989	0.567	0.125	- 0.252	- 0.348	0.937	0.976	1.000	0.997	0.973
9y	0.974	0.627	0.199	- 0.179	- 0.277	0.961	0.989	0.997	1.000	0.953
10y	0.997	0.362	- 0.107	- 0.468	- 0.555	0.832	0.899	0.973	0.953	1.000
11y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
12y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
13y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
14y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
15y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
16y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
17y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
18y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641
19y	0.701	0.947	0.694	0.378	0.283	0.959	0.912	0.801	0.843	0.641

Correlations oscillate occasionally between positive and negative values. At the same time, from a certain index on, they become constant. This is too a weird behaviour to be trusted. Terminal correlations computed through Formula (13) are in this case, after ten years,

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y
10y	1.000	0.574	0.625	0.525	0.602	0.500	0.505	0.445	0.418	0.413
11y	0.574	1.000	0.801	0.976	0.805	0.926	0.750	0.760	0.646	0.641
12y	0.625	0.801	1.000	0.712	0.917	0.653	0.745	0.590	0.588	0.597
13y	0.525	0.976	0.712	1.000	0.697	0.906	0.622	0.721	0.550	0.576
14y	0.602	0.805	0.917	0.697	1.000	0.655	0.866	0.570	0.666	0.540
15y	0.500	0.926	0.653	0.906	0.655	1.000	0.627	0.855	0.524	0.641
16y	0.505	0.750	0.745	0.622	0.866	0.627	1.000	0.597	0.846	0.487
17y	0.445	0.760	0.590	0.721	0.570	0.855	0.597	1.000	0.556	0.825
18y	0.418	0.646	0.588	0.550	0.666	0.524	0.846	0.556	1.000	0.516
19y	0.413	0.641	0.597	0.576	0.540	0.641	0.487	0.825	0.516	1.000

and they still look erratic, although to a lesser degree. Finally, let us have a look at the time evolution of caplet volatilities. We know that the model reproduces exactly the initial caplet volatility structure observed in the market. However, as time passes, the above ψ and Φ parameters imply the evolution shown in Figure 1. This evolution shows that the structure loses the "humped shape" after a short time. Moreover, it becomes somehow "noisy". What is one to learn from such an example? Well, the fitting quality is not the only criterion by which a calibration session has to be judged. A trader has to decide whether he is willing to sacrifice part of the fitting quality for a better evolution in time of the key structures.

We have tried several other calibrations with the S-P-C parameterization. We tried to impose more stringent constraints on the angles θ, and we even fixed them both to typical and atypical values, leaving the calibration to the volatility parameters. We also let all instantaneous correlations go to one, so as to have a one-factor LFM to be calibrated only through its instantaneous volatility parameters. Details and Figures in Brigo and Mercurio (2001). We reached the following conclusions. In order to have a good calibration to swaptions data we need to allow for at least partially oscillating patterns in the correlation matrix. If we force a given "smooth/monotonic" correlation matrix into the calibration and rely upon volatilities, the results are the same as in the case of a one-factor LFM model where correlations are all set to one. This kind of results

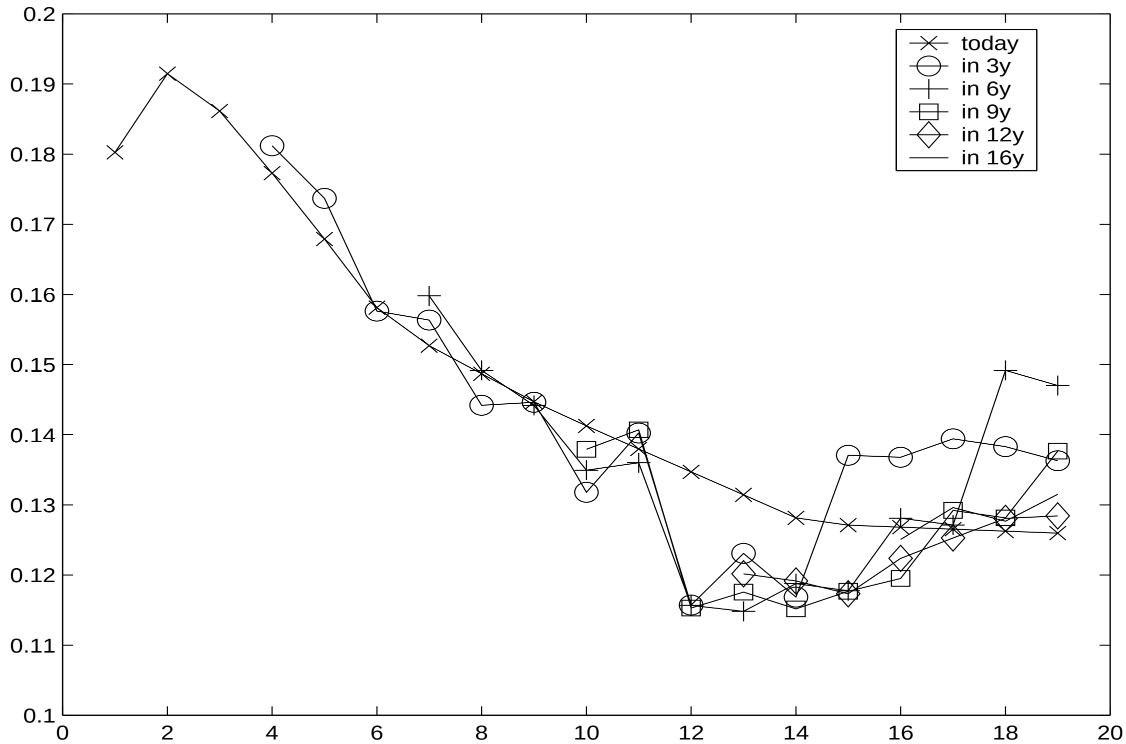

Figure 1: Evolution of the term structure of caplet volatilities

suggests some considerations. Since by fixing rather different instantaneous correlations the calibration does not change that much, probably instantaneous correlations do not have a strong link with European swaptions prices. Therefore swaptions volatilities do not always contain clear and precise information on instantaneous correlations of forward rates. This was clearly stated also in Rebonato (1999). On the other hand, one suspects that this permanence of "bad results", no matter the particular "smooth" choice of fixed instantaneous correlation, might reflect an impossibility of a low rank formulation to decorrelate quickly the forward rates in a steeper initial pattern. It might happen that columns of an instantaneous correlations matrix of a low rank formulation have a sigmoid like shape that cannot decrease quickly initially. Instantaneous correlations in the rank two model are then trying to mimic something like a steep initial pattern by a sigmoid like shape through an oscillating behaviour. The obvious remedy would be to increase drastically the number of factors, but this is computationally undesirable. We have tried experiments with three factor correlation matrices, but we have obtained results analogous to the two factor case. And resorting to a nineteen dimensional model is not desirable in terms of implementation issues when pricing exotics with Monte Carlo. See also Rebonato (1998) on the "sigmoidal" correlation structure typical of models with a low number of factors.

3.2 Joint calibration with the L-E parameterization

We try a local algorithm of minimization for finding the fitted parameters a, b, c, d and $\theta_1, \ldots, \theta_{19}$ starting from the initial guesses $a = 0.0285$, $b = 0.20004$, $c = 0.1100$, $d = 0.0570$, and initial θ components ranging from $\theta_1 = 0$ to $\theta_{19} = 2\pi$ and equally spaced. The Φ's are obtained as functions of a, b, c, d through caplet volatilities according to Formula (10), and this is the caplet calibration part. As for swaptions, we compute swaptions prices as functions of a, b, c, d and the θ's by using

Rebonato's Formula (19). To this end, the lengthy computation of terms such as

$$\int_0^T \psi(T_{i-1} - t; a, b, c, d)\ \psi(T_{j-1} - t; a, b, c, d)\ dt$$

has to be carried out. This can be done easily with software for formal manipulations such as for example MAPLE, with a command line such as

```
int(((a*(S-t)+d)*exp(-b*(S-t))+c)*((a*(T-t)+d)*exp(-b*(T-t))+c),t=t1..t2);
```

We also impose the constraints $-\pi/3 < \theta_i - \theta_{i-1} < \pi/3$, $0 < \theta_i < \pi$ to the correlation angles. Finally, the local minimization is constrained by the requirement

$$1 - 0.1 \leq \Phi_i(a, b, c, d) \leq 1 + 0.1\ ,$$

for all i. This constraint ensures that all Φ's will be close to one, so that the qualitative behaviour of the term structure should be preserved in time. Moreover, with this parameterization we can expect a smooth shape for the term structure of volatilities at all instants, since with linear/exponential functions we avoid the typical erratic behaviour of piecewise constant formulations. For the actual calibration we use only volatilities in the swaptions matrix corresponding to the 2y, 5y and 10y columns, in order to speed up the constrained optimization. The local optimization routine produced the following parameters:

$$a = 0.29342753, \quad b = 1.25080230, \quad c = 0.13145869, \quad d = 0.00000000.$$

$$\theta_{1\div 10} = \ [1.75411\ 0.57781\ 1.68501\ 0.58176\ 1.53824\ 2.43632\ 0.88011\ 1.89645\ 0.48605\ 1.28020],$$

$$\theta_{11\div 19} = \ [2.44031\ 0.94480\ 1.34053\ 2.91133\ 1.99622\ 0.70042\ 0\ 0.81518\ 2.38376].$$

Notice that $d = 0$ has reached the lowest value allowed by the positivity constraint, meaning that possibly the optimization would have improved with a negative d. We allowed d to go negative in other cases in Brigo and Mercurio (2001). The instantaneous correlations resulting from this calibration are again oscillating and non-monotonic. We find some repeated oscillations between positive and negative values that are not desirable. Terminal correlations share part of this negative behaviour. However, this time the evolution in time of the term structure of caplet volatilities looks good, as shown in Figure 2. There remains to see the fitting quality. Recall that caplets are fitted exactly, whereas we have fitted only the 2y, 5y and 10y columns of the swaptions volatilities.

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	2.28%	-3.74%	-3.19%	-4.68%	2.46%	1.50%	0.72%	1.33%	-1.42%
2y	-1.23%	-7.67%	-9.97%	2.10%	0.49%	1.33%	1.56%	-0.44%	1.88%
3y	2.23%	-6.20%	-1.30%	-1.32%	-1.43%	1.86%	-0.19%	2.42%	1.17%
4y	-2.59%	9.02%	1.70%	0.79%	3.22%	1.19%	4.85%	3.75%	1.21%
5y	-3.26%	-0.28%	-8.16%	-0.81%	-3.56%	-0.23%	-0.08%	-2.63%	2.62%
7y	0.10%	-2.59%	-10.85%	-2.00%	-3.67%	-6.84%	2.15%	1.19%	0.00%
10y	0.29%	-3.44%	-11.83%	-1.31%	-4.69%	-2.60%	4.07%	1.11%	0.00%

In such columns percentage differences reach at most 4.68%, and are usually quite smaller. The remaining data are also reproduced with small errors, even if they have not been included in the calibration, with a number of exceptions. Notably, the 10×4 swaption features for example a difference of 11.83%. However, the last columns show relatively small errors, so that the 7y, 8y and 9y columns seem to be rather aligned with the 5y and 10y columns . On the contrary, the 3y and 4y columns seem to be rather misaligned with the 2y and 5y columns, since they show larger errors.

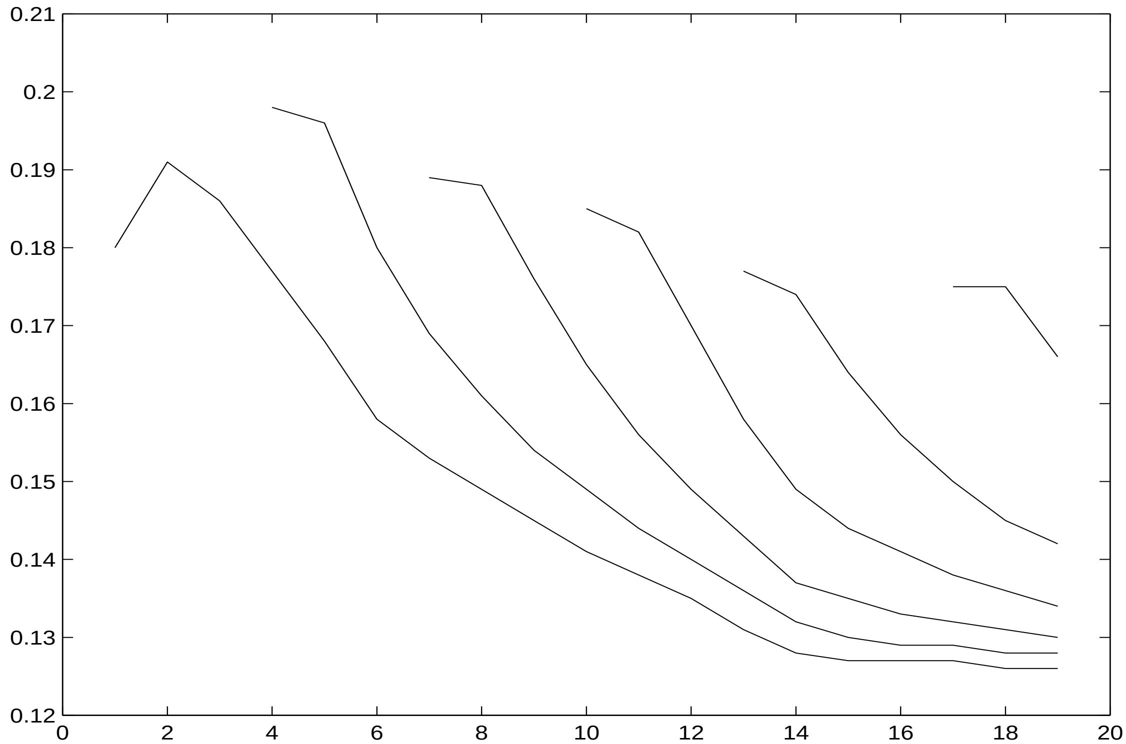

Figure 2: Evolution of the term structure of caplet volatilities

We have performed many more experiments with this choice of volatility, and some are reported in Brigo and Mercurio (2001). We have also tried rank three correlations structures, less or more stringent constraints on the angles and on the Φ's, and so on. A variety of results have been obtained. In general we can say that the fitting to the whole swaption matrix can be improved, but at the cost of an erratic behaviour of both correlations and of the evolution of the term structure of volatilities in time. The three factor choice does not seem to help that much. The above example is sufficient to let one appreciate both the potential and the disadvantages of this parameterization with respect to the piecewise constant case. In general, this parameterization allows for an easier control of the evolution of the term structure of volatilities, but produces more erratic correlation structures, since most of the "noise" in the swaptions data now ends up in the angles, due to the fact that we have only four volatility parameters a, b, c, d that can be used to calibrate swaption volatilities. A different possible use of the model, however, is to limit the calibration to act only on swaption prices, by ignoring the cap market, or by keeping it for testing the caps/swaptions misalignment a posteriori. With this approach the Φ's become again free parameters to be used in the swaption calibration, and are no longer functions of a, b, c, d imposed by the caplet volatilities. In this case we obtained a good fitting to market data, not so good instantaneous correlations, interesting terminal correlations and relatively satisfactory evolution of the term structure of volatilities in time.

3.3 Calibration with general piecewise constant volatilities

Now we examine the structure with the largest number of parameters. One would expect this structure to lead to a complex calibration routine, requiring optimization in a space of huge dimension. Instead, it turns out that with this structure and by assuming exogenously given instantaneous correlations ρ, *the calibration can be carried out through closed form formulas having as inputs the exogenous correlations and the swaption volatilities.* In our case Formula (19) reads

$$(v_{\alpha,\beta})^2 \approx \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{T_\alpha \; S_{\alpha,\beta}(0)^2} \sum_{h=0}^{\alpha}(T_h - T_{h-1}) \; \sigma_{i,h+1} \; \sigma_{j,h+1} \; . \tag{20}$$

Keep in mind that the weights w are specific of the swaption being considered, i.e. they depend on α and β. In order to effectively illustrate the calibration in this case without getting lost in notation and details, let us work on an example with just six swaptions. The procedure will be generalized later to an arbitrary number of swaptions.

Suppose we start from the swaptions volatilities in the upper half of the swaption matrix:

Length / Maturity	1y	2y	3y
$T_0 = 1$y	$v_{0,1}$	$v_{0,2}$	$v_{0,3}$
$T_1 = 2$y	$v_{1,2}$	$v_{1,3}$	-
$T_2 = 3$y	$v_{2,3}$	-	-

Let us now move along this table. Let us start from the $(1,1)$ entry $v_{0,1}$. Use the approximated Formula (20) and compute, after straightforward simplifications, $(v_{0,1})^2 \approx \sigma_{1,1}^2$.

This formula is immediately invertible and provides us with the volatility parameter $\sigma_{1,1}$ as a function of the swaption volatility $v_{0,1}$. Now move on to the right, to entry $(1,2)$, containing $v_{0,2}$. The same formula gives, this time,

$$\begin{aligned} S_{0,2}(0)^2(v_{0,2})^2 \;\; &\approx \;\; w_1(0)^2F_1(0)^2\sigma_{1,1}^2 + w_2(0)^2F_2(0)^2\sigma_{2,1}^2 \\ &\quad +2\rho_{1,2}w_1(0)F_1(0)w_2(0)F_2(0)\sigma_{1,1}\sigma_{2,1} \; . \end{aligned}$$

Everything in this formula is known, except $\sigma_{2,1}$. We then solve the elementary-school algebraic second order equation in $\sigma_{2,1}$, and recover analytically $\sigma_{2,1}$ in terms of the previously found $\sigma_{1,1}$ and of the known swaptions data. Now move on to the right again, to entry $(1,3)$, containing $v_{0,3}$. The same formula gives, this time,

$$\begin{aligned} S_{0,3}(0)^2(v_{0,3})^2 \;\; &\approx \;\; w_1(0)^2F_1(0)^2\sigma_{1,1}^2 + w_2(0)^2F_2(0)^2\sigma_{2,1}^2 \\ &\quad +w_3(0)^2F_3(0)^2\sigma_{3,1}^2 + 2\rho_{1,2}w_1(0)F_1(0)w_2(0)F_2(0)\sigma_{1,1}\sigma_{2,1} \\ &\quad +2\rho_{1,3}w_1(0)F_1(0)w_3(0)F_3(0)\sigma_{1,1}\sigma_{3,1} + 2\rho_{2,3}w_2(0)F_2(0)w_3(0)F_3(0)\sigma_{2,1}\sigma_{3,1}. \end{aligned}$$

Once again, everything in this formula is known, except $\sigma_{3,1}$. We then solve in $\sigma_{3,1}$. Now move on to the second row of the swaptions matrix, entry (2,1), containing $v_{1,2}$. Our formula gives now

$$T_1 \; v_{1,2}^2 \approx T_0 \; \sigma_{2,1}^2 + (T_1 - T_0) \; \sigma_{2,2}^2 \; .$$

This time everything is known except $\sigma_{2,2}$. Once again, we solve explicitly this equation for $\sigma_{2,2}$, being $\sigma_{2,1}$ known from previous passages. Now move on to the right, entry (2,2), containing $v_{1,3}$. Our formula gives now

$$\begin{aligned} T_1 \; S_{1,3}(0)^2v_{1,3}^2 \;\; &\approx \;\; w_2(0)^2F_2(0)^2(T_0\sigma_{2,1}^2 + (T_1 - T_0)\sigma_{2,2}^2) \\ &\quad +w_3(0)^2F_3(0)^2(T_0\sigma_{3,1}^2 + (T_1 - T_0)\sigma_{3,2}^2) \\ &\quad +2\rho_{2,3}w_2(0)F_2(0)w_3(0)F_3(0)(T_0\sigma_{2,1}\sigma_{3,1} + (T_1 - T_0)\sigma_{2,2}\sigma_{3,2}). \end{aligned}$$

Here everything is known except $\sigma_{3,2}$. Once again, we solve explicitly this equation for $\sigma_{3,2}$. Finally, we move to the only entry (3,1) of the third row, containing $v_{2,3}$. The usual formula gives

$$T_2\ v_{2,3}^2 \approx T_0\sigma_{3,1}^2 + (T_1 - T_0)\sigma_{3,2}^2 + (T_2 - T_1)\sigma_{3,3}^2\ .$$

The only entry unknown at this point is $\sigma_{3,3}$, that can be easily found by explicitly solving this last equation.

We have been able to find all instantaneous volatilities. A table summarizing the dependence of the swaptions volatilities v from the instantaneous forward volatilities σ is the following.

Length Maturity	1y	2y	3y
$T_0 = 1$y	$v_{0,1}$ $\sigma_{1,1}$	$v_{0,2}$ $\sigma_{1,1}, \sigma_{2,1}$	$v_{0,3}$ $\sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}$
$T_1 = 2$y	$v_{1,2}$ $\sigma_{2,1}, \sigma_{2,2}$	$v_{1,3}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}$	- -
$T_2 = 3$y	$v_{2,3}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$	-	-

In this table we have put in each entry the related swaption volatility and the instantaneous volatilities upon which it depends. In reading the table left to right and top down, you realize that each time only one new σ appears, and this makes the relationship between the v's and the σ's invertible (analytically).

We now give the general method for calibrating our volatility formulation of Table 1 to the upper-diagonal part of the swaption matrix when an arbitrary number s of rows of the matrix is given. So we generalize the case just seen with $s = 3$ to a generic positive integer s. Rewrite Formula (20) as follows:

$$\begin{aligned} T_\alpha\ S_{\alpha,\beta}(0)^2\ v_{\alpha,\beta}^2 &= \sum_{i,j=\alpha+1}^{\beta-1} w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j} \sum_{h=0}^{\alpha}(T_h - T_{h-1})\ \sigma_{i,h+1}\ \sigma_{j,h+1} \\ &+ 2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0)w_j(0)F_\beta(0)F_j(0)\rho_{\beta,j} \sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\ \sigma_{\beta,h+1}\ \sigma_{j,h+1} \\ &+ 2\sum_{j=\alpha+1}^{\beta-1} w_\beta(0)w_j(0)F_\beta(0)F_j(0)\rho_{\beta,j}\ (T_\alpha - T_{\alpha-1})\ \boxed{\sigma_{\beta,\alpha+1}}\ \sigma_{j,\alpha+1} \\ &+ w_\beta(0)^2F_\beta(0)^2\sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\ \sigma_{\beta,h+1}^2 \\ &+ w_\beta(0)^2F_\beta(0)^2(T_\alpha - T_{\alpha-1})\ \boxed{\sigma_{\beta,\alpha+1}^2}\ . \end{aligned} \tag{21}$$

In turn, by suitable definition of the coefficients A, B and C this equation can be rewritten as:

$$A_{\alpha,\beta}\sigma_{\beta,\alpha+1}^2 + B_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C_{\alpha,\beta} = 0,$$

which can be solved analytically by the usual elementary formula. Since $A\,B > 0$, its smaller solution, when real, is always negative. We take the greater solution (hoping for it to be real and positive). It is important to realize (as from the above $s = 3$ example) that when one solves this equation, all quantities are indeed known with the exception of $\sigma_{\beta,\alpha+1}$ *if* the swaption matrix is visited from left to right and top down.

We may wonder about what happens when in need to recover the whole matrix and not only the upper diagonal part. After calibrating the upper diagonal part as above, we continue our visit of the empty boxes in the table from left to right and top down. At the same time, we still apply formula (21), but in certain positions of the table we will have more than one unknown. However, we can still manage by assuming these unknowns to be equal. Then we can still solve the second order equation analytically and move on. Details in Brigo and Mercurio (2001).

We now present some numerical results. We calibrate our σ's to the same swaptions data as in the previous cases. We now add swaption volatilities for missing maturities of 6,8, and 9 years by linear interpolation, just to ease our calibration routine. We assume a typical nice rank 2 correlation structure given exogenously, corresponding to the angles

$$\begin{aligned}\theta_{1\div 9} &= [\ 0.0147\ 0.0643\ 0.1032\ 0.1502\ 0.1969\ 0.2239\ 0.2771\ 0.2950\ 0.3630\ ],\\ \theta_{10\div 19} &= [\ 0.3810\ 0.4217\ 0.4836\ 0.5204\ 0.5418\ 0.5791\ 0.6496\ 0.6679\ 0.7126\ 0.7659\ ].\end{aligned}$$

The resulting correlations are all positive and decrease when moving away for the diagonal. By applying the above explicit method, we obtain the σ's in an instant given by the following Table analogous to Table 1:

0.1800	-	-	-	-	-	-	-	-	-
0.1548	0.2039	-	-	-	-	-	-	-	-
0.1285	0.1559	0.2329	-	-	-	-	-	-	-
0.1178	0.1042	0.1656	0.2437	-	-	-	-	-	-
0.1091	0.0988	0.0973	0.1606	0.2483	-	-	-	-	-
0.1131	0.0734	0.0781	0.1009	0.1618	0.2627	-	-	-	-
0.1040	0.0984	0.0502	0.0737	0.1128	0.1633	0.2633	-	-	-
0.0940	0.1052	0.0938	0.0319	0.0864	0.0969	0.1684	0.2731	-	-
0.1065	0.0790	0.0857	0.0822	0.0684	0.0536	0.0921	0.1763	0.2848	-
0.1013	0.0916	0.0579	0.1030	0.1514	- 0.0316	0.0389	0.0845	0.1634	0.2777
0.0916	0.0916	0.0787	0.0431	0.0299	0.2088	- 0.0383	0.0746	0.0948	0.1854
0.0827	0.0827	0.0827	0.0709	0.0488	0.0624	0.1561	- 0.0103	0.0731	0.0911
0.0744	0.0744	0.0744	0.0744	0.0801	0.0576	0.0941	0.1231	- 0.0159	0.0610
0.0704	0.0704	0.0704	0.0704	0.0704	0.1009	0.0507	0.0817	0.1203	- 0.0210
0.0725	0.0725	0.0725	0.0725	0.0725	0.0725	0.1002	0.0432	0.0619	0.1179
0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0736	0.0551	0.0329
0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0708	0.0702
0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0680
0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663

This "real-market" calibration shows several negative signs in instantaneous volatilities. Recall that these undesirable negative entries might be due to "temporal misalignments" caused by illiquidity in the swaption matrix. In Brigo and Mercurio (2001) we discuss a toy calibration and show what can cause negative and sometimes even complex volatilities. Here we just say that the model parameters might reflect misalignments in the swaption data. To avoid this, we smooth the above market swaption matrix of May 16, 2000 with the following parametric form:

$$\mathrm{vol}(S,T) = \gamma(S) + \left(\frac{\exp(f \cdot \ln(T))}{e \cdot S} + D(S)\right) \cdot \exp(-\beta \cdot \exp(p \cdot \ln(T))),$$

where

$$\gamma(S) = c + (\exp(h \cdot \ln(S)) \cdot a + d) \cdot \exp(-b \cdot \exp(m \cdot \ln(S))),$$
$$D(S) = (\exp(g \cdot \ln(S)) \cdot q + r) \cdot \exp(-s \cdot \exp(t \cdot \ln(S))) + \delta,$$

and S, T are respectively the tenor and the maturity vectors in the swaption matrix. So, for example, vol(2, 3) is the volatility of the swaption whose underlying swap rate resets in two years and lasts three years (entry (2, 3) of the swaption matrix). We do not claim that this form has any appealing characteristic or that it always yields the precision needed by a trader, but we use it to point out the effect of smoothing. We obtain the following values of the parameters, corresponding to the smoothed matrix

a	-0.00016	g	-0.10002
b	0.376284	h	-4.18228
c	0.201927	m	0.875284
d	0.336238	p	0.241479
e	5.21409	q	-6.37843
f	0.193324	r	5.817809
δ	0.809365	s	0.048161
β	0.840421	t	1.293201

The percentage difference between the market and the smoothed matrices is as follows:

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-2.64%	3.09%	2.30%	1.26%	0.07%	0.12%	-0.43%	-1.46%	-1.22%	-1.22%
2y	-2.63%	2.61%	0.71%	-0.12%	-1.44%	-0.77%	-0.01%	0.21%	0.89%	1.27%
3y	1.04%	4.14%	1.44%	-1.10%	-2.70%	-1.21%	-0.47%	-0.30%	0.38%	0.73%
4y	0.88%	2.54%	-0.39%	-2.86%	-4.06%	-2.71%	-1.16%	-1.17%	-0.60%	-0.36%
5y	-1.13%	0.79%	-0.66%	-1.84%	-1.36%	2.76%	1.34%	1.25%	1.74%	1.91%
6y	-0.99%	1.40%	-0.50%	-2.06%	-2.34%	1.35%	1.20%	1.50%	2.45%	3.08%
7y	-0.83%	2.07%	-0.34%	-2.31%	-3.43%	-0.26%	1.03%	1.77%	3.22%	4.35%
8y	-0.30%	1.93%	-0.87%	-2.72%	-3.73%	-0.57%	0.67%	1.69%	3.12%	4.22%
9y	0.18%	1.87%	-1.33%	-3.29%	-4.04%	-0.89%	0.28%	1.73%	3.13%	4.20%
10y	0.77%	1.71%	-1.94%	-3.77%	-4.37%	-1.24%	-0.14%	1.64%	3.01%	4.06%

If we run our algebraic calibration with the smoothed swaptions data as input, the instantaneous volatility values are all real and positive, as we can see below:

0.1848	-	-	-	-	-	-	-	-	-
0.1403	0.2221	-	-	-	-	-	-	-	-
0.1282	0.1337	0.2424	-	-	-	-	-	-	-
0.1212	0.1124	0.1329	0.2542	-	-	-	-	-	-
0.1164	0.1003	0.1082	0.1317	0.2620	-	-	-	-	-
0.1126	0.0920	0.0950	0.1049	0.1305	0.2721	-	-	-	-
0.1095	0.0853	0.0859	0.0902	0.1009	0.1332	0.2771	-	-	-
0.1065	0.0798	0.0792	0.0809	0.0858	0.1030	0.1325	0.2904	-	-
0.1046	0.0752	0.0740	0.0742	0.0760	0.0870	0.1002	0.1405	0.2977	-
0.1025	0.0716	0.0701	0.0694	0.0695	0.0774	0.0841	0.1083	0.1429	0.3017
0.0853	0.0853	0.0686	0.0664	0.0651	0.0710	0.0741	0.0916	0.1087	0.1423
0.0778	0.0778	0.0778	0.0649	0.0618	0.0662	0.0670	0.0811	0.0908	0.1072
0.0720	0.0720	0.0720	0.0720	0.0599	0.0626	0.0617	0.0737	0.0796	0.0890
0.0669	0.0669	0.0669	0.0669	0.0669	0.0607	0.0580	0.0683	0.0721	0.0777
0.0630	0.0630	0.0630	0.0630	0.0630	0.0630	0.0549	0.0637	0.0659	0.0694
0.0596	0.0596	0.0596	0.0596	0.0596	0.0596	0.0596	0.0613	0.0616	0.0636
0.0573	0.0573	0.0573	0.0573	0.0573	0.0573	0.0573	0.0573	0.0576	0.0588
0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552
0.0531	0.0531	0.0531	0.0531	0.0531	0.0531	0.0531	0.0531	0.0531	0.0531

We conclude that irregularity and illiquidity in the input swaption matrix can cause negative or even imaginary values in the calibrated instantaneous volatilities. However, by smoothing the input data before calibration, usually this undesirable features can be avoided.

In the following we show the terminal correlations and the evolution of the term structure of volatility for the smoothed case. The non-smoothed case is slightly worse: It shows terminal correlations that at some points deviate from monotonic behaviour, although not of a large amount, and also features a slightly erratic evolution of the term structure of volatilities compared to the the smoothed case, see Brigo and Mercurio (2001) for the details.

Below we show the terminal correlation matrix obtained by using the smoothed swaption volatilities as inputs.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
10y	1.000	0.534	0.418	0.363	0.330	0.304	0.283	0.263	0.248	0.233
11y	0.534	1.000	0.594	0.480	0.419	0.375	0.341	0.311	0.287	0.266
12y	0.418	0.594	1.000	0.621	0.513	0.447	0.400	0.362	0.331	0.306
13y	0.363	0.480	0.621	1.000	0.635	0.525	0.459	0.408	0.370	0.341
14y	0.330	0.419	0.513	0.635	1.000	0.636	0.527	0.456	0.407	0.371
15y	0.304	0.375	0.447	0.525	0.636	1.000	0.641	0.527	0.458	0.412
16y	0.283	0.341	0.400	0.459	0.527	0.641	1.000	0.633	0.522	0.458
17y	0.263	0.311	0.362	0.408	0.456	0.527	0.633	1.000	0.636	0.530
18y	0.248	0.287	0.331	0.370	0.407	0.458	0.522	0.636	1.000	0.642
19y	0.233	0.266	0.306	0.341	0.371	0.412	0.458	0.530	0.642	1.000

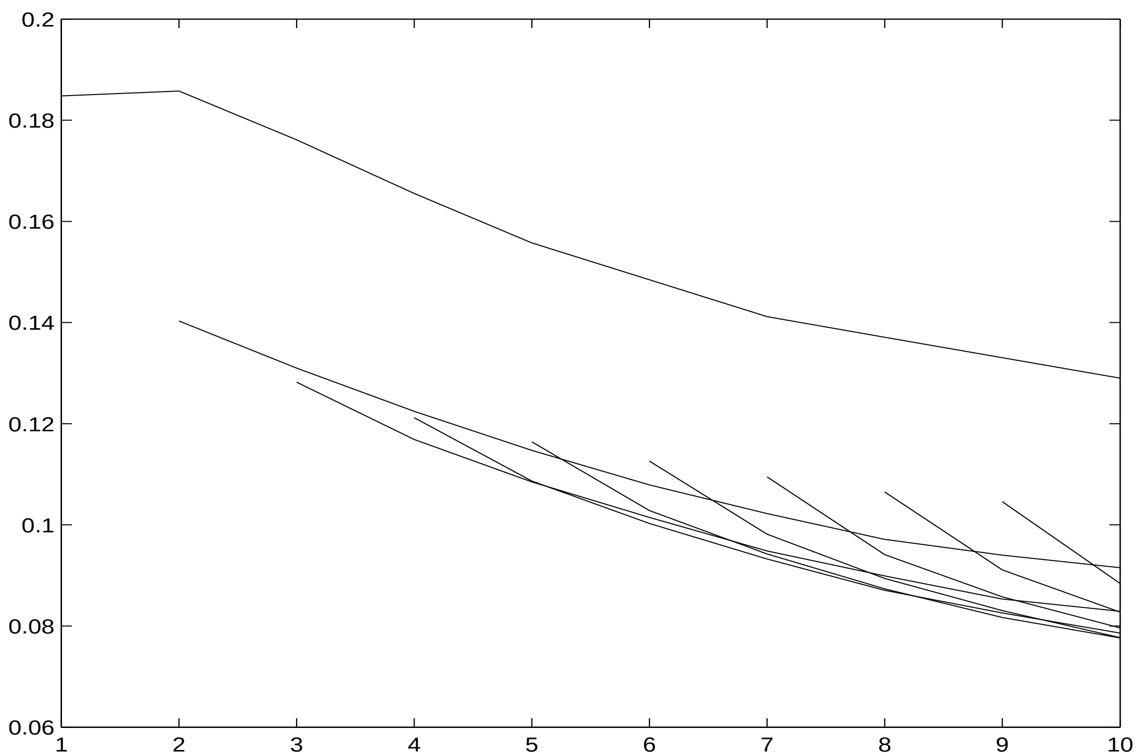

Figure 3: Term structure evolution corresponding to the smoothed volatility swaption data

As we can see in figure 3, with this smoothed swaption prices in input, the evolution of the term structure of volatility is interesting, and the related terminal correlations are decreasing and non-negative.

3.4 Conclusions: Where now?

Far from being conclusive or even systematic, our examples have served the purposed of pointing out the actual problems involved in the LFM market model calibration, with some discussion on possible solutions. In conclusion, the features one would like a calibration of the LFM model to have are:

1. A small calibration error, i.e. a small matrix of percentage differences

   100*(Market swaptions vol - LFM swaption vol)/Market swaptions vol

2. Regular instantaneous correlations. One would like to avoid large oscillations along the columns of the correlation matrix, and one would appreciate a monotonically decreasing pattern when moving away from a diagonal term of the matrix along the related row or column.

3. Regular terminal correlations.

4. Smooth and qualitatively stable evolution of the term structure of caplet volatilities over time.

One would like to accomplish the above points by a low number of factors. Clearly the above points are related each other. None of the structures we proposed can accomplish perfectly all these points at the same time. This shows once again that the core of the LFM market model is the right choice of the covariance function: The rest are just mathematical details. You may try and combine many of the ideas presented here to come up with a different approach that might work better. Research in this issue is still quite open, and probably the next months/years will show some significant evolution. We just dare say that Formulation of Table 1, with its "automatic closed form algebraic" exact calibration is probably the most promising, since one can impose exogenously a decent instantaneous correlation matrix and hope to obtain decent terminal correlations and evolution of the term structure of volatilities. This method maps in a one-to-one correspondence swaptions volatilities into pieces of instantaneous volatilities of forward rates. This can help also in computing sensitivities with respect to swaptions volatilities, since one knows on which σ's one needs to act in order to influence a single swaption volatility. By a kind of chain rule we can translate sensitivity with respect to the σ's in sensitivity with respect to the swaption volatilities used in the calibration.

Finally, we recall that we have used the swaption matrix as we found it in the market from one broker. However, this can be dangerous. Further calibration tests are needed with smoothed or adjusted data, before concluding for sure that limitations concern only the model formulations and not "noise" in the data. One can in fact decide that the model is detecting a misalignment in the market, instead of concluding that the model is not suitable for the joint calibration. We have done this for the last piecewise constant formulation of Table 1, and we have actually found that with smoothed data in input the outputs improve.

References

[1] BRACE, A., DUN, T., and BARTON, G. (1998). Towards a Central Interest Rate Model. *FMMA notes* working paper.

[2] BRACE, A., GATAREK, D., and MUSIELA, M. (1997). The market model of interest rate dynamics. *Mathematical Finance*, Vol. 7, 127–154.

[3] BRIGO, D., MERCURIO, F. (2001) Interest Rate Models: Theory and Practice, Springer - Verlag, Berlin.

[4] DE JONG, F., DRIESSEN, J., and PELSSER, A. (1999). Libor and Swap Market Models for the Pricing of Interest Rate Derivatives: An Empirical Analysis. Preprint.

[5] HULL, J., and WHITE, A. (1999). Forward rate volatilities, swap rate volatilities, and the implementation of the Libor market model. Preprint.

[6] JAMSHIDIAN, F. (1997). Libor and swap market models and measures. *Finance and Stochastics*, 4, pp. 293–330.

[7] REBONATO, R. (1998). *Interest Rate Option Models*. Second Edition. Wiley, Chichester, 1998.

[8] REBONATO, R. (1999) Volatility and Correlation. Wiley, Chichester.