CDS Market Formulas and Models

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September 2005

Abstract

In this work we analyze market payoffs of Credit Default Swaps (CDS) and we derive rigorous standard market formulas for pricing options on CDS. Formulas are based on modelling CDS spreads which are consistent with simple market payoffs, and we introduce a subfiltration structure allowing all measures to be equivalent to the risk neutral measure.

Then we investigate market CDS spreads through change of measure and consider possible choices of rates for modelling a complete term structure of CDS spreads. We also consider approximations and apply them to pricing of specific market contracts. Results are derived in a probabilistic framework similar to that of Jamshidian (2004).

*Speaker. We are grateful to Massimo Masetti for his help with the CMCDS examples.

1 Introduction

The importance of the credit derivatives market has increased remarkably during recent years. According to the 2004 survey of Merrit et al., the outstanding notional has reached \$3 trillion, increasing by 71% in one year. In particular, Credit Default Swaps (CDS) have a clear prominence among all credit derivatives. The market of CDS represents almost two thirds (\$1.9 trillion) of the global credit derivatives outstanding. It increased by 100% from the end of 2002. Along with this impressive development of the CDS market, also options on CDS are becoming a more popular product.

In spite of this development of the market, no standard market model has emerged yet. With reference to the other, long-established financial markets, a standard market model for options is commonly intended to be a model enabling to define the implied volatility of a market option, positing a lognormal dynamics of the underlying under an equivalent pricing measure. The seminal example is the Black and Scholes (1973) model for the equity market. Another example is the Black (1976) model for commodity options, used for years by market operators also as a pricing formula for interest rates options. By the work of Jamshidian (1997) and Brace, Gatarek and Musiela (1997), the formula was embedded into rigorous models of the term structure, called Swap Market Model (SMM) and Libor Market Model (LMM). These models, based on lognormality of the underlying under a natural equivalent measure, allow pricing reference options by Black and Scholes market valuation formulas and therefore allow for rigorous definition and computation of implied volatility. In addition, by providing also the joint dynamics of different underlying rates under a common pricing measure, they can be consistently used for more advanced products.

A first important contribution to the development of models of this kind for credit derivatives is given in Schonbucher (1999). The model allows for Black and Scholes formulas but differs from standard market models, since it is based on using probability measures which are not equivalent to the risk neutral probability measure.

In Hull and White (2003) Black and Scholes formulas for CDS options are tested on market data, and the importance of the development of a market model for improving liquidity of the CDS options market is further pointed out.

In a similar context Wu (2005) considers an alternative definition of fundamental bond prices, including recovery.

A very important theoretical advance in this trail is given in Jamshidian (2004). This work develops a probabilistic framework that naturally lends itself to the development of standard market models based on probability measures equivalent to the risk neutral probability measure. Yet in this work standard (Black and Scholes) market formulas are considered only as possible approximations.

Brigo (2005) on the other hand develops, starting from market definition of CDS, an exact standard market pricing formula for CDS options under an equivalent change of measure in a Cox process setting. This is a fundamental result and a natural starting point for the development of a candidate market model.

In this work we analyze market payoffs of Credit Default Swaps (CDS) and we derive rigorous standard market formulas for pricing options on CDS, in a more general setting than Cox processes. Formulas are based on modelling CDS spreads which are consistent with simple market payoffs, and we introduce a subfiltration structure allowing all measures to be equivalent to the risk neutral measure.

Then we investigate market CDS spreads through change of measure and consider possible choices of rates for modelling a complete term structure of CDS spreads. We apply the model to pricing specific market contracts (Constant Maturity CDS) and consider approximations allowing to increase tractability of pricing formulas. Results are derived in a probabilistic framework similar to that of Jamshidian (2004). We point out under which conditions pricing formulas are equivalent to that of Brigo (2005).

In Section 2 we present foundations on CDS pricing in a market model context. In Section 3 we derive standard market pricing formulas. In Section 4 we illustrate how the definition of CDS rates can be expressed via change of measure and consider possibilities for a term structure model. In Section 5 a term structure market model and its dynamics are presented and applied to the valuation of Constant Maturity CDS. Then we illustrate a model based on a different payoff definition and point out relations and differences with previous literature, before concluding. Some theorems and properties are proved in the Appendix.

2 Credit Default Swaps and Options

A *Credit Default Swap* is an agreement between two parties, called the protection buyer and the protection seller, typically designed to transfer to the protection seller the financial loss that the protection buyer would suffer if a particular default event happened to a third party, called the reference entity.

The protection buyer pays rate R at times $T_{a+1}, \ldots, T_b$, ending payments in case of default. The protection seller agrees to make a single protection payment LGD in case the pre-specified default event happens between T_a and T_b. These contracts, with some possible variations in the exact definition of the payoff, represent by far the most liquid credit derivative market. It is natural to define a market model in credit risk starting from a conventional definition of CDS.

The initial steps for a rigorous and market motivated derivation of a market model for Credit Default Swaps are close to the steps one follows to define the Swap Market Model of Jamshidian (1997). The main goal in the latter case is pricing swaptions. Swaptions are options on interest rate swaps. One starts from one specification of the payoff and the price of the swap to detect the value of the fixed rate making the swap fair. This defines the swap rate which is also the underlying of the swaption. Then one has to detect the probability measure under which the swap rate is a martingale and the pricing formula simplifies. With deterministic percentage volatility assumptions for the underlying swap rate one recovers the standard market Black formula. This is the approach in Brigo (2004), a fundamental point for then developing a model of the entire term structure model to be consistent with this pricing formula.

2.1 Par CDS spread

Indicate the default time by τ, the year fraction between T_{i-1} and T_i with α_i, and the bank-account by B_t, so that the usual bank-account discount factor is

$$D(t,T) = \frac{B_t}{B_T}.$$

The general buyer CDS discounted payoff, with unit notional and protection payment LGD, is at $t \leq T_a$

$$\mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\mathrm{LGD} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbf{1}_{\{\tau>T_i\}} - D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a<\tau<T_b\}}$$

where $T_{\beta(\tau)}$ is the first of the T_i's following τ. Modifications to this basic structure are then possible, for example the protection buyer can pay an upfront fee.

For developing a market model a conventional definition of the payoff must be considered. Brigo (2004) analyzes different possible market specifications of the CDS payoff. Following Brigo (2004) we mainly consider the payoff

$$CDS\Pi_t(R) = \mathrm{LGD} \sum_{i=a+1}^{b} D(t,T_i)\, \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i \mathbf{1}_{\{\tau>T_i\}} R, \quad (1)$$

which is simple enough for developing a market model but realistic enough for practical applications. Here the protection payment is made at the first T_i following default and there is no payment of the protection buyer for the period T_{i-1} - T_i when default happens. The payment dates remain the same for all multiperiod CDS throughout the paper, therefore we do not indicate them in the symbols for payoffs and prices.

A payoff from and approximation. Also payoffs which do not correspond to viable real world CDS payoffs can be considered for modelling purposes. We illustrate below an approximated payoff which takes into account that, when a contract provides protection payment at default and τ is much closer to $T_{\beta(\tau)-1}$ than to $T_{\beta(\tau)}$, then the postponement of the protection payment can have a relevant impact. In such a case a better approximation for the discounted protection leg is

$$\mathrm{LGD} \sum_{i=a+1}^{b} D(t,T_{i-1})\, \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}},$$

while when τ is closer to $T_{\beta(\tau)}$ the discounted protection leg

$$\mathrm{LGD} \sum_{i=a+1}^{b} D(t,T_i)\, \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}$$

is better. Thus, for suitable ε_i, a generally good approximation is

$$\text{LGD} \sum_{i=a+1}^{b} \left[D(t,T_{i-1}) \mathbf{1}_{\{T_{i-1}<\tau\le T_{i-1}+\varepsilon_i\}} + D(t,T_i) \mathbf{1}_{\{T_{i-1}+\varepsilon_i<\tau\le T_i\}} \right] =$$

$$\text{LGD} \sum_{i=a+1}^{b} \left[D(t,T_{i-1}) \mathbf{1}_{\{\tau>T_{i-1}\}} + \mathbf{1}_{\{\tau>T_{i-1}+\varepsilon_i\}} \left(D(t,T_i) - D(t,T_{i-1}) \right) - D(t,T_i) \mathbf{1}_{\{\tau>T_i\}} \right]$$

$$\approx \text{ LGD} \sum_{i=a+1}^{b} \left[D(t,T_{i-1}) \mathbf{1}_{\{\tau>T_{i-1}\}} - D(t,T_i) \mathbf{1}_{\{\tau>T_i\}} \right] =: CDS\Pi_t^S(R) \quad (2)$$

where the final approximation amounts to assuming $D(t,T_i) - D(t,T_{i-1})$ to be negligible. Differently from the above payoffs, $CDS\Pi_t^S(R)$ is not a real world CDS payoff, but it can sometimes represent a good approximation of a general CDS payoff. We will see in Section 5.4 that, when considering one single period contracts, $CDS\Pi_t^S(R)$ leads to a definition of the CDS par spread which resembles the definition of the defaultable forward rate in Schonbucher (1999).

As usual in no-arbitrage pricing the price of a CDS is given by the risk neutral expectation of its discounted payoff. Considering our reference payoff

$$\mathbf{CDS}_t(R) = \mathbb{E}^Q \left[CDS\Pi_t(R) | \mathcal{F}_t \right] \quad (3)$$

where $\mathbb{Q}$ is the risk-neutral equivalent martingale measure and the filtration $\mathcal{F}_t$ represents all available information up to t. Default is modelled as an $\mathcal{F}_t$-stopping time.

Subfiltration Structure. In credit risk valuation it is often convenient to express prices making use of a *subfiltration structure*. Following Jeanblanc and Rutkowski (2000) we define $\mathcal{F}_t = \mathcal{F}_t^\tau \vee \mathcal{H}_t$, where

$$\mathcal{F}_t^\tau = \sigma\left(\{\tau > u\}, u \le t \right),$$

basically the subfiltration generated by τ, while $\mathcal{H}_t$ is a subfiltration representing the flow of all information except default itself (default-free information).[1] A market operator observing only this second filtration can have information on the probability of default but cannot say exactly when, or even if, default has happened. This structure is typical for instance of the Cox process setting, where default is defined as the first jump of a Cox Process. The definition of Cox Process hinges on assuming default intensity λ_t of τ to be $\mathcal{H}_t$-adapted. In fact, if for instance the intensity dropped to zero after default, the default jump process conditional on the path followed by λ_t would not be an inhomogeneous Poisson Process, since the time of its first jump would be known.

This subfiltration structure allows to define pricing formulas in terms of conditional survival probability $\mathbb{Q}(\tau > t | \mathcal{H}_t)$ which can be assumed to be strictly positive in any state of the world. This is the assumption we make in this work (so excluding standard structural models, in which default is a predictable stopping time). The subfiltration structure is useful also in settings more general than intensity models, for instance when the conditional default probability is not necessarily absolutely continuous but it is allowed to be a general semimartingale. We will see below examples of the advantages yielded by considering a subfiltration structure.

[1] We adopt Jamshidian (2004) notation. Notice that $\mathcal{F}_t$ is called $\mathcal{G}_t$ in Brigo (2004, 2005, 2005b) while $\mathcal{H}_t$ is called $\mathcal{F}_t$.

With subfiltrations as in Jeanblanc and Rutkowski (2000), using the definition of conditional expectation and recalling that $CDS\Pi_t(R) = \mathbf{1}_{\{\tau>t\}}CDS\Pi_t(R)$, the price of the above CDS can be expressed as

$$\mathbf{CDS}_t(R) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\mathbb{E}^Q\left[CDS\Pi_t(R)|\mathcal{H}_t\right]. \tag{4}$$

Following the standard SMM case, we set expression (4) of the price to zero and solve in R to derive the expression for the fair rate $R_{a,b}(t)$, also called fair or par CDS spread, which will be also the underlying of the CDS option. For the rest of this section we follow Brigo (2004).

Par CDS Spread

$$\begin{aligned} R_{a,b}(t) &= \text{LGD}\frac{\sum_{i=a+1}^{b}\mathbb{E}^Q\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{H}_t\right]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{E}^Q\left[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_t\right]} \\ &= \text{LGD}\frac{\sum_{i=a+1}^{b}\mathbb{E}^Q\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{H}_t\right]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau>t|\mathcal{H}_t)\bar{P}(t,T_i)}, \end{aligned}$$

$$\bar{P}(t,T) = \frac{\mathbb{E}^Q\left[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{H}_t\right]}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}.$$

Remark 1 *Notice that $\bar{P}(t,T)$ coincides before default with the price of a T-maturity* zero-coupon defaultable bond

$$\mathbb{E}^Q\left[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\right] = \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T).$$

Remark 2 *Notice a specific advantage of using the subfiltration $\mathcal{H}_t$ in our setting. Using expectation conditional on information $\mathcal{F}_t$ including the default time, the denominator of the par spread may jump to zero at default, so that the spread definition would not be valid in all states of the world. Instead, by expressing prices in terms of partial information $\mathcal{H}_t$, our definition holds globally.*

2.2 Pricing a CDS Option

Now we can consider CDS options. The CDS option to enter a CDS with fixed rate K at future time T_a has discounted payoff

$$D(t,T_a)\left[CDS_{T_a}(K)\right]^+ = D(t,T_a)\left[CDS_{T_a}(K) - \underbrace{CDS_{T_a}(R_{a,b}(T_a))}_{0}\right]^+$$

which from (1) inserted in (4) is

$$
\begin{aligned}
& CDSOption\Pi_t (K) \;= \\
= \;\; & D(t,T_a)\,\frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{H}_{T_a})}\mathbb{E}^Q\left[\sum_{i=a+1}^{b}\alpha_i D(T_a,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right](R_{a,b}(T_a)-K)^+ \\
= \;\; & D(t,T_a)\,\frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{H}_{T_a})}\left\{\sum_{i=a+1}^{b}\alpha_i \mathbb{Q}(\tau>T_a|\mathcal{H}_{T_a})\,\bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+ . \\
= \;\; & D(t,T_a)\,\mathbf{1}_{\{\tau>T_a\}}\left\{\sum_{i=a+1}^{b}\alpha_i \bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+
\end{aligned}
$$

Using the pricing formula conditional on $\mathcal{H}_t$

$$
\begin{aligned}
& \mathbf{CDSOption}_t (K) \;= \\
= \;\; & \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\mathbb{E}^Q\left[D(t,T_a)\,\mathbf{1}_{\{\tau>T_a\}}\left\{\sum_{i=a+1}^{b}\alpha_i \bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+\middle|\mathcal{H}_t\right] \\
= \;\; & \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\mathbb{E}^Q\left[\mathbb{E}^Q\left[D(t,T_a)\,\mathbf{1}_{\{\tau>T_a\}}\left\{\sum_{i=a+1}^{b}\alpha_i \bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+\middle|\mathcal{H}_{T_a}\right]\middle|\mathcal{H}_t\right] \\
= \;\; & \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\mathbb{E}^Q\left[D(t,T_a)\left\{\sum_{i=a+1}^{b}\alpha_i \bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+\,\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>T_a\}}\middle|\mathcal{H}_{T_a}\right]\middle|\mathcal{H}_t\right]
\end{aligned}
$$

$$
= \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\mathbb{E}^Q\left[D(t,T_a)\left\{\sum_{i=a+1}^{b}\alpha_i \mathbb{Q}(\tau>T_a|\mathcal{H}_{T_a})\,\bar{P}(T_a,T_i)\right\}(R_{a,b}(T_a)-K)^+\middle|\mathcal{H}_t\right]. \tag{5}
$$

This rather complicated formula can be reduced to a simple standard formula by changing the numeraire. This issue is analyzed in the next section.

3 Standard Market Formula for CDS

In this section we obtain standard market formulas under a general probabilistic framework analogous to part of Jamshidian (2004).

In this context we are in a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where filtration $\mathcal{F}_t$ satisfies the usual hypothesis and we set $\mathcal{F}_0 = (\Omega, \emptyset)$ and $\mathcal{F} = \mathcal{F}_{\bar{T}}$ for a terminal date $\bar{T}$. $\mathcal{H}_t$ is a subfiltration of $\mathcal{F}_t$ satisfying the usual hypothesis and we set $\mathcal{H}_0 = (\Omega, \emptyset)$ and $\mathcal{H} = \mathcal{H}_{\bar{T}}$.

Given the standard bank-account numeraire with price process B_t and an equivalent risk neutral measure $\mathbb{Q} \sim P$, we define a claim X as an $\mathcal{F}$-measurable random variable such that $\frac{X}{B}$ is $\mathbb{Q}$-integrable, writing B for $B_{\bar{T}}$. The price process of any claim is given by

$$
X_t = B_t \mathbb{E}^Q\left[\frac{X}{B}|\mathcal{F}_t\right]
$$

Any different numeraire β is a claim such that $\beta > 0$ almost surely. The measure P^β associated with numeraire β is defined by the Radon-Nikodym derivative

$$\frac{dP^\beta}{dQ} = \frac{B_0 \beta}{\beta_0 B}$$

so that we have the standard change of numeraire formula

$$X_0 = B_0 \mathbb{E}^Q \left[\frac{X}{B} \right] = \mathbb{E}^Q \left[\frac{X \beta_0}{\beta} \frac{dP^\beta}{dQ} \right] = \beta_0 \mathbb{E}^\beta \left[\frac{X}{\beta} \right].$$

extended as usual to prices X_t (see Appendix).

3.1 A numeraire for CDS

The quantity of interest in (5) is obviously the quantity between curly brackets, that we call $C_{a,b}(T_a)$ according to

Definition 3

$$C_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t | \mathcal{H}_t) \bar{P}(t, T_i) = \sum_{i=a+1}^{b} \alpha_i \mathbb{E}^Q \left[D(t, T_i) \mathbf{1}_{\{\tau > T_i\}} | \mathcal{H}_t \right].$$

Before default this coincides with the price of a portfolio of defaultable bonds called ***defaultable present value per basis point****.*

The accrued value

$$\mathbb{C}^{a,b} = C_{a,b}(T_a) \frac{B}{B_{T_a}}$$

is a claim and $\mathbb{C}^{a,b} > 0$ almost surely[2], therefore it is a numeraire associated to a measure $\bar{\mathbb{Q}}^{a,b} := P^{\mathbb{C}^{a,b}}$. Notice that $\mathbb{C}_t^{a,b} = B_t \mathbb{E}^Q \left[\frac{\mathbb{C}^{a,b}}{B} | \mathcal{F}_t \right]$.

In (5) we change measure to $\bar{\mathbb{Q}}^{a,b}$ and we obtain

$$\textbf{CDSOption}_t(K) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}(\tau > t | \mathcal{H}_t)} \mathbb{C}_t^{a,b} \bar{\mathbb{E}}^{a,b} \left[(R_{a,b}(T_a) - K)^+ \middle| \mathcal{H}_t \right]. \qquad (6)$$

This formula is similar but not equivalent to that in Brigo (2005), since in Brigo (2005) $\mathbb{C}_t^{a,b}$ is replaced by $C_{a,b}(t)$. Namely there a $\mathcal{H}_t$ conditioning replaces our iterated $\mathcal{H}_{T_a}, \mathcal{F}_t$ conditioning in the numeraire price. The formula of Brigo (2005) holds in the particular setting of that work, the setting of Cox Processes, while (6) is valid under the more general hypothesis of this work.

We recall now the definition of *conditional independence for subfiltrations* (Jamshidian (2004)), a property called martingale invariance in Jeanblanc and Rutkowski (2000). Given the numeraire β, $\mathcal{H}_t$ is a P^β-conditionally independent subfiltration of $\mathcal{F}_t$ if under P^β every process which is a martingale when conditioning on $\mathcal{H}_t$ is a martingale

[2] This fact is proven in the Appendix.

also when conditioning on $\mathcal{F}_t$. For our CDS pricing, the most relevant way of expressing this property is the following: for X bounded and $\mathcal{H}$-measurable,

$$\mathbb{E}^{\beta}\left[X|\mathcal{H}_t\right] = \mathbb{E}^{\beta}\left[X|\mathcal{F}_t\right],\ \forall t.$$

Therefore we also have

$$\begin{aligned}
\mathbb{C}_t^{a,b} &= B_t\mathbb{E}^Q\left[\frac{1}{B}C_{a,b}\left(T_a\right)\frac{B}{B_{T_a}}|\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{B_t}{B_{T_a}}C_{a,b}\left(T_a\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}^Q\left[\frac{B_t}{B_{T_a}}C_{a,b}\left(T_a\right)|\mathcal{H}_t\right] \\
&= \mathbb{E}^Q\left[D\left(t,T_a\right)\mathbb{E}^Q\left[\sum_{i=a+1}^{b}\alpha_i D\left(T_a,T_i\right)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right]|\mathcal{H}_t\right] \\
&= \mathbb{E}^Q\left[\sum_{i=a+1}^{b}\alpha_i D\left(t,T_i\right)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_t\right] \\
&= C_{a,b}\left(t\right).
\end{aligned}$$

Hence if $\mathcal{H}_t$ is $\mathbb{Q}$-conditionally independent (6) simplifies to

$$\textbf{CDSOption}_t\left(K\right) = \mathbf{1}_{\{\tau>t\}}\sum_{i=a+1}^{b}\alpha_i\bar{P}\left(t,T_i\right)\bar{\mathbb{E}}^{a,b}\left[\left(R_{a,b}\left(T_a\right)-K\right)^+\middle|\mathcal{H}_t\right]. \quad (7)$$

which is equivalent to the formula in Brigo (2005). Yet notice that we are not constrained to assume the conditional survival probability to be absolutely continuous and given by $\exp\left(-\int_0^t\lambda_s ds\right)$ for a $\mathcal{H}_t$-adapted intensity process λ_t. The formula is valid in the more general context of conditionally independent subfiltration, of which Cox Processes are a special case. Analogously, in developing the Swap Market Model the existence of an instantaneous spot interest rate process is not required.

Remark and Assumption 4 *In financial terms, the assumption that $\mathcal{H}_t$ is $\mathbb{Q}$-conditionally independent implies that, if a claim (or a terminal payoff) can be known based only on total (terminal) default-free information, its current price can be computed based only on current default-free information. Although, as Jamshidian (2004) correctly remarks, "this somewhat degrades the role played by subfiltration $\mathcal{H}_t$, for all $\mathcal{H}_t$ conditional expectations of $\mathcal{H}$-measurable random variable become replaceable with corresponding $\mathcal{F}_t$-conditional expectations", it appears to us that this property makes the financial meaning of a subfiltration setting more clear and understandable. Considering also the computational (and notational) ease it can grant,* ***from now on we assume conditional independence to hold****.*

3.2 The dynamics of the underlying spread

Obviously formula (7) will be specified by giving a dynamics for $R_{a,b}\left(t\right)$ under $\bar{\mathbb{Q}}^{a,b}$. In the next computations LGD$=1$ for easing notation. Notice first that

$$R_{a,b}\left(t\right) = \frac{\sum_{i=a+1}^{b}\mathbb{E}^Q\left[D\left(t,T_i\right)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{H}_t\right]}{C_{a,b}\left(t\right)}$$

coincides before default with the price of a CDS payed upfront, with a single initial payment, divided by the numeraire.
Consider the claim

$$R^C = \sum_{i=a+1}^{b} \mathbb{E}^Q \left[D\left(T_a, T_i\right) \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} | \mathcal{H}_{T_a} \right] \frac{C_{a,b}}{C_{a,b}\left(T_a\right)}.$$

At time $t \leq T_a$

$$\begin{aligned} R_t^C &= C_{a,b}\left(t\right) \bar{\mathbb{E}}^{a,b} \left[\frac{R^C}{C_{a,b}} \middle| \mathcal{F}_t \right] \\ &= C_{a,b}\left(t\right) \bar{\mathbb{E}}^{a,b} \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^Q \left[D\left(T_a, T_i\right) \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} | \mathcal{H}_{T_a} \right]}{C_{a,b}\left(T_a\right)} \middle| \mathcal{F}_t \right] \\ &= C_{a,b}\left(t\right) \bar{\mathbb{E}}^{a,b} \left[R_{a,b}\left(T_a\right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Furthermore

$$\begin{aligned} R_t^C &= B_t \mathbb{E}^Q \left[\frac{R^C}{B} \middle| \mathcal{F}_t \right] = \\ &= B_t \mathbb{E}^Q \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^Q \left[D\left(T_a, T_i\right) \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} | \mathcal{H}_{T_a} \right] C_{a,b}\left(T_a\right) \frac{B}{B_{T_a}}}{B\ C_{a,b}\left(T_a\right)} \middle| \mathcal{F}_t \right] \\ &= B_t \mathbb{E}^Q \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^Q \left[D\left(T_a, T_i\right) \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} | \mathcal{H}_{T_a} \right]}{B_{T_a}} \middle| \mathcal{F}_t \right] \\ &= B_t \mathbb{E}^Q \left[\sum_{i=a+1}^{b} \mathbb{E}^Q \left[\frac{\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}}{B_{T_i}} | \mathcal{H}_{T_a} \right] \middle| \mathcal{F}_t \right]. \end{aligned}$$

When $\mathcal{H}_t$ is $\mathbb{Q}$-conditionally independent,

$$\begin{aligned} R_t^C &= B_t \mathbb{E}^Q \left[\sum_{i=a+1}^{b} \mathbb{E}^Q \left[\frac{\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}}{B_{T_i}} | \mathcal{H}_{T_a} \right] \middle| \mathcal{H}_t \right] \\ &= B_t \mathbb{E}^Q \left[\sum_{i=a+1}^{b} \frac{\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}}{B_{T_i}} \middle| \mathcal{H}_t \right] \\ &= \sum_{i=a+1}^{b} \mathbb{E}^Q \left[D\left(t, T_i\right) \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} | \mathcal{H}_t \right] = R_{a,b}\left(t\right) C_{a,b}\left(t\right). \end{aligned}$$

Thus we have that

$$R_{a,b}\left(t\right) = \bar{\mathbb{E}}^{a,b} \left[R_{a,b}\left(T_a\right) \middle| \mathcal{F}_t \right],$$

and $R_{a,b}(t)$ is an $\mathcal{F}_t$-martingale under $\bar{\mathbb{Q}}^{a,b}$ for $t \leq T_a$. Since $\mathcal{H}_t$ is $\bar{\mathbb{Q}}^{a,b}$-conditionally independent as well,[3] $R_{a,b}(t)$ is also an $\mathcal{H}_t$-martingale under $\bar{\mathbb{Q}}^{a,b}$ for $t \leq T_a$

[3] That this actually holds is proven in the Appendix.

Assumption 5 *We assume that, as in standard market models, the instantaneous percentage volatility is deterministic*

$$dR_{a,b}(t) = \bar{\sigma}_{a,b} R_{a,b}(t)\, dV^{a,b}, \quad t \leq T_a$$

where $V^{a,b}$ is a brownian motion under $\bar{\mathbb{Q}}^{a,b}$.

Then
Standard Market Formula for CDS Options

$$\mathbf{CDSOption}_t = \mathbf{1}_{\{\tau > t\}} \sum_{i=a+1}^{b} \alpha_i \bar{P}(t, T_i)\, \mathbf{Black}\left(R_{a,b}(t), K, \bar{\sigma}_{a,b}\sqrt{T_a - t} \right), \quad (8)$$

$$\mathbf{Black}(F, K, v) = F\ N(d_1(F,K,v)) - K\ N(d_2(F,K,v))$$

$$d_1(F,K,v) = \frac{\ln(\frac{F}{K}) + \frac{1}{2}v^2}{v}, \qquad d_2(F,K,v) = \frac{\ln(\frac{F}{K}) - \frac{1}{2}v^2}{v}.$$

Remark 6 *The distributional assumption is obviously inspired by the analogy with standard market models in equity and interest rate markets. However, this assumption for the CDS market is specifically underpinned by the empirical analysis in Schonbucher (2004).*

3.3 Empirical application

When the market is not very liquid, a market model is not easily calibrated to market quotations to be used for pricing, although it plays an important role. It allows to translate the prices of different options into implied volatilities, making the understanding of quotations much better. Compare the tables below

	$R_{a,b}(0)$	K	$Mid\ Opt\ quote$
Option 1	61	60	**32.5**
Option 2	43.4	43	**24.5**

	$R_{a,b}(0)$	K	$Mid\ implied\ \sigma_{a,b}$
Option 1	61	60	**62.16%**
Option 2	43.4	43	**63.71%**

The possibility to compute implied volatility also allows to assess the implications of different models on the classic strike volatility curve (smile or skew). And the numeraire martingale framework is general and we can assume for $R_{a,b}(t)$ an alternative local or stochastic volatility dynamics

$$dR_{a,b}(t) = \nu_{a,b} R_{a,b}^{PR}(t) dW^{a,b}(t)$$

as it may be required in the market.

Now we show some examples of CDS implied volatilities, to see how they change when modifying some inputs or assumptions. The corporates considered are:

C1 = Deutsche Telecom; C2 = Daimler Chrysler; C3 = France Telecom.

The data are Euro market CDS options quotes as of March 26, 2004; $REC = 0.4$; T_a =June 20, 2004 and T'_a =December 20, 2004; T_b = June 20, 2009;

Values obtained for volatility appear high compared to interest rate default-free swaptions, for example, but they have the same order of magnitude as some of those found on the CDS market by Hull and White (2003) and in particular by Schonbucher (2004) via historical estimation.

Below we see that changing the definition of the CDS rate both CDS forward rates and implied volatilities are almost unchanged. $R^B_{a,b}(0)$ and $\sigma^B_{a,b}$ refer to a CDS payoff including one more payment of the protection buyer, for the period when default happens. For more details on these tests see Brigo (2004).

	Option: bid	mid	ask	$R_{0,b}(0)$	$R_{a,b}(0)$	$R^B_{a,b}(0)$	K	$\sigma_{a,b}$	$\sigma^B_{a,b}$
C1(T_a)	14	24	34	60	61.497	61.495	60	50.31	50.18
C2	32	39	46	94.5	97.326	97.319	94	54.68	54.48
C3	18	25	32	61	62.697	62.694	61	52.01	51.88
C1(T'_a)	28	35	42	60	65.352	65.344	61	51.45	51.32

In the next table we check the impact of the recovery rate on implied volatilities and CDS forward rates. Again the impact is very reduced.

	REC = 20%	REC = 30%	REC = 40%	REC = 50%	REC = 60%
$\sigma_{a,b}$:					
C1(T_a)	50.02	50.14	50.31	50.54	50.90
C2	54.22	54.42	54.68	55.05	55.62
C3	51.71	51.83	52.01	52.25	52.61
C1(T'_a)	51.13	51.27	51.45	51.71	52.10
$R_{a,b}$:					
C1(T_a)	61.488	61.492	61.497	61.504	61.514
C2	97.303	97.313	97.326	97.346	97.374
C3	62.687	62.691	62.697	62.704	62.716
C1(T'_a)	65.320	65.334	65.352	65.377	65.415

In the next table we check the impact of a shift in the simply compounded rates of the zero coupon interest rate curve.These shifts have still a small impact, even though it is larger than the impact of recovery. We have volatilities on the left, rates on the right.

	shift -0.5%	0	$+0.5\%$	shift -0.5%	0	$+0.5\%$
C1(T_a)	49.68	50.31	50.93	61.480	61.497	61.514
C2	54.02	54.68	55.34	97.294	97.326	97.358
C3	51.36	52.01	52.65	62.677	62.697	62.716

4 Variables for a general CDS Market Model

In the previous sections we have shown the fundamental steps for building a standard CDS Market Model. Although various complications in the credit setting required specific attention, steps are analogous to those required in the definition of the Swap Market Model of Jamshidian (1997). Obviously, the above results immediately translate for the case of a CDS Market Model designed along the definition of the Libor Market Model of Brace, Gatarek and Musiela (1997).

4.1 One period CDS forward rates

The LMM is designed for pricing caplets, the building blocks of caps. A caplet is an option on a forward rate agreement, which is a one-period swap. One sets the price of this one-period swap to zero and recovers the value of the fixed rate making it fair. This defines the forward rate (one-period swap rate) which is also the underlying of the caplet. Then one detects the equivalent measure under which a forward rate is a martingale and assumes it has deterministic percentage volatility. Therefore up to this point the steps are the same as those seen for the SMM (the relevant differences arise in giving a term structure model), but applied to a one-period swap.

Analogously, we can express the results of the previous sections with reference to a one-period CDS, as done in Brigo (2005).

We consider the measure $\bar{\mathbb{Q}}^j := \bar{\mathbb{Q}}^{j-1,j}$ associated with numeraire $\mathbb{C}^j = \mathbb{C}^{j-1,j}$. The *forward CDS spread* $R_j(t) := R_{j-1,j}(t)$, martingale under $\bar{\mathbb{Q}}^j$, is

$$\begin{aligned} R_j(t) &= \frac{\text{LGD}\,\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]} \qquad (9)\\ &= \text{LGD}\frac{\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]-\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}\\ &= \frac{\text{LGD}}{\alpha_j}\frac{\left\{\frac{\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]}{Q(\tau>t|\mathcal{H}_t)}\right\}-\bar{P}(t,T_j)}{\bar{P}(t,T_j)}, \end{aligned}$$

and pricing formulas for one-period CDS options are trivially derived by a change of notation. The relationship between $R_j(t)$ and $R_{a,b}(t)$ is

$$R_{a,b}(t) = \sum_{j=a+1}^{b}\frac{\alpha_j\bar{P}(t,T_j)}{\sum_{i=a+1}^{b}\alpha_i\bar{P}(t,T_i)}R_j(t). \qquad (10)$$

Notice that in the definition of a CDS forward rate given in (9) we have complex quantity between curly brackets. The definition of the rate can be better understood via a change of measure. Therefore in the next section we carry out an analysis, based on change of measure, of the nature of the fundamental variables.

4.2 CDS spreads as forward conditional probability ratios

Another advantage of our subfiltration structure approach is that now all quantities are explicitly defined and represented as conditional expectations. This allows us to investigate their nature more deeply. It is a well known result in change-of-measure theory that for a σ-subalgebra $\mathcal{N}$ of σ-algebra $\mathcal{M}$ and an $\mathcal{M}$-measurable X, integrable under the measures considered, we have[4]

$$\mathbb{E}^{P2}[X|\mathcal{N}] = \mathbb{E}^{P1}\left[X\frac{dP2}{dP1}\frac{1}{\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]}\middle|\mathcal{N}\right]. \qquad (11)$$

[4] A proof of this result is provided in the Appendix.

This implies that, for a $\mathcal{F}$-measurable Y and a measure P^Z associated to numeraire Z

$$\mathbb{E}^Z\left[Y|\mathcal{H}_t\right]=\mathbb{E}^Q\left[Y\frac{\frac{Z}{B}}{\mathbb{E}^Q\left[\frac{Z}{B}|\mathcal{H}_t\right]}\middle|\mathcal{H}_t\right]. \tag{12}$$

A one period CDS rate is

$$R_j(t)=\text{LGD}\frac{\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]-\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}.$$

Consider

$$Y=\mathbf{1}_{\{\tau>T_j\}}$$

and apply (12) with P^Z equal to the T_j forward measure $\mathbb{Q}^j$ associated to numeraire $Z=B/B_{T_j}$ having price process $P(t,T_j)$, $t\le T_j$, namely the T_j-maturity default-free zero-coupon bond. We obtain that

$$\begin{aligned}\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right] &= P(t,T_j)\,\mathbb{E}^j\left[\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]\\ &= P(t,T_j)\,\mathbb{Q}^j(\tau>T_j|\mathcal{H}_t)\end{aligned}$$

and analogously

$$\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]=P(t,T_j)\,\mathbb{Q}^j(\tau>T_{j-1}|\mathcal{H}_t).$$

Therefore the CDS spread is

$$\begin{aligned}R_j(t) &= \text{LGD}\frac{\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]-\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}{\alpha_j\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}\\ &= \text{LGD}\frac{P(t,T_j)\,\mathbb{Q}^j(\tau>T_{j-1}|\mathcal{H}_t)-P(t,T_j)\,\mathbb{Q}^j(\tau>T_j|\mathcal{H}_t)}{\alpha_jP(t,T_j)\,\mathbb{Q}^j(\tau>T_j|\mathcal{H}_t)}\end{aligned}$$

$$=\frac{\text{LGD}}{\alpha_j}\left(\frac{\mathbb{Q}^j(\tau>T_{j-1}|\mathcal{H}_t)}{\mathbb{Q}^j(\tau>T_j|\mathcal{H}_t)}-1\right) \tag{13}$$

Thus we obtain that the basic real world CDS rate is actually a ratio of survival probabilities, if the right probability measure and information flow are selected in defining conditional default probabilities. In particular, differently from their usual representation as in (9), no direct presence of default-free interest rates and discount factors appears in (13).

This result has a simple financial meaning. Consider for instance a classic toy intensity model with constant intensity. It is well known that if we assume a continuous instantaneous premium, a CDS rate can be seen as equivalent, by no-arbitrage, to the credit spread of the same reference entity over the instantaneous spot interest rate r. In fact this is often called CDS spread. It is also well known that in this case the spread is given by the instantaneous intensity, namely it is determined by survival probability.

The results above show that, in the much more complex context of real market discrete-tenor CDS spreads, a dependence of the one-period CDS spread from default probability is maintained, but to detect this link separating out default-free interest rates from default probabilities one needs to consider a probability measure associated to discrete tenor interest rates, namely a forward measure. This is analogous to the computation of

$$\mathbb{E}^{Q}\left[D\left(t,T_k\right)F\left(T_{k-1};T_{k-1},T_k\right)|\mathcal{F}_t\right]=P\left(t,T_k\right)F\left(t;T_{k-1},T_k\right),$$

where actually a change to the T_k-forward measure is performed

$$\mathbb{E}^{Q}\left[D\left(t,T_k\right)F\left(T_{k-1};T_{k-1},T_k\right)|\mathcal{F}_t\right]=P\left(t,T_k\right)\mathbb{E}^{T_k}\left[F\left(T_{k-1};T_{k-1},T_k\right)\right].$$

Notice that actually (13) gives a representation of the CDS spread analogous to that of a Libor rate, where bond prices are replaced by forward default probabilities.

Now we move to consider multi-period CDS spreads. From (13) we have

$$R_{a,b}\left(t\right)=\text{LGD}\sum_{j=a+1}^{b}\frac{P\left(t,T_j\right)\mathbb{Q}^j\left(\tau>T_{j-1}|\mathcal{H}_t\right)-P\left(t,T_j\right)\mathbb{Q}^j\left(\tau>T_j|\mathcal{H}_t\right)}{\sum_{i=k+1}^{b}\alpha_i P\left(t,T_i\right)\mathbb{Q}^j\left(\tau>T_i|\mathcal{H}_t\right)}. \tag{14}$$

Here we have an expression where both forward default probabilities and default-free bonds appear explicitly, as one can expect. But due to the change of measure bond prices and probabilities are separated, something which is usually done by assuming independence of default probabilities and default free interest rates.

4.3 Dynamics under different measures

Up to now we have given dynamics and formulas involving one rate at a time under the associated measure. Therefore the practical usefulness of our results is analogous to that of the Black formula justified by the change of numeraire in the interest rate market. We do not have yet a general model of the CDS term structure like the SMM and LMM for the default-free term structure. A major contribution of the Swap and Libor Market Models is giving the joint distribution of different interest rates under a single convenient pricing measure.

For a CDS market model one needs defining a plurality of CDS spreads covering the required tenor structure. A plurality of CDS spreads $R_{a,b}$ are associated with a plurality of natural measures $\bar{\mathbb{Q}}^{a,b}$. One needs to know the dynamics of CDS spreads jointly under a single $\bar{\mathbb{Q}}^{a,b}$ measure.

We can span a complete tenor structure $\Upsilon=\{T_0,T_1,\ldots,T_M\}$ choosing for instance the one-period forward CDS rates $R_j\left(t\right)$, $j=1,\ldots,M$. This choice is analogous to the LMM of Brace, Gatarek and Musiela (1997) and is the main choice suggested in Brigo (2004, 2005). The definition of the model requires in this case computing the dynamics of each R_j CDS spread under any of the $\bar{\mathbb{Q}}^i$ measures for the T_i's in the tenor structure. This implies defining the dynamics of R_j under $\bar{\mathbb{Q}}^i$ when $i\neq j$. However Brigo (2004, 2005) points out problems in defining the dynamics because of some fundamental differences between CDS forward spreads and default-free interest rates. We analyze these differences in the following.

Recall the change of numeraire rule for diffusions, given in more detail in Section 5. Suppose we know the dynamics of X under a measure $\mathbb{Q}_1$, associated with numeraire N^1, then through Girsanov's Theorem we know that the dynamics of X under the equivalent measure $\mathbb{Q}_2$, associated with N^2, differs in the drift from the dynamics of X under $\mathbb{Q}_1$. In order to compute the new drift we need to consider the dynamics of the logarithm of the ratio of N_t^1 over N_t^2, in particular its diffusion coefficient, which is invariant under equivalent measures.

In case of the Libor Market Model the variables modelled are forward rates with expiry T_{j-1} and maturity T_j

$$F_j\left(t\right) = \frac{1}{\alpha_j}\left(\frac{P(t,T_{j-1})}{P(t,T_j)} - 1\right) \quad j = 1, \ldots, M. \tag{15}$$

In this case the probability measures to consider are $\mathbb{Q}_1 = \mathbb{Q}^k$, the T_k-forward measure associated to the T_k-maturity zero-coupon bond $P\left(t, T_k\right)$, and $\mathbb{Q}_2 = \mathbb{Q}^i$, the T_i-forward measure associated to $P\left(t, T_i\right)$. If for example $i > k$, the numeraire ratio is

$$\frac{P\left(t,T_k\right)}{P\left(t,T_i\right)} = \prod_{j=k+1}^{i} \frac{P(t,T_{j-1})}{P(t,T_j)} = \prod_{j=k+1}^{i} \left(1 + \alpha_j F_j(t)\right)$$

The numeraire ratio is a function of the state variables being modelled in the LMM and the diffusion coefficient of its logarithm is easily computed.

CDS numeraire ratios. Considering CDS forward rates R_j, the relevant numeraire ratios have the form

$$\frac{\mathbb{C}_t^k}{\mathbb{C}_t^i} = \frac{\alpha_k \bar{P}\left(t,T_k\right)}{\alpha_i \bar{P}\left(t,T_i\right)}. \tag{16}$$

Notice that

$$R_j\left(t\right) = \frac{\text{LGD}}{\alpha_j} \frac{\left\{\frac{\mathbb{E}^Q\left[D(t,T_j)\mathbf{1}_{\{\tau > T_{j-1}\}}|\mathcal{H}_t\right]}{Q(\tau>t|\mathcal{H}_t)}\right\}}{\bar{P}\left(t,T_j\right)} - 1,$$

where in the quantity between curly brackets the discount factor refers to a time different from the time in the default indicator. Thus we do not have a defaultable bond numeraire, so $R_j\left(t\right)$ cannot be expressed as a function of only numeraire ratios. In turn numeraire ratios cannot be expressed only in terms of $R_j\left(t\right)$'s.

This is even clearer making use of the change of measure approach. Express the numeraire ratio via

$$\frac{\bar{P}\left(t,T_k\right)}{\bar{P}\left(t,T_i\right)} = \frac{P\left(t,T_k\right)\mathbb{Q}^k\left(\tau > T_k|\mathcal{H}_t\right)}{P\left(t,T_i\right)\mathbb{Q}^i\left(\tau > T_i|\mathcal{H}_t\right)} = \prod_{j=k+1}^{i} \frac{P\left(t,T_{j-1}\right)}{P\left(t,T_j\right)} \frac{\mathbb{Q}^{j-1}\left(\tau > T_{j-1}|\mathcal{H}_t\right)}{\mathbb{Q}^j\left(\tau > T_j|\mathcal{H}_t\right)}$$

supposing $i > k$.

Thanks to this change of measure, we can apply the definition of forward Libor rates (15) and find

$$\frac{\bar{P}\left(t,T_k\right)}{\bar{P}\left(t,T_i\right)} = \prod_{j=k+1}^{i} \left(1 + \alpha_j F_j(t)\right) \frac{\mathbb{Q}^{j-1}\left(\tau > T_{j-1}|\mathcal{H}_t\right)}{\mathbb{Q}^j\left(\tau > T_j|\mathcal{H}_t\right)}.$$

But we cannot express the probability ratio in terms of defaultable forward rates, since

$$R_j(t) = \frac{\text{LGD}}{\alpha_j}\left(\frac{\mathbb{Q}^j(\tau > T_{j-1}|\mathcal{H}_t)}{\mathbb{Q}^j(\tau > T_j|\mathcal{H}_t)} - 1\right)$$

and

$$\frac{\mathbb{Q}^j(\tau > T_{j-1}|\mathcal{H}_t)}{\mathbb{Q}^j(\tau > T_j|\mathcal{H}_t)} \neq \frac{\mathbb{Q}^{j-1}(\tau > T_{j-1}|\mathcal{H}_t)}{\mathbb{Q}^j(\tau > T_j|\mathcal{H}_t)}$$

because of a mismatch in the measure under which the probabilities at numeraire are taken.

The presence of default risk besides interest rates risk adds degrees of freedom, and, in order to build a model for a complete term structure, additional assumptions need to be done compared to a standard default free market model.

One possibility, considered by Brigo (2004), is modelling also additional rates, for example two-period CDS rates $R_{j-2,j}(t)$, $j = 2, \ldots, M$. The choice appears appropriate since, thanks to (10), we have

$$\frac{\bar{P}(t, T_k)}{\bar{P}(t, T_i)} = \frac{\alpha_i}{\alpha_k}\prod_{j=i+1}^{k}\frac{R_{j-1}(t) - R_{j-2,j}(t)}{R_{j-2,j}(t) - R_j(t)}$$

and from this all required numeraire ratios are specified, so that we could in principle complete the model, modelling both $R_j(t)$ and $R_{j-2,j}(t)$ rates as lognormal martingales under their natural measures.

Financial Behaviour. Unfortunately, as noticed by Brigo (2004), this holds mathematically but does not hold in financial terms. We are not free to model the two-period rate as we find more convenient, since the defaultable bonds must remain positive and decreasing in the maturity:

$$0 < \frac{\bar{P}(t, T_j)}{\bar{P}(t, T_{j-1})} < 1.$$

This translates into the following constraints

$$\min\left(R_{j-1}(t), \frac{R_{j-1}(t) + R_j(t)}{2}\right) < R_{j-2,j}(t) < \max\left(R_{j-1}(t), \frac{R_{j-1}(t) + R_j(t)}{2}\right),$$

which require specific, nonstandard dynamics whose existence is still to be demonstrated. Analogous problems arise if considering one-period CDS rates and multiperiod co-terminal CDS rates, namely with common final date $T_{M.}$

The analysis previously carried out indicates that the problem can be also interpreted as a measure mismatch. This mismatch disappears if, as in Hull and White (2003), Wu (2005) and in some parts of Schonbucher (1999) we assume independence of interest rates and default probability. This is shown in the next section, where we also present how the CDS rate definitions and CDS option formulas here presented can be applied to develop closed-form formulas for other exotic credit derivatives, gaining tractability via standard approximations.

5 Dynamics and Constant Maturity CDS

Notice first that, when interest rates are independent of the default event, our CDS forward spread becomes

$$\begin{aligned} R_j(t) &= \mathrm{LGD}\frac{P(t,T_j)\,\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]-P(t,T_j)\,\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}{P(t,T_j)\,\alpha_j\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]} \\ &= \frac{\mathrm{LGD}}{\alpha_j}\left(\frac{\mathbb{Q}(\tau>T_{j-1}|\mathcal{H}_t)}{\mathbb{Q}(\tau>T_j|\mathcal{H}_t)}-1\right) \end{aligned}$$

and the numeraire ratio is

$$\frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)}=\frac{\mathbb{E}^Q\left[D(t,T_{j-1})\,\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{H}_t\right]}{\mathbb{E}^Q\left[D(t,T_j)\,\mathbf{1}_{\{\tau>T_j\}}|\mathcal{H}_t\right]}=\frac{P(t,T_{j-1})\,\mathbb{Q}(\tau>T_{j-1}|\mathcal{H}_t)}{P(t,T_j)\,\mathbb{Q}(\tau>T_j|\mathcal{H}_t)}$$

Hence there is no measure mismatch

$$\begin{aligned} \frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} &= \left(R_j(t)\frac{\alpha_j}{\mathrm{LGD}}+1\right)(F_j(t)\,\alpha_j+1)\,, \\ \frac{\bar{P}(t,T_k)}{\bar{P}(t,T_i)} &= \prod_{j=k+1}^{i}\left(R_j(t)\frac{\alpha_j}{\mathrm{LGD}}+1\right)(F_j(t)\,\alpha_j+1)\,,\quad i>k \end{aligned}$$

Financial behaviour. With this choice of modelling variables the numeraire ratios are specified and we are free to model each variable according to a standard lognormal dynamics under the corresponding natural measure. The resulting behaviour of defaultable bonds satisfies all required financial regularities, in fact

$$\frac{\bar{P}(t,T_j)}{\bar{P}(t,T_{j-1})}=\frac{1}{\left(R_j(t)\frac{\alpha_j}{\mathrm{LGD}}+1\right)(F_j(t)\,\alpha_j+1)},$$

with

$$0<\frac{1}{\left(R_j(t)\frac{\alpha_j}{\mathrm{LGD}}+1\right)(F_j(t)\,\alpha_j+1)}<1$$

whenever the rates in the model are bounded away from zero. Moreover

$$\begin{aligned} \frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} &= \frac{P(t,T_{j-1})}{P(t,T_j)}\left(R_j(t)\frac{\alpha_j}{\mathrm{LGD}}+1\right), \\ \frac{1}{\bar{P}(T_{j-1},T_j)} &= \frac{1}{P(T_{j-1},T_j)}\left(R_j(T_{j-1})\frac{\alpha_j}{\mathrm{LGD}}+1\right), \\ \bar{P}(T_{j-1},T_j) &= \frac{P(T_{j-1},T_j)}{\left(R_j(T_{j-1})\frac{\alpha_j}{\mathrm{LGD}}+1\right)}<P(T_{j-1},T_j) \end{aligned}$$

We are now ready to define a complete model.

5.1 A Libor and CDS Market Model under independence

Change of numeraire for dynamics. Assume that under a measure Q_1 associated with numeraire N^1 the dynamics of the process X is given by

$$dX_t = \mu_t dt + \sigma_t dW_t^1,$$

where X, μ are M-dimensional vectors and W^1 is standard M-dimensional brownian motion under $\mathbb{Q}_1$ with instantaneous correlation Ξ. The matrix σ_t is $M \times M$ and diagonal. Then the dynamics of X under an equivalent measure $\mathbb{Q}_2$ associated with N^2 is

$$dX_t = \left(\mu_t - \sigma_t \Xi \Sigma_{1,2}(t)'\right) dt + \sigma_t dW_t^2, \tag{17}$$

where $\Sigma_{1,2}(t)$, or $\mathrm{DC}\left(\ln \frac{N_t^1}{N_t^2}\right)$, where DC stand for diffusion coefficient, is defined by

$$d\left(\ln \frac{N_t^1}{N_t^2}\right) = U_t^x dt + \Sigma_{1,2}(t)\, dW_t^x, \quad x = 1, 2.$$

In terms of stochastic shocks the equivalent formula is

$$dW_t^1 = dW_t^2 - \Xi \Sigma_{1,2}(t)' \, dt \tag{18}$$

CDS and Libor Market Model Assume a tenor structure $\{T_0, T_1, \ldots, T_M\}$. For $k = 1, \ldots, M$, our variables are: $F_k(t)$, the simply compounded forward rate resetting at T_{k-1} and with maturity T_k, and $R_k(t)$, the CDS par spread for period from T_{k-1} to T_k. $\mathbb{Q}^k$ is the equivalent T_k-forward measure associated with the numeraire bond $P(t, T_k)$, and $\bar{\mathbb{Q}}^k$ is the equivalent measure associated with the credit numeraire $\alpha_k \bar{P}(t, T_k)$.
For $k = 1, \ldots, M$,

$$dR_k(t) = \bar{\sigma}_k(t) R_k(t) dV_k^k(t), \ \ t \le T_{k-1},$$

where $V_k^k(t)$ is the k-th component of an M-dimensional Brownian motion $V^k(t)$ under $\bar{\mathbb{Q}}^k$ and

$$dF_k(t) = \sigma_k(t) F_k(t) dZ_k^k(t), \ \ t \le T_{k-1},$$

where $Z_k^k(t)$ is the k-th component of an M-dimensional Brownian motion $Z^k(t)$ under $\mathbb{Q}^k$. The correlation structure is

$$\begin{aligned} dV_i dV_j &= \rho_{ij} dt, \\ dZ_i dZ_j &= \delta_{ij} dt, \\ dV_i dZ_j &= 0. \end{aligned}$$

since we have assumed independence of default probabilities and default-free interest rates.
Define $h_i^j := 1_{\{j<i\}} - 1_{\{j>i\}}$.
The dynamics of $R_i(t)$ under $\bar{\mathbb{Q}}^j$ is

$$dR_i(t) = h_i^j \bar{\sigma}_i(t) R_i(t) \sum_{h=(i\wedge j)+1}^{i \vee j} \rho_{i,h} \frac{\bar{\sigma}_h(t) R_h(t)}{R_h(t) + \frac{\mathrm{LGD}}{\alpha_h}} dt + \bar{\sigma}_i(t) R_i(t) dV_i^j(t).$$

for $t \leq T_{(i-1)\wedge j}$.
Proof Apply (18) to

$$\begin{aligned} X &= (R_1, \ldots, R_M, F_1, \ldots, F_M)', \\ W &= \begin{bmatrix} V \\ Z \end{bmatrix}, \\ \sigma &= diag(\bar{\sigma}_1, \ldots, \bar{\sigma}_M, \sigma_1, \ldots, \sigma_M), \\ \Xi &= \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix}, \end{aligned}$$

recalling that with $i > j$

$$\frac{\bar{P}(t, T_j)}{\bar{P}(t, T_i)} = \prod_{h=j+1}^{i} \left(R_h(t) \frac{\alpha_h}{\text{LGD}} + 1 \right) (F_h(t) \alpha_h + 1).$$

For the dynamics of CDS spread $R_i(t)$ under $\bar{\mathbb{Q}}^j$, $i > j$, set $d\bar{Z}^k$ for dZ under $\bar{\mathbb{Q}}^k$, so

$$\begin{aligned} d \begin{bmatrix} V^j \\ \bar{Z}^j \end{bmatrix} &= d \begin{bmatrix} V^i \\ Z^i \end{bmatrix} - \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix} \text{DC} \left(\ln \frac{\mathbb{C}_t^j}{\mathbb{C}_t^i} \right) dt \\ = \quad & d \begin{bmatrix} V^i \\ \bar{Z}^i \end{bmatrix} - \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix} \text{DC} \left(\ln \frac{\bar{P}(t, T_j)}{\bar{P}(t, T_i)} \right)' dt \\ = \quad & d \begin{bmatrix} V^i \\ \bar{Z}^i \end{bmatrix} - \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix} \sum_{h=j+1}^{i} \left[\text{DC} \left(\ln \left(R_h(t) \frac{\alpha_h}{\text{LGD}} + 1 \right) \right)' + \text{DC} \left(\ln \left(F_h(t) \alpha_h + 1 \right) \right)' \right] dt \\ = \quad & d \begin{bmatrix} V^i \\ \bar{Z}^i \end{bmatrix} - \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix} \sum_{h=j+1}^{i} \left[\frac{\text{DC}(R_h(t))'}{R_h(t) + \frac{\text{LGD}}{\alpha_h}} + \frac{\text{DC}(F_h(t))'}{F_h + \frac{1}{\alpha_h}} \right] dt, \end{aligned}$$

$$dV_i^j = dV_i^i - \sum_{h=j+1}^{i} \rho_{i,h} \frac{\bar{\sigma}_h(t) R_h(t)}{R_h(t) + \frac{\text{LGD}}{\alpha_h}} dt,$$

and

$$\begin{aligned} dR_i(t) &= \bar{\sigma}_i(t) R_i(t) dV_i^i(t) = \bar{\sigma}_i(t) R_i(t) \sum_{h=j+1}^{i} \rho_{i,h} \frac{\bar{\sigma}_h(t) R_h(t)}{R_h(t) + \frac{\text{LGD}}{\alpha_h}} dt + \bar{\sigma}_i(t) R_i(t) dV_i^j(t) \\ &=: \bar{\sigma}_i(t) R_i(t) \mu_i^j(R(t), t) dt + \bar{\sigma}_i(t) R_i(t) dV_i^j(t). \end{aligned}$$

For Libor rates the derivation is analogous, and consistent with the standard Libor Market Model of Brace, Gatarek and Musiela (1997) and Jamshidian (1997).

5.2 Constant maturity CDS

We see now application of the model to Constant Maturity CDS (CMCDS). In a CM-CDS two parties, called the protection buyer and the protection seller agree that if a third company, called the reference entity, defaults at time τ, $T_a < \tau \leq T_b$, then the protection seller pays to the protection buyer an amount LGD at the first T_i following default time. In exchange the protection buyer pays periodically at all T_i before default the $(c+1)$-long CDS rate $R_{i-1,i+c}(T_{i-1})$ times α_i

This product, developed for giving investors the flexibility of a CDS rate adapting periodically to changing market conditions, is particularly suitable for being evaluated via market models. Also the Constant Maturity Swap of the default-free interest rate market is easily evaluated in a Libor Market Model, in particular closed-form formulas are easily obtained via well-established approximation procedures. Now we see a simple application in the context of CDS market models.

The first approximation we apply is known as drift freezing approximation: in the dynamics of $R_i(t)$ we assume $\mu_i^j(R(t),t) \approx \mu_i^j(R(0),t)$. This approximation, tested via Monte Carlo simulation for the LMM, can be justified noticing the presence of rates in both numerator and denominator of $\mu_{i,k}$, so that their volatilities tend to partially cancel out. With this approximation $R_i(t)$ is a geometric brownian motion, so

$$\begin{aligned}\bar{\mathbb{E}}^j\left[R_i(T_{j-1})\right] &\approx R_i(0)\exp\left[\int_0^{T_{j-1}} \mu_i^j(R(0),t)\,dt\right]\\ &= R_i(0)\exp\left[\sum_{h=j+1}^{i}\frac{R_h(0)}{R_h(0)+\frac{\text{LGD}}{\alpha_h}}\rho_{i,h}\int_0^{T_{j-1}}\sigma_h(t)\,\sigma_i(t)\,dt\right]\end{aligned}$$

The second approximation, also typical of the default free market model and justified in related literature, represents swap rates as linear combinations of forward rates, assuming

$$R_{j-1,j+c}(T_{j-1}) \approx \sum_{i=j}^{j+c}\bar{w}_i^j(0)\,R_i(T_{j-1}), \qquad \bar{w}_i^j(t) = \frac{\alpha_i\bar{P}(t,T_i)}{\sum_{h=j}^{j+c}\alpha_h\bar{P}(t,T_h)}$$

Since the protection leg is like in a standard CDS, valuing a CMCDS amounts to valuing the premium leg, computing

$$\begin{aligned}&\sum_{j=a+1}^{b}\alpha_j\mathbb{E}_0^Q[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}}R_{j-1,j+c}(T_{j-1})]\\ \approx\ &\sum_{j=a+1}^{b}\sum_{i=j}^{j+c}\alpha_j\bar{w}_i^j(0)\mathbb{E}_0^Q[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}}R_i(T_{j-1})]\\ =\ &\sum_{j=a+1}^{b}\sum_{i=j}^{j+c}\alpha_j\bar{w}_i^j(0)\mathbb{E}_0^Q[D(0,T_j)R_i(T_{j-1})\mathbb{E}^Q(\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_{T_j})]\\ =\ &\sum_{j=a+1}^{b}\sum_{i=j}^{j+c}\bar{w}_i^j(0)\mathbb{E}_0^Q\left[D(0,T_j)(R_i(T_{j-1})\,\mathbb{C}_{T_j}^j)\right]\\ =\ &\sum_{j=a+1}^{b}\sum_{i=j}^{j+c}\bar{w}_i^j(0)\alpha_j\bar{P}(0,T_j)\bar{\mathbb{E}}_0^j[R_i(T_{j-1})].\end{aligned}$$

Now we apply the drift freezing approximation and can easily compute the CMCDS price with a closed-form formula. Assuming time-constant volatility we find the same formula obtained, modelling with different assumptions, in Brigo (2004b), giving the price at time 0 as

$$\mathbf{CMCDS}_{a,b,c}(0) \;=$$
$$= \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j) \left\{ \sum_{i=j}^{j+c} \bar{w}_i^j(0) R_i(0) \exp\left[T_{j-1}\sigma_i \sum_{h=j+1}^{i} \frac{\sigma_h R_h(0)}{R_h(0) + \frac{\mathrm{LGD}}{\alpha_h}} \rho_{i,h} \right] - R_j(0) \right\}.$$

5.3 Empirical application

We consider the FIAT car company CDS market quotes of December 20, 2004, with $REC = 0.4$. We start by giving a table for

$$conv(\sigma,\rho) := \mathbf{CMCDS}_{a,b,c}(0;\sigma,\rho) - \mathbf{CMCDS}_{a,b,c}(0;\rho=0).$$

with $\sigma_i = \sigma$ and $\rho_{i,j} = \rho$. The second term is the value where no correction due to CDS forward rate dynamics is accounted for: $\sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j)\,\{R_{j-1,j+c}(0) - R_j(0)\}$. This difference gives the impact of volatilities and correlations of CDS rates on the CMCDS price. We take $a = 0$, $b = 20$, $c = 20$ (resetting quarterly).

$conv(\sigma,\rho)$	ρ: 0.7	0.8	0.9	0.99
σ: 0.1	0.000659	0.000754	0.000848	0.000933
0.2	0.002662	0.003047	0.003435	0.003784
0.4	0.011066	0.012742	0.014442	0.015995
0.6	0.026619	0.030964	0.035464	0.039652

The "convexity difference" increases with respect both to correlation and volatility, as expected. The next table reports the so called "participation rate" $\phi_{a,b,c}(\sigma,\rho)$

$$\phi_{0,20,20}(\sigma,\rho) = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS”}} = \frac{\sum_{j=1}^{20} \alpha_j \bar{P}(0,T_j) R_{0,20}(0)}{\sum_{j=1}^{20} \alpha_j \bar{\mathbb{E}}_0^{j-1,j}[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_{j-1,j+20}(T_{j-1})]},$$

$\phi_{0,20,20}(\sigma,\rho)$	ρ: 0.7	0.8	0.9	0.99
σ: 0.1	0.71358	0.71325	0.71292	0.71262
0.2	0.70664	0.70532	0.704	0.70281
0.4	0.67894	0.67368	0.66842	0.66368
0.6	0.63302	0.62128	0.60957	0.59907

The participation rate decreases with volatility and correlation.

In the following section we abandon the hypothesis of independence of default-free interest rates and default probabilities. We signal that, rather than adding other rates or assuming independence, one can also consider modelling more fundamental quantities to complete the model. For consistency with previous literature, in particular Schonbucher (1999), here we develop this possibility starting from the expression (2) in defining a conventional CDS contract. This allows to underline similarities with previous approaches but also the differences and the advantages of the current approach. However a similar solution can be applied also starting from the feasible market payoff (1), an alternative that will be addressed in subsequent research.

5.4 A different model with approximated payoff

As we mentioned in the introduction, the first important development in the field of market models for credit derivatives is given in Schonbucher (1999). Therefore we show in this last sections relationships and differences with the approach of our work.

Building blocks. Schonbucher (1999) builds a market model based on the following fundamental quantities, in the original notation:

1. T_k-maturity default free bond $B_k(t)$
2. T_k-maturity defaultable bond $\mathbf{1}_{\{\tau>t\}}\bar{B}_k(t)$
3. A ratio defined as
$$D_k(t) = \frac{\bar{B}_k(t)}{B_k(t)}$$

Since no subfiltration structure is introduced in Schonbucher (1999), no other representation of the quantity $\bar{B}_k(t)$ is given. In our subfiltration setting, we can give a precise description of this quantity, in terms of its constituent components, since defaultable bonds can be expressed as

$$\mathbf{1}_{\{\tau>t\}}\bar{P}(t,T_k) = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{E}^Q\left[D(t,T_k)\,\mathbf{1}_{\{\tau>T_k\}}|\mathcal{H}_t\right]}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}.$$

The nature of $\bar{P}(t,T_k)$ is therefore not opaque, and we can exploit its definition for tractability. An application is describing $D_k(t)$, which is here[5]

$$\begin{aligned}\frac{\bar{P}(t,T_k)}{P(t,T_k)} &= \frac{P(t,T_k)\,\mathbb{Q}^k(\tau>T_k|\mathcal{H}_t)}{P(t,T_k)\,\mathbb{Q}(\tau>t|\mathcal{H}_t)} = \frac{\mathbb{Q}^k(\tau>T_k|\mathcal{H}_t)}{\mathbb{Q}(\tau>t|\mathcal{H}_t)}\\ &= \frac{\mathbb{Q}^k(\tau>T_k|\mathcal{H}_t)}{\mathbb{Q}^k(\tau>t|\mathcal{H}_t)} =: Q_H^k(t)\end{aligned}$$

easily interpreted as a forward conditional (Bayes) survival probability based on default-free information.

Schonbucher assumes for the quantity $D_k(t)$ a dynamics under a generic measure. With subfiltrations is the dynamics of the above probability

$$\frac{dQ_H^k(t)}{Q_H^k(t-)} = \ldots dt - \sigma_H^k dW$$

for a generic covariance vector process σ_H^k and a vector W of uncorrelated standard brownian motions.

[5] Notice the following property

$$\mathbb{Q}^k(\tau>t|\mathcal{H}_t) = \mathbb{E}^k\left[\mathbf{1}_{\{\tau>t\}}|\mathcal{H}_t\right] = \frac{\mathbb{E}^Q\left[D(t,T_k)\mathbf{1}_{\{\tau>t\}}|\mathcal{H}_t\right]}{\mathbb{E}^Q\left[D(t,T_k)|\mathcal{H}_t\right]}$$

$$= \frac{\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>t\}}\mathbb{E}^Q\left[D(t,T_k|\mathcal{F}_t)\right]|\mathcal{H}_t\right]}{\mathbb{E}^Q\left[\mathbb{E}^Q\left[D(t,T_k)\mathcal{F}_t\right]|\mathcal{H}_t\right]} = \frac{P(t,T_k)\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>t\}}|\mathcal{H}_t\right]}{P(t,T_k)} = \mathbb{Q}(\tau>t|\mathcal{H}_t)$$

Obviously modelling this quantity can allow further development of a model built on the synthetic real-world payoff (1) as in the previous sections. But in this work we prefer to show possible relationships with Schonbucher (1999).

Rates. Schonbucher (1999) defines the following rates (with a slightly different notation for maturities)

1. Default free Libor rate

$$F_k(t) = \frac{1}{\alpha_k}\left(\frac{B_{k-1}(t)}{B_k(t)} - 1\right)$$

2. Defaultable Libor rate defined as

$$\bar{F}_k(t) = \frac{1}{\alpha_k}\left(\frac{\bar{B}_{k-1}(t)}{\bar{B}_k(t)} - 1\right),$$

3. Discrete-tenor forward intensity defined as

$$H_k(t) = \frac{1}{\alpha_k}\left(\frac{B_k(t)\,\bar{B}_{k-1}(t)}{\bar{B}_k(t)\,B_{k-1}(t)} - 1\right)$$

Adapting the notation for bonds, the default free Libor rate $F_k(t)$ is equivalent to our default free forward rate and to that of the standard market model.
As for the defaultable forward rate, we signal that in a subfiltration setting it is possible to find an analogous quantity setting the price of the contract to zero. One needs to consider contract (2) for a one-period interval:

$$\begin{aligned}
\mathbf{CDS}_t^S(R) &= \textsc{Lgd}\left\{\mathbb{E}^Q\left[D(t,T_{k-1})\,\mathbf{1}_{\{\tau>T_{k-1}\}}|\mathcal{H}_t\right] - \mathbb{E}^Q\left[D(t,T_k)\,\mathbf{1}_{\{\tau>T_k\}}|\mathcal{H}\right]\right\} \\
&\quad -\mathbb{E}^Q\left[D(t,T_k)\,\mathbf{1}_{\{\tau>T_k\}}|\mathcal{H}\right]\alpha_k R, \\
R_k^S(t) &= \frac{\textsc{Lgd}}{\alpha_k}\frac{\mathbb{E}^Q\left[D(t,T_{k-1})\,\mathbf{1}_{\{\tau>T_{k-1}\}}|\mathcal{H}_t\right] - \mathbb{E}^Q\left[D(t,T_k)\,\mathbf{1}_{\{\tau>T_k\}}|\mathcal{H}\right]}{\mathbb{E}^Q\left[D(t,T_k)\,\mathbf{1}_{\{\tau>T_k\}}|\mathcal{H}\right]} \\
&= \frac{\textsc{Lgd}}{\alpha_k}\left\{\frac{\bar{P}(t,T_{k-1})}{\bar{P}(t,T_k)} - 1\right\}
\end{aligned}$$

Here the nature of such a rate is

$$R_k^S(t) = \frac{\textsc{Lgd}}{\alpha_k}\left\{\frac{P(t,T_{k-1})\,\mathbb{Q}^{k-1}(\tau > T_{k-1}|\mathcal{H}_t)}{P(t,T_k)\,\mathbb{Q}^k(\tau > T_k|\mathcal{H}_t)} - 1\right\}.$$

In our context, we can perform the following simplification in defining the analogous of $H_k(t)$:

$$\frac{1}{\alpha_k}\left(\frac{P(t,T_k)\,\bar{P}(t,T_{k-1})}{\bar{P}(t,T_k)\,P(t,T_{k-1})} - 1\right) = \frac{1}{\alpha_k}\left(\frac{\mathbb{Q}^{k-1}(\tau > T_{k-1}|\mathcal{H}_t)}{\mathbb{Q}^k(\tau > T_k|\mathcal{H}_t)} - 1\right) =: \tilde{R}_k(t)$$

very close to R that we presented in the previous sections, and equal to it in case of independence, when however probabilities can be taken also under the risk neutral measure.

Looking for a measure under which $\bar{F}$ is a martingale, Schonbucher (1999) introduces the T_k survival measure associated to the defaultable bond price numeraire $\mathbf{1}_{\{\tau>t\}}\bar{B}_k(t)$. This numeraire is not strictly positive, and this leads to a measure which is not equivalent to the risk-neutral and real world probability measures.

In our setting, based on Jamshidian (2004), one can change instead to the measure $\bar{\mathbb{Q}}^k$ associated with $\alpha_k\bar{P}(t,T_k)$. The fundamental martingale properties are maintained and in addition the measure is equivalent to the risk-neutral and real world probability measures as in standard mathematical finance, and change of measure and dynamics is performed as usual.

With the building blocks of this section the numeraire ratios are easily expressed in terms of $R_k^S(t)$'s. However, with these building blocks, by modelling the $R_k^S(t)$'s directly as lognormal random variables we may incur into an unacceptable financial behaviour. $H_k(t)$ can rather be modelled as having deterministic percentage volatility.

We have under $\bar{\mathbb{Q}}^k$

$$dR_k^S(t) = \sigma_k^S R_k^S(t)\, d\bar{W}^k(t)$$

where σ_k^S is a generic covariance vector process and $\bar{W}$ a vector of uncorrelated standard brownian motions under $\bar{\mathbb{Q}}^k$.

As for the default-free forward rate, applying (17),

$$dF_k(t) = -F_k(t)\,\sigma_k^F\sigma_H^{k\prime}dt + F_k(t)\,\sigma_k^F d\bar{W}^k(t)\,.$$

where σ_k^F is a deterministic covariance vector process. Then the dynamics of $\tilde{R}_k(t)$ can be computed through Ito's Formula:

$$d\tilde{R}_k(t) = \frac{F_k(t)\,\sigma_k^F}{1+\alpha_k F_k(t)}\left(\left(1+\alpha_k\tilde{R}_k(t)\right)\sigma_H^{k\prime} - \alpha_k\tilde{R}_k(t)\,\tilde{\sigma}_k\right)dt + \tilde{R}_k(t)\,\tilde{\sigma}_k d\bar{W}^k(t)\,,$$

where $\tilde{\sigma}_k$ is a deterministic covariance vector process.

6 Conclusion

In this work we analyze CDS pricing in a probabilistic setting equipped with a subfiltration structure. We derive pricing formulas consistent with the standard market model framework, with particular attention to the case of conditional independence for subfiltrations. We consider possibilities for a term structure model and analyze the nature of CDS spreads via change of measure. We compute a term structure model dynamics and apply it to deriving an approximated formula for Constant Maturity CDS. We conclude showing a term structure model in the same setting derived from an alternative definition of conventional CDS contracts pointing out relations and differences with previous literature.

7 Appendix

Proposition 7 $\mathbb{C}^{a,b} = C_{a,b}(T_a)\,\frac{B}{B_{T_a}} = \sum_{i=a+1}^{b}\alpha_i\mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right]\frac{B}{B_{T_a}} > 0$ *almost surely.*

Proof. We need only to prove that $\mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right]>0$ a.s. Notice that

$$\begin{aligned}\mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right] &= \mathbb{E}^Q\left[\mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{\tau>T_i\}}\middle|\mathcal{H}_{T_i}\right]\middle|\mathcal{H}_{T_a}\right]\\ &= \mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbb{E}^Q\left[\mathbf{1}_{\{\tau>T_i\}}\middle|\mathcal{H}_{T_i}\right]\middle|\mathcal{H}_{T_a}\right]\\ &= \mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbb{Q}\left(\tau>T_i|\,\mathcal{H}_{T_i}\right)\middle|\mathcal{H}_{T_a}\right].\end{aligned}$$

We assumed that $\mathbb{Q}\left(\tau>t|\,\mathcal{H}_t\right)>0, \forall t$. Since $B_t>0, \forall t$, $\mathbb{E}^Q\left[\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_{T_a}\right]>0$ a.s. ■

We prove now that

Proposition 8 *$\mathcal{H}_t$ is $\mathbb{Q}$-conditionally independent implies that $\mathcal{H}_t$ is $\bar{\mathbb{Q}}^{a,b}$-conditionally independent, namely that for any Y bounded and $\mathcal{H}$-measurable*

$$\bar{\mathbb{E}}^{a,b}\left[Y|\mathcal{H}_t\right]=\bar{\mathbb{E}}^{a,b}\left[Y|\mathcal{F}_t\right],\ \forall t.$$

Proof. Notice that $\frac{\mathbb{C}_t^{a,b}}{B_t}$ is $\mathcal{H}_t$-adapted. In fact $\frac{\mathbb{C}^{a,b}}{B}=\frac{C_{a,b}(T_a)}{B_{T_a}}$ is $\mathcal{H}$-measurable, so given that $\mathcal{H}_t$ is $\mathbb{Q}$-conditionally independent

$$\frac{\mathbb{C}_t^{a,b}}{B_t}=\mathbb{E}^Q\left[\frac{\mathbb{C}^{a,b}}{B}\middle|\mathcal{H}_t\right]. \tag{19}$$

Let X to be a claim such that $\frac{X}{\mathbb{C}^{a,b}}$ is $\mathcal{H}$-measurable. Then also $\frac{X}{\mathbb{B}}$ is $\mathcal{H}$-measurable and, as above, $\frac{X_t}{B_t}$ is $\mathcal{H}_t$-adapted. This implies that $\frac{X_t}{\mathbb{C}_t^{a,b}}=\frac{X_t}{B_t}\frac{B_t}{\mathbb{C}_t^{a,b}}$ is $\mathcal{H}_t$-adapted (by (19)).So

$$\bar{\mathbb{E}}^{a,b}\left[\frac{X}{\mathbb{C}^{a,b}}\middle|\mathcal{H}_t\right]=\frac{X_t}{\mathbb{C}_t^{a,b}}=\bar{\mathbb{E}}^{a,b}\left[\frac{X}{\mathbb{C}^{a,b}}\middle|\mathcal{F}_t\right],\ \forall t$$

(see also Jamshidian (2004) and the concept of coadaptedness). ■

Now consider a σ-subalgebra $\mathcal{N}$ of σ-algebra $\mathcal{M}$ and an $\mathcal{M}$-measurable X, integrable under the measures considered. We have the following results.

Lemma 9 *If X is $\mathcal{N}$-measurable*

$$\mathbb{E}^{P2}\left[X\right]=\mathbb{E}^{P1}\left[X\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]\right].$$

In fact $\mathbb{E}^{P1}\left[\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]X\right]=\mathbb{E}^{P1}\left[\mathbb{E}^{P1}\left[\frac{dP2}{dP1}X|\mathcal{N}\right]\right]=\mathbb{E}^{P1}\left[\frac{dP2}{dP1}X\right]=\mathbb{E}^{P2}\left[X\right].$

This implies that if X is $\mathcal{N}$-measurable and $A\in\mathcal{N}$

$$\mathbb{E}^{P2}\left[1_AX\right]=\mathbb{E}^{P1}\left[1_AX\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]\right],$$

namely

$$\int_A XdP2=\int_A X\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]dP1.$$

Theorem 10 *When X is $\mathcal{M}$-measurable*

$$\mathbb{E}^{P2}\left[X|\mathcal{N}\right] = \mathbb{E}^{P1}\left[X\frac{dP2}{dP1}\frac{1}{\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]}\middle|\mathcal{N}\right].$$

Proof. The RHS is by definition $\mathcal{N}$-measurable. We apply Lemma 9. For $A \in \mathcal{N}$

$$\begin{aligned} & \int_A \mathbb{E}^{P1}\left[X\frac{dP2}{dP1}\frac{1}{\mathbb{E}^{P1}\left[\frac{dP2}{dP1}|\mathcal{N}\right]}\middle|\mathcal{N}\right] dP2 \\ = & \int_A \mathbb{E}^{P1}\left[X\frac{dP2}{dP1}\middle|\mathcal{N}\right] dP1 \end{aligned}$$

By definition of conditional expectation

$$\int_A \mathbb{E}^{P1}\left[X\frac{dP2}{dP1}\middle|\mathcal{N}\right] dP1 = \int_A X\frac{dP2}{dP1}dP1 = \int_A X dP2.$$

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