# Lois: credit and liquidity

*The spread between Libor and overnight index swap rates used to be negligible – until the crisis. Its behaviour since can be explained theoretically and empirically by a model driven by typical lenders' liquidity and typical borrowers' credit risk. By* ***Stéphane Crépey*** *and* ***Raphaël Douady***



Most fixed-income derivatives reference Libor or the euro interbank offered rate (Euribor), calculated daily at tenors up to a year as an average of the rates at which a panel of banks believe they can obtain unsecured funding. In the past, these were nearly identical to the overnight indexed swap (OIS) rates, which are calculated by compounding some rate reflecting the cost of unsecured overnight interbank lending, such as the federal funds rate or the euro overnight index average (Eonia) rate (see figure 1).

However, since the 2007 subprime crisis, trust has broken down between banks and the rates have diverged. As more derivatives became collateralised, their effective funding rate became the OIS, while the reference rate remains Libor or Euribor. This creates a situation where the value of a product is dependent on the corresponding spread.

The Libor-OIS spread – sometimes known as the Lois – is commonly explained as a combination of credit and liquidity risk premiums, but the second of these is rarely precisely defined or examined carefully. This article presents a stylised equilibrium model to find breakeven rates at which banks will lend for a given tenor rather than rolling over an overnight rate. In this setup, the Lois is a consequence of the skew of a Libor panel representative's credit curve, and the volatility of the spread of its funding rate over the overnight rate. The model is calibrated to euro data between July 7, 2005 and April 16, 2012.

**Equilibrium model**

We assume the funding rate of a bank raising €$x$ to roll over its current short-term debt $D_t$ at time $t$ is given by an increasing random function $\rho_t(D_t + x)$. This represents the annualised rate charged to the bank for the last of the $x$ borrowed euros. It is likely to be a complicated function of various factors such as the overnight rate $r_t$, the bank's credit default swap (CDS) spread, and its leverage. The total refinancing cost for a loan of €$N$ is given by integrating this function in the $x$ variable from zero to $N$. As the integral of an increasing function this total cost is convex, implying the value of the lender's option not to renew the overnight loan on a day-to-day basis, in contrast to a fixed tenor counterpart.

We denote by $\mathbb{P}$ and $\mathbb{E}$ the actuarial probability measure and expectation. Let $n_t$ represent the amount of notional that the bank is willing to lend at the overnight rate $r_t$ between $t$ and $t + dt$. The problem for the bank lending overnight is to maximise its expected profit over the whole stochastic process $n$, which in mathematical form is:

$$\mathcal{U}(r;n) := \frac{1}{T}\mathbb{E}\left(\int_0^T n_t r_t\,dt - \int_0^T \int_0^{n_t} \rho_t\left(D_t + x\right)dx\,dt\right) \leftarrow \max n \qquad (1)$$

In contrast, when lending at Libor over a period of length $T$, a bank cannot modify the notional amount $N$. As the composition of the Libor panel is updated at regular time intervals, during the life of a Libor loan there is increasing credit risk compared with an overnight loan rolled over the same period. The refreshment mechanism of the panel guarantees a sustained credit quality of the names underlying the rolling overnight loan, whereas the Libor loan is contracted once and for all with the initial panellists (see Filipović & Trolle, 2011, for a detailed analysis, and further developments following (6) below). Accordingly, the default time $\tau$ of the borrower reflects the deterioration of the average credit quality of Libor contributors over the tenor. Overnight lending is considered default-free, and we assume zero discount rates for ease of tractability.

The related optimisation problem for the bank is then maximising the expected profit of the bank over the constant amount $N$, which we put in mathematical form as:

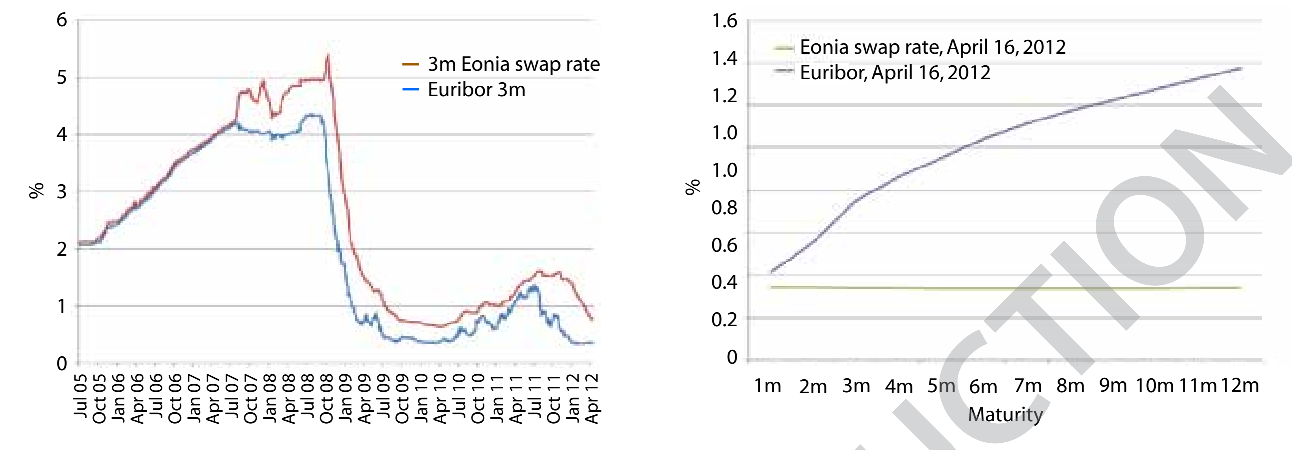
$$\mathcal{V}(L;N) := \frac{1}{T}\mathbb{E}\Big(NL(T \wedge \tau) - \int_0^{T\wedge\tau} \int_0^{N} \rho\left(D_t + x\right)dx\,dt - \mathbf{1}_{\tau<T} N\Big) \leftarrow \max N \qquad (2)$$

As we are dealing with short-term debt, we assume no recovery in case of default.

We stress that $r$ and $n$ represent stochastic processes in (1) whereas $L$ and $N$ are constants in (2). The utility functions of the bank that are implicit in (1) and (2) are taken in a standard economic equilibrium formalism as Legendre transforms of the OIS and Libor cost functions, represented by the integrals over $x$ in the right-hand side of (1) and (2) (see Karatzas & Shreve, 1998). These utility functions are linear to reflect the general risk-neutral behaviour of banks when lending, in which gains and losses are assessed in terms of actuarial expectations. In other words, the choice of banks to lend is driven less by preferences than by an optimisation of the cost-of-capital and credit protection. One could incorporate a concavely distorted utility function to account for risk aversion. Such a distortion would appear in our model as an increased volatility of capital needs and of the corresponding borrowing rate. However, we believe that short-term lending decisions are driven more by the estimated cost-of-capital than by a trade-off between interest returns and default risk.

Letting $U(r) = \max_n \mathcal{U}(r; n)$ and $V(L) = \max_N \mathcal{V}(L; N)$ represent the best utilities a bank can achieve by lending OIS or Libor, respectively, our approach for explaining the Lois consists, given the overnight rate process $r$, in solving the following equation for $L$:

1 The Euribor-Eonia basis




$$V(L) = U(r) \qquad (3)$$

This equation expresses an equilibrium relation between the utility of lending rolling overnight versus Libor for a bank involved in both markets.

To summarise, the analysis consists in comparing the deterioration in creditworthiness of a representative Libor borrower with the funding liquidity of a representative Libor lender. Note that other issues such as central banks' policies or possible manipulations of the rates are not explicitly stated in the analysis. However, these can be reflected to some extent in the model parameterisation that we will specify. Later we shall see that formula (3) equating the optimal expected profits implies a square-root dependence of the Lois on the tenor $T$.

**Credit and funding costs specification.** For tractability we assume henceforth that the funding rate $\rho$ is linear in $x$, that is:

$$\rho_t(D_t + x) = \alpha_t + \beta_t x \qquad (4)$$

where $\alpha_t = \rho_t(D_t)$ is the time-$t$ cost-of-capital of the lending bank, already with €$D_t$ worth of debt, and the coefficient $\beta_t$ represents the marginal cost of borrowing one more unit of notional. For instance, $\alpha_t$ = 2% and $\beta_t$ = 50 basis points means that the last euro borrowed by the bank was charged an annualised interest rate of two cents, whereas if the bank was indebted by €100 more, the next euro to be borrowed would be at an annualised interest charge of 2.5 cents.

By (4), we have:

$$\mathcal{U}(r;n) = \frac{1}{T}\mathbb{E}\int_0^T\left((r_t - \alpha_t)n_t - \frac{1}{2}\beta_t n_t^2\right)dt \qquad (5)$$

Denoting by $\lambda_t$ the intensity of $\tau$ and letting $\gamma_t = \alpha_t + \lambda_t$ and $\ell_t = e^{-\int_0^t \lambda_s ds}$, we also have:

$$\mathcal{V}(L;N) = \frac{1}{T}\mathbb{E}\int_0^T\left((L - \gamma_t)N - \frac{1}{2}\beta_t N^2\right)\ell_t dt \qquad (6)$$

where a standard credit risk calculation was used to get rid of the default indicator functions in (6) (see, for instance, Bielecki & Rutkowski, 2002). As explained after (2), the default time $\tau$ reflects the deterioration of the average credit quality of a Libor representative borrower during the length of the tenor. Recall the classical argumentation of Merton (1974), according to which a high-quality credit name has a decreasing CDS curve reflecting the expected deterioration of his credit. Consistent with this interpretation, the intensity $\lambda_t$ of $\tau$ can be proxied by the slope of the credit curve of the Libor representative (and therefore high-quality) borrower. This is given by the difference between the borrower's one-year CDS spread and the spread of its short-term certificate deposits, currently 10 to a few tens of basis points for major banks. Accordingly we call $\lambda_t$ the credit skew of the Libor representative borrower.

Connections with exogenous or endogenous variables can be considered. Central banks' liquidity policies will be reflected in the $\alpha_t$ and $\beta_t$ components of the cost-of-liquidity $\rho$ in (4). A manipulation effect, or incentive for a Libor contributor to bias its borrowing rate estimate in order to appear in a better condition than it is in reality (Wheatley, 2012), could be modelled by a spread in the borrower's credit risk skew component $\lambda$.

## Lois formula

Problems (1) and (5), and (2) and (6) are respectively solved for given $r$ and $L$ as follows. Writing $c_t := \alpha_t - r_t$, the OIS problem (1), (5) is resolved independently at each date $t$ according to:

$$u_t(r_t; n_t) = c_t n_t - \frac{1}{2}\beta_t n_t^2 \leftarrow \max n_t$$

hence the maximum is attained at:

$$n_t^* = \frac{c_t}{\beta_t} \quad \text{and} \quad u_t(r_t; n_t^*) = \frac{c_t^2}{2\beta_t}$$

The expected profit of the bank over the period $[0, T]$ is:

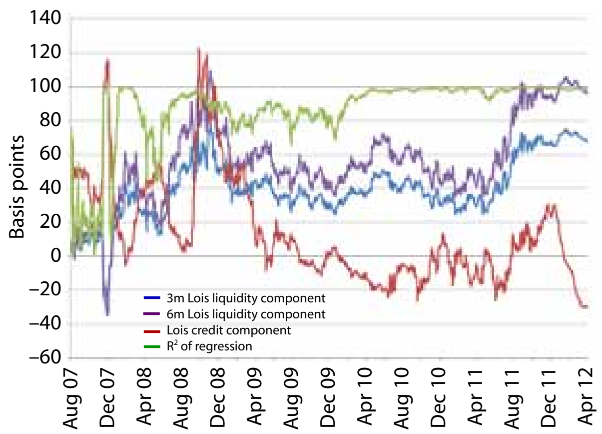
$$U(r) = \mathcal{U}(r; n^*) = \mathbb{E}\left(\frac{1}{T}\int_0^T \frac{c_t^2}{2\beta_t}dt\right)$$

In the Libor problem (2), (6), we must solve:

$$\mathcal{V}(L;N) = \frac{N}{T}\mathbb{E}\int_0^T (L - \gamma_t)\ell_t\, dt - \frac{1}{2}\frac{N^2\mathbb{E}}{T}\int_0^T \beta_t \ell_t\, dt \leftarrow \max N$$

hence:

2 Credit and liquidity components of Lois, and $R^2$ of regression




$$N^* = \frac{\mathbb{E}\frac{1}{T}\int_0^T (L-\gamma_t)\ell_t\,dt}{\mathbb{E}\frac{1}{T}\int_0^T \beta_t \ell_t\,dt}$$

and:

$$V(L) = \mathcal{V}(L;N^*) = \frac{\left(\mathbb{E}\frac{1}{T}\int_0^T (L-\gamma_t)\ell_t\,dt\right)^2}{2\mathbb{E}\frac{1}{T}\int_0^T \beta_t \ell_t\,dt} \quad (7)$$

We define $R = \mathbb{E}\frac{1}{T}\int_0^T r_t dt$. The rest of the article is devoted to the calculation of a stylised Lois defined as $(L^* - R)$, where $L^*$ is the solution to (3) given the process $r$. This is assumed to exist; note that the function $V$ is continuous and increasing in $L$, so that a solution $L^*$ to (3) is unique.

First note that in the case $\lambda = 0$, one necessarily has $U(r) \geq V(R)$, since the constant $N^*$ solving the Libor maximisation problem (2) is a particular strategy (constant process $n_t = N^*$) of the OIS maximisation problem (1). As $V$ is an increasing function, the indifference pricing equation (3) in turn yields that $L^* \geq R$.

Let $\mathcal{V}_0(\cdot; N)$ be the utility of lending Libor in the case $\lambda = 0$. When $\lambda > 0$, for each given amount $N$, one has, via $\lambda$ that is present in $\gamma$ in (6), that $\mathcal{V}(R; N) \leq \mathcal{V}_0(R; N)$ up to the second-order impact of $\ell_t$. Hence $V(R) \leq V_0(R) \leq U(r)$ follows from the inequality in the $\lambda = 0$ case and the fact that $\tau$ doesn't appear in $\mathcal{U}(r; n)$. We conclude that $L^* \geq R$ as before.

For notational convenience, let us introduce the time-space probability measures on $\Omega \times [0, T]$, $\overline{\mathbb{P}}$, given by the product of $\mathbb{P}$ times $dt/T$, along with $\hat{\mathbb{P}}$ given by the Radon-Nikodym derivative $d\hat{\mathbb{P}}/d\overline{\mathbb{P}} \propto \ell$. For a process $f = f_t(\omega)$, we denote the corresponding time-space averages by:

$$\overline{f} = \overline{\mathbb{E}}f = \mathbb{E}\frac{1}{T}\int_0^T f_t\,dt, \quad \hat{f} = \hat{\mathbb{E}}f = \overline{\mathbb{E}}\left[f\frac{\ell}{\overline{\ell}}\right]$$

(so, in this notation, $R = \overline{r}$). Similarly for processes $f$, $g$:

$$\overline{\mathrm{Cov}}(f,g) = \overline{\mathbb{E}}(fg) - \overline{\mathbb{E}}f\overline{\mathbb{E}}g, \quad \hat{\sigma}_f^2 = \hat{\mathbb{E}}\left(f-\hat{f}\right)^2 \quad (8)$$

Since:

$$U(r) = \overline{\mathbb{E}}\left[\frac{c^2}{2\beta}\right] \quad \text{and} \quad V(L) = \frac{\overline{\ell}^2(L-\hat{\gamma})^2}{2\overline{\mathbb{E}}[\beta\ell]}$$

equating $V(L^*) = U(r)$ yields:

$$\overline{\ell}^2\left(L^* - \hat{\gamma}\right)^2 = \overline{\mathbb{E}}[\beta\ell]\overline{\mathbb{E}}\left[\frac{c^2}{\beta}\right] \quad (9)$$

in which:

$$\overline{\mathbb{E}}[\beta\ell]\overline{\mathbb{E}}\left[\frac{c^2}{\beta}\right] = \overline{\mathbb{E}}\left[c^2\ell\right] - \overline{\mathrm{Cov}}\left[\beta\ell, \frac{c^2}{\beta}\right]$$

So:

$$\overline{\ell}^2\left(L^* - \hat{\gamma}\right)^2 = \overline{\ell}\hat{E}\left[c^2\right] - \overline{\mathrm{Cov}}\left[\beta\ell, \frac{c^2}{\beta}\right]$$

An interesting condition is when the funding rate equals the average overnight rate, when $R = \hat{\alpha} = \hat{\gamma} - \hat{\lambda}$, that is, $\hat{c} = \hat{\alpha} - \hat{r} = \overline{r} - \hat{r}$. Then the previous formula reads:

$$\overline{\ell}\left(L^* - R - \hat{\lambda}\right)^2 = \hat{\sigma}_c^2 + (\overline{r} - \hat{r})^2 - \overline{\mathrm{Cov}}\left[\beta\frac{\ell}{\overline{\ell}}, \frac{c^2}{\beta}\right] \quad (10)$$

A reasonable first approximation is that $(\overline{r} - \hat{r})^2$ and the covariance are negligible in the right-hand side. In particular, these terms vanish when the intensity $\lambda$ is zero and the marginal cost-of-capital coefficient $\beta$ is constant. For the sake of argument, suppose a diffusive behaviour of the instantaneous funding spread process $c_t = \alpha_t - r_t$, that is, $dc_t = \sigma^* dW_t$ for some reference volatility $\sigma^*$ and a Brownian motion $W$. Let us also assume a constant credit skew $\lambda_t = \lambda^*$ of the borrower and a constant marginal cost of borrowing $\beta$. Neglecting the impact of $\ell_t = e^{-\lambda^* t} \approx 1 - \lambda^* t$ in (10) (so that $\hat{\mathbb{P}} \approx \overline{\mathbb{P}}$), it follows that:

$$\hat{\sigma}_c^2 \approx \overline{\sigma}_c^2 = \left(\sigma^*\right)^2 \mathbb{E}\frac{1}{T}\int_0^T W_t^2 dt = \left(\sigma^*\right)^2 T/2$$

and our Lois formula follows from the above as:

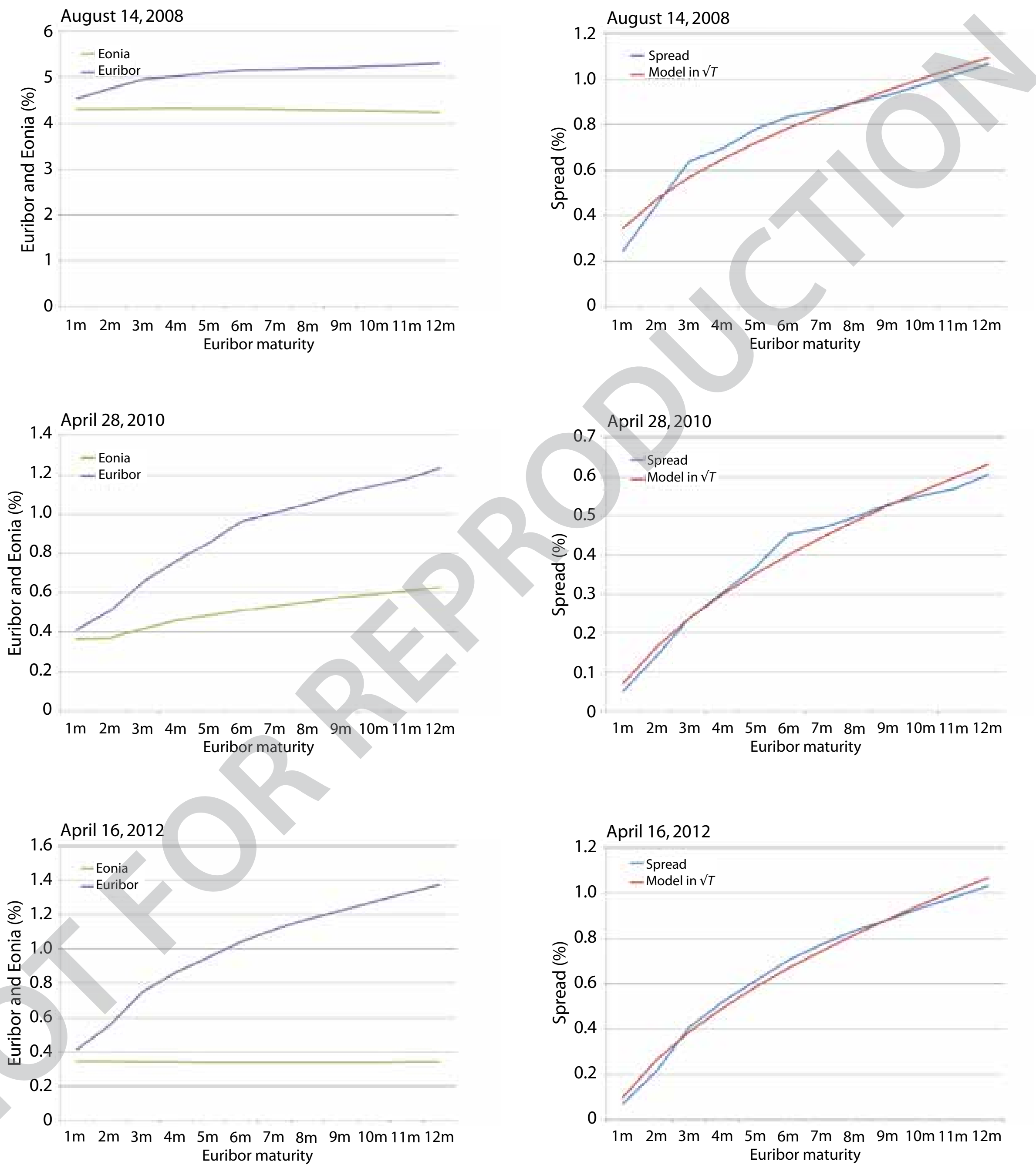
$$L^* - R \approx \lambda^* + \sigma^*\sqrt{T/2} \quad (11)$$

where we take the positive root of (10), $L^* - R - \hat{\lambda} \geq 0$. This is a natural assumption as Libor should at least compensate for the credit deterioration of the panel over the tenor. From a broader perspective, according to formula (11), the two key drivers of the Lois are:

- A suitable average $\lambda^*$ of the borrower's credit skew $\lambda$, which can be seen as the intrinsic value component of the Lois and is a borrower's credit component.
- A suitable volatility $\sigma^*$ of the instantaneous funding spread process $c_t$. This second component can be seen as the time-value of the Lois, interpreted as a lender's liquidity component.

From a quantitative trading perspective, the formula (11) can be used for implying the value $\sigma^*$ priced in by the market from an observed Lois $L^* - R$, and a borrower's CDS slope taken as a proxy for $\lambda^*$. The value $\sigma^*$ implied through (11) can be compared by a bank with an internal estimate of its realised funding spread volatility, so that the bank can decide whether it should lend Libor or OIS, much like with going long or short an equity option depending on the relative position of the implied and

3 Euribor/Eonia swap rates (left) and square-root fit of the Lois (right)






realised volatilities of the underlying stock. Another possible application of the formula (11) is for the calibration of the volatility $\sigma^*$ of the funding spread process $c_t$ in a stochastic model for the latter, for example, in the context of credit valuation adjustment computations (see Crépey, 2012b).

**Numerical analysis.** Figure 2 shows euro market time series for August 15, 2007 to April 16, 2012 of the intercept, slope and $R^2$ coefficients of a linear regression of the Lois term structure against $\sqrt{T/3\text{m}}$ or $\sqrt{T/6\text{m}}$, for $T$ varying from one month to one year. Choosing $\sqrt{T/3\text{m}}$ or $\sqrt{T/6\text{m}}$, the two most liquid tenors, as a regressor only affects the slope coefficient of the regression, by a factor of $\sqrt{2}$, with the inputs chosen as the most liquid part of the curve. In our financial interpretation, the intercept represents the credit component of the Lois (of any tenor $T$), while the slope coefficients represent the liquidity component of the Lois for $T$ = 3m or 6m. The red curve on the figure can be viewed as the credit component while the blue and purple curves represent the liquidity components of the three- and six-month Lois. Before August 2007, the Lois is negligible so that the regression (not displayed on the figure) is insignificant. Since the advent of the Lois in mid-August 2007, we can distinguish three market regimes. In a first phase, until first-quarter 2009, the market adapts, with the $R^2$ becoming significant together with very large and volatile credit and liquidity Lois components. Note in particular the spike of both components at the turn of the credit crisis following Lehman's default in September 2008, during which liquidity in the interbank market dried up. Between second-quarter 2009 and mid-2011, the situation seems stabilised with an $R^2$ close to one, a liquidity component of the order of 30bp on the three-month or 45bp on the six-month and a much smaller credit component. The ongoing eurozone crisis and the US downgrade of mid-2011 prompts a third phase with a much higher liquidity component, of the order of 60bp on the three-month or 90bp on the six-month. This shows banks' increasing difficulties in funding, for instance due to stricter repo eligibility requirements. To illustrate the three market regimes in this analysis, figure 3 shows the fit between a square-root term structure and the empirical Lois term structure corresponding to the Euribor/Eonia-swap data of August 14, 2008, April 28, 2010 and April 16, 2012 (data of the right panel in figure 1). The last two terms that we neglected in (10) to deduce (11) are a possible explanation for (minor) departures of the actual Lois spread curve from the theoretical square-root term structure implied by (11).

The simplicity of the model means the credit component of Lois, the red line in figure 2, is somewhat volatile, but appears broadly reasonable for a credit skew, as interpreted as the difference between the one-year CDS spread and the short-term certificate deposit credit spread of a major bank. The estimate for $\sigma^*$ – the coefficient of the regression against $\sqrt{T/2}$, obtained by doubling the corresponding coefficient for that against $\sqrt{T/6\text{m}}$, in purple on figure 2 – ranges between 100bp and 200bp. This is reasonably in line with the volatilities of major banks' CDS spreads – certainly a reasonable lower bound, as funding spreads are complex and may depend on other less volatile inputs.

**Conclusion**

Since the 2007 subprime crisis, OIS and Libor markets (Eonia and Euribor in the euro market) have diverged. We have shown that this can be explained in a simple model in which banks optimise their lending between Libor and OIS markets, and so are led to apply a spread over the OIS rate when lending at Libor. This Lois can be considered as consisting of two components: one corresponding to the credit skew $\lambda_t$ of a representative Libor borrower in an interbank loan, and one corresponding to liquidity – in the sense of the volatility of the funding spread – of a representative Libor lender over the overnight interbank rate. Assuming a diffusive evolution of this funding spread $c_t$, the above-mentioned optimisation results in a square-root term structure of the Lois given by the formula (11). The intercept $\lambda^*$ can be proxied by the slope of a representative Libor credit curve and the coefficient $\sigma^*$ is a volatility of $c_t$. These theoretical developments are corroborated by empirical data from second-half 2007 to second-half 2012 on the euro market studied in this article. The Lois is explained by credit and liquidity until the beginning of 2009, and dominantly explained by liquidity since then. Residual discrepancies between the theory and the data can be explained by the existence of other features such as Libor manipulations, which could be included in the methodology in future work. The equilibrium approach of this article allows a bank to in principle monetise the Lois, by preferably lending Libor (respectively OIS) whenever its internally estimated funding spread is statistically found less (respectively more) volatile than $\sigma^*$ implied from the market through the Lois formula (11). Another application of this formula is for the calibration or estimation of the volatility $\sigma^*$ of the funding spread process $c_t$ in a stochastic model for the latter. ■

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