Fixed Income Analysis

Estimation of the Term Structure
Part I

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Basic problem

- Our basic building blocks are prices of zero-coupon bonds (also called discount factors), that is $d(t)$ or $R(t) = -\log d(t)/t$
  - We need $d(t)$ for all payment dates, $t_1, t_2, \ldots, t_m$, to price coupon bonds consistently. Yield-to-maturity cannot be used to spot pricing errors.
  - Stochastic term-structure models rely on zero-coupon rates — never yield-to-maturity (YTM) on specific bonds.
- In principle, we need $d(t)$ or $R(t)$ for all t, i.e. as a continuous function, or at least for all (relevant) payment dates.
- However, the available data typically consists of prices of coupon bonds (not zeros), and the number of different payment dates exceeds the number of bond prices.
- Prices of some bonds may be noisy due to stale prices (non-synchronous trading), bid-ask spread, rounding, tax effects, etc.

Estimating zero-coupon rates
Overview of different techniques

1. Bootstrapping (with linear interpolation)
   - The number of bonds (prices) must equal the number of payment dates.
   - Typical setup: par yield curve (YTM on bonds trading at par), perhaps extended with linear interpolation, or another interpolation scheme.
   - We get an exact fit to the data. Noise in the bond prices translates into noisy estimates of the spot and (especially) forward rates.
2. Statistical techniques based on non-linear regression **(next week)**
   - No restriction on the number of bonds vs. payment dates. We can use prices of coupon bonds directly without prior interpolation.
   - Spot curve, $R(t)$, discount factor, $d(t)$, or perhaps instantaneous forward curve, $f(t)$, is specified as some sufficiently flexible functional form of maturity t, and the parameters are estimated by non-linear regression.
   - Does not produce an exact fit — some smoothing involved.

Bootstrapping technique

- **Not** related to the statistical technique with the same name....
- Mainly used in the **swap market**. The swap rate of a fixed-for-floating swap (plain vanilla) with maturity t corresponds to the par yield for a bullet bond with the same maturity.
- Recursice computations of the discount factor $d(n)$ using the basic pricing equation for a n-maturity par bond with YTM — and coupon rate — $c(n)$

$$100 = c(n) \sum_{i=1}^{n-1} d(i) + \{100 + c(n)\} d(n) \tag{1}$$

- Solving for $d(n)$ in (1) gives

$$d(n) = \frac{100 - c(n) \sum_{i=1}^{n-1} d(i)}{100 + c(n)} \tag{2}$$

First numerical example

Assumptions:

- Discrete, annual compounding. Exactly one year until next payment.
- All maturities up to 10 years observed (no interpolation required).

n	$c(n)$	$d(n)$	$\sum_{i=1}^{n} d(i)$	$R(n)$	$100 \times R(n)$
1	6.0000	0.9434	0.9434	0.0600	6.0000
2	7.0000	0.8729	1.8163	0.0704	7.0353
3	8.0000	0.7914	2.6076	0.0811	8.1111
4	9.5000	0.6870	3.2947	0.0984	9.8398
5	9.0000	0.6454	3.9400	0.0915	9.1528
6	10.5000	0.5306	4.4706	0.1114	11.1410
7	11.0000	0.4579	4.9285	0.1181	11.8062
8	11.2500	0.4005	5.3290	0.1212	12.1182
9	11.5000	0.3472	5.6762	0.1247	12.4713
10	11.7500	0.2980	5.9742	0.1287	12.8690

Second numerical example – 1

- If some maturities are missing, the requisite coupon rate, $c(n)$, is replaced by interpolation between $c(n_1)$ and $c(n_2)$, where the adjacent maturities n_1 and n_2 satisfy $n_1 < n < n_2$.
- In general, linear interpolation is used

$$\hat{c}(n) = \frac{n_2 - n}{n_2 - n_1} c(n_1) + \frac{n - n_1}{n_2 - n_1} c(n_2) \tag{3}$$

- The missing values are replaced by $\hat{c}(n)$, and the recursive bootstrap computations are performed as before.
- Other interpolation schemes are possible, for example cubic spline functions.
- However, effect usually minor except at those maturities n, where interpolation is needed (where $c(n)$ is replaced by $\hat{c}(n)$).

Second numerical example – 2

- Below, the par yields are missing for 6, 8 and 9 years to maturity. This is the common situation in most swap markets, e.g. US.
- Example: $\hat{c}(8) = (2/3) \times c(7) + (1/3)c(10)$.

n	$c(n)$	$c(n)$ / $\hat{c}(n)$	$d(n)$	$\sum_{i=1}^{n} d(i)$	$100 \times R(n)$
1	3.0000	3.0000	0.9709	0.9709	3.0000
2	4.0000	4.0000	0.9242	1.8951	4.0202
3	4.5000	4.5000	0.8753	2.7704	4.5384
4	5.0000	5.0000	0.8205	3.5909	5.0718
5	6.0000	6.0000	0.7401	4.3310	6.2031
6		**6.5000**	0.6746	5.0056	6.7797
7	7.0000	7.0000	0.6071	5.6127	7.3895
8		**7.3333**	0.5482	6.1609	7.8035
9		**7.6667**	0.4901	6.6510	8.2465
10	8.0000	8.0000	0.4333	7.0843	8.7240

Statistical techniques — introduction

- Data consist of N bonds, with payments b_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, m_i$, and the respective payment dates are t_{ij}.
- Pricing equation, allowing for measurement error ε_i

$$P_i + A_i = \sum_{j=1}^{m_i} b_{ij} \cdot d(t_{ij}) + \varepsilon_i, \qquad i = 1, 2, \ldots, N \tag{4}$$

- If $d(t)$ in (4) is parameterized using some functional form with K parameters, and $K < N$, these parameters can be estimates by non-linear regression analysis.
- Various approaches differ as to whether they parameterize $d(t)$ directly, or indirectly via spot rates $R(t)$ or forward rates $f(t)$, and which parameterization is used (often cubic spline functions).