Modeling burnout and borrower heterogeneity

The price-rate function
Duration and convexity
Cost of convexity

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Modeling burnout — 1

• Consider two scenarios:
  1. Interest rates decrease by 2% after the first period, and increase by 1% between first and second period.
  2. Interest rates increase by 1% after the first period, and decrease by 2% between first and second period.

• In both cases, interest rates have dropped by 1% at time $n = 2$, compared to time $n = 0$.
• Should we expect the same level of prepayments then?
  • Probably not — in the first scenario there has been some prepayment at time $n = 1$, and the borrowers with the lowest transaction costs are the first to prepay their loans.
  • Thus, in scenario 1, the remaining borrowers face higher transaction costs (on average) at $n = 2$.
  • This problem is known as burnout.
Modeling burnout – 2

- If the prepayment function, \( \lambda(n,s) \), depends on past interest rates, the pricing problem is no longer Markovian (but path-dependent).
- We want to use binomial models (and not Monte Carlo simulation), so we cannot model burnout in this way.
- Alternative approach [Jakobsen (1994)]
  - Assume we have \( N \) mortgage (sub)pools with different prepayment functions, \( \lambda_i(n,s) \), \( i = 1,\ldots,N \).
  - Within each (sub)pool, there are no path-dependencies.
  - The different pools could be determined by loan size (this information is now available in Denmark).
  - In Jakobsen (1994), \( N = 2 \) and the two pools consists of households and firms (corporations). At that time (1994), the loan size information was not available.

Modeling burnout – 3

- **Within each pool**, we use the binomial model and the MBS backward equation to calculate the price of the MBS, \( V_i(0,0) \).
- With relative weights of each (sub)pool, \( w_i(0,0) \), the price of the MBS today is given by
  \[
  V(0,0) = \sum_{i=1}^{N} w_i(0,0)V_i(0,0)
  \]  
  (1)
- For pools with above-average prepayment rates (typically corporate borrowers), the relative weight will be reduced over time.
- This means that aggregate prepayment will be reduced (since the low-prepayment pools get a greater weight) if there has been prepayment in the past.
- Thus, even though there are no path-dependencies within each pool, we incorporate the burnout feature in the model.
Introduction to risk management

- The importance of this topic cannot be understated.
- Before we can talk about risk management, we must talk about risk measurement. Today, we concentrate on the latter.
- Risk measurement is also important for hedging (reducing risk) or selective risk exposure (hedge funds). For example, buying a MBS and hedging the general interest-rate risk by shorting T-bonds.
- For securities with fixed payments (non-callable bonds), duration is the most widely used measure of risk.
- However, for many fixed-income securities the cash flows are stochastic (depend on the evolution of interest rates).
- In general, we need a term-structure model to compute “something like duration” for these securities.

The price-rate function $P(y) - 1$

- Basic assumption: the term-structure is governed by a one-factor model with state variable $y$.
- The price of a given fixed-income security is a function of $y$, denoted $P(y)$.
- We call $P(y)$ the price-rate function — since $y$ is an interest rate in most cases.
- What happens to the price if $y$ changes to, say, $y + \Delta$?
- First order Taylor-series approximation:

$$P(y + \Delta) \approx P(y) + P'(y)\Delta$$

(2)

- Computation of $P(y)$ and its derivative:
  - Valuation techniques discussed earlier (forward-risk adjusted measure, binomial and trinomial trees, Monte Carlo simulation, etc).
  - Empirical approaches (curve-fitting using historical data).
The price-rate function $P(y) - 2$

- We can also express (2) in relative terms

$$\frac{P(y + \Delta) - P(y)}{P(y)} \approx \frac{P'(y)}{P(y)} \Delta = -D(y) \Delta$$  \hspace{1cm} (3)

where $D(y) = -P'(y)/P(y)$ is the new **duration** measure.

- Simple example: coupon-bearing bond with fixed payments, $\{c_i\}$, and $y$ is the yield-to-maturity on the bond

$$P(y) = \sum_{i=1}^{n} c_i \exp[-yt_i]$$  \hspace{1cm} (4)

- Here, duration is given by the well-know formula

$$D(y) = \sum_{i=1}^{n} c_i t_i \exp[-yt_i]/P(y)$$  \hspace{1cm} (5)

Hedging with duration

- Duration is a relative measure, but $P'(y)$ (sometimes called dollar duration when the sign is changed) is more useful for calculating **hedge ratios**.

- Suppose that we have a portfolio of two assets (the number of each asset is $w_1$ and $w_2$) and that both prices depend on $y$,

$$V(y) = w_1 P_1(y) + w_2 P(y)$$  \hspace{1cm} (6)

- First-order approximation to the change in value

$$V(y + \Delta) - V(y) = V'(y) \Delta = \left\{w_1 P'_1(y) + w_2 P'_2(y)\right\} \Delta$$  \hspace{1cm} (7)

- The portfolio is riskless (approximately) if

$$w_2/w_1 = -P'_1(y)/P'_2(y) \equiv H_{12}(y).$$  \hspace{1cm} (8)

- $H_{12}(y)$ is called the hedge ratio between securities 1 and 2.
Convexity

• In general, a second-order approximation is more accurate

\[
\frac{P(y + \Delta) - P(y)}{P(y)} \approx -D(y)\Delta + \frac{1}{2}C(y)\Delta^2
\]  

(9)

• In equation (9), \( C(y) \) is **convexity**, 

\[
C(y) = \frac{d^2P(y)/dy^2}{P(y)}
\]

(10)

• If \( C(y) > 0 \), the second term on the RHS of (9) is always positive — no matter the sign of \( \Delta \).

• This means that positive convexity is desirable — other things equal.

• You don’t get anything for free — we need to look at the **cost of convexity**.

Cost of convexity – 1

• Assumptions:
  – The current term structure is flat, and \( r = 0.10 \).
  – The term-structure is governed by the Ho-Lee model with \( \sigma = 0.02 \).

• Prices of zero-coupon bonds today,

\[
P(0,T) = \exp[-rT] = \exp[-0.1 \cdot T]
\]

(11)

• Duration and convexity of a zero:

\[
D(0,T) = \frac{-dP(0,T)}{dr}/P(0,T) = T
\]

(12)

\[
C(0,T) = \frac{-d^2P(0,T)}{dr^2}/P(0,T) = T^2
\]

(13)

• Consider two portfolios: **Portfolio A** has 100% in the 15Y zero, and **Portfolio B** has 50% each in the 5Y and 25Y zeros.
Cost of convexity – 2

- Duration and convexity for the two portfolios:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>5Y</th>
<th>15Y</th>
<th>25Y</th>
<th>Duration</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>0</td>
<td>50</td>
<td>15</td>
<td>325</td>
</tr>
</tbody>
</table>

- The duration and convexity on portfolio B are calculated as follows: $D = 0.5 \cdot (5 + 25) = 15$ and $C = 0.5 \cdot (5^2 + 25^2) = 325$.

- Portfolio B has the same duration as A, but higher convexity.

- According to (9) — and Figure 1 — this means that the return on portfolio B is greater than on A — no matter whether interest rates go up or down (positive or negative $\Delta$).

![Figure 1: value of A and B at t=0 as a function of yields (static analysis)](image)
Cost of convexity – 3

- **Question:** why don’t we all buy portfolio B and short A? According to Figure 1, the worst than can happen is that we don’t gain anything (if interest rates don’t move).

- **Answer:** we are misinterpreting a *static* analysis.

- We really need a *dynamic* analysis — which will show that the yield curve cannot continue to be flat (as we assume in Figure 1).

- Absence of arbitrage requires that for all bonds (all $T$)
  \[ P(t,T) = \frac{P(0,T)}{P(0,t)} \exp \left[ -\frac{\sigma^2}{2} (T-t)^2 t - (T-t) (r_t - f(t)) \right]. \tag{14} \]

- In Figure 2 we use this formula to compute the value (at $t = 1$) of portfolios B and A for different future short rates, $r_t$.

- The risk of B is clear now — if rates *don’t change enough*, we lose money compared to A.

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**Figure 2:** difference btw. $V_B$ and $V_A$ at time $t=1$ as a function of $r(1)$