Fixed Income Analysis

Pricing Term-Structure Derivatives

The forward-risk adjusted measure
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Introduction to mortgage-backed securities

Jesper Lund
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Motivation

- Consider a fixed-income derivative with payoff V_T a time T.
- The price today ($t = 0$) is given by

$$V_0 = E_0^Q \left[e^{-\int_0^T r_s ds} V_T \right]. \qquad (1)$$

- Problem: we are calculating the expectation of the product of two **dependent** random variables.
- In general, it is easier to calculate V_0 using the so-called **forward-risk adjusted measure** technique.
- This means that V_0 is given by

$$V_0 = P(0,T) \cdot E_0^{Q^T} (V_T), \qquad (2)$$

where Q^T is a new probability measure.
- Basic idea: change probabilities of different events (again).

Model setup

- The general results are derived for a one-factor HJM model.
- Risk-neutral forward-rate dynamics:

$$df(t,T) = -\sigma(t,T)\sigma_P(t,T)dt + \sigma(t,T)dW_t^Q, \tag{3}$$

where

$$\sigma_P(t,T) = -\int_t^T \sigma(t,u)du. \tag{4}$$

- Bond prices evolve according to the SDE

$$dP(t,T) = r_t P(t,T)dt + \sigma_P(t,T)P(t,T)dW_t^Q. \tag{5}$$

- Thus, $\sigma_P(t,T)$ is the time t volatility of the zero-coupon bond maturing at time T.

Forward-risk adjusted measure – 1

- The price of the derivative at time t is denoted V_t.
- Under the risk-neutral distribution we have

$$dV_t = r_t V_t dt + \sigma_V(t) V_t dW_t^Q. \tag{6}$$

- Note: we do not know V_t or $\sigma_V(t)$, but the only important thing right now is the **form** of the SDE (6).
- Define the relative (deflated) price of the derivative

$$F_t \;\equiv\; V_t \,/\, P(t,T), \qquad \text{for } t \in [0,T]. \tag{7}$$

- SDE for F_t can be obtained from Ito's lemma

$$dF_t = \sigma_P(\sigma_P - \sigma_V)F_t dt + (\sigma_V - \sigma_P)F_t dW_t^Q \tag{8}$$

- Shorthand notation: $\sigma_P \equiv \sigma_P(t,T)$ and $\sigma_V \equiv \sigma_V(t)$.

Forward-risk adjusted measure – 2

- Define a new probability measure, denoted Q^T, such that

$$W_t^{Q^T} = W_t^Q - \int_0^t \sigma_P(u,T)du, \quad t \in [0,T], \tag{9}$$

is a Brownian motion under Q^T.

- The differential form of (9) is

$$dW_t^{Q^T} = dW_t^Q - \sigma_P(t,T)dt. \tag{10}$$

- If we substitute (9) into the SDE for F_t, we obtain the process for F_t under the forward-risk adjusted measure,

$$\begin{aligned} dF_t &= -\sigma_P(\sigma_V - \sigma_P)F_t dt + (\sigma_V - \sigma_P)F_t \left(dW_t^{Q^T} + \sigma_P dt\right) \\ &= (\sigma_V - \sigma_P)F_t dW_t^{Q^T}. \end{aligned} \tag{11}$$

Forward-risk adjusted measure – 3

- **Key result:** under the new measure (distribution) Q^T, the relative price F_t is a martingale — since the drift is zero.
- Step 1: because of the martingale property, we have

$$F_t = E_t^{Q^T}(F_T), \quad \text{for } t \le T. \tag{12}$$

- Step 2: since $P(T,T) = 1$, we get $F_T = V_T$.
- Step 3: the time $t = 0$ price can be calculated as

$$\begin{aligned} V_0 &= P(0,T)F_0 = P(0,T)E_0^{Q^T}(F_T) \\ &= P(0,T)E_0^{Q^T}(V_T). \end{aligned} \tag{13}$$

- Only remaining problem: determine distribution of payoff under the forward-risk adjusted measure, Q^T.

Forward-risk adjusted measure – 4

- Forward-rate dynamics under the new measure Q^T,

$$
\begin{aligned}
df(t,T) &= -\sigma(t,T)\sigma_P(t,T)dt + \sigma(t,T)\left(dW_t^{Q^T} + \sigma_P(t,T)dt\right) \\
&= \sigma(t,T)dW_t^{Q_T}. \qquad (14)
\end{aligned}
$$

- Thus, the T-maturity forward rate is a martingale under Q^T.
- Integrating (14) from $t = 0$ to $t = T$,

$$f(T,T) = f(0,T) + \int_0^T \sigma(t,T)dW_t^{Q^T} \qquad (15)$$

- Since the expectation of the second term in (15) is zero, and since $f(T,T) = r_T$, we get

$$f(0,T) = E_0^{Q^T}(r_T). \qquad (16)$$

Options on zero-coupon bonds – 1

- Notation:

K	exercise price of the call option.
T	maturity of the call option.
T_1	maturity of the underlying zero-coupon bond.
$C(T,T_1,K)$	price of the call option at time $t = 0$.

- For concreteness, we use the extended Vasicek model which corresponds to the HJM model with

$$
\begin{aligned}
\sigma(t,T) &= \sigma e^{-\kappa(T-t)} \qquad (17) \\
\sigma_P(t,T) &= \sigma \frac{e^{-\kappa(T-t)} - 1}{\kappa}. \qquad (18)
\end{aligned}
$$

- The following results apply to any Gaussian HJM model, however.

Options on zero-coupon bonds – 2

- The price of the option is given by:

$$C(T, T_1, K) = P(0, T) E_0^{Q^T} \left[\max\{P(T, T_1) - K, 0\} \right]. \qquad (19)$$

- In order to calculate this expectation, we must determine the distribution of $P(T, T_1)$ under Q^T.
- Since $P(T, T) = 1$, the distribution of $P(T, T_1)$ can be obtained from the relative price

$$F(t, T, T_1) = P(t, T_1)/P(t, T). \qquad (20)$$

- Note that (20) is the forward price of the T_1-maturity bond.
- SDE under Q^T for $F(t, T, T_1)$:

$$\begin{aligned} dF(t, T, T_1) &= \{\sigma_P(t, T_1) - \sigma_P(t, T)\} F(t, T, T_1) dW_t^{Q^T} \\ &\equiv \sigma_F(t, T, T_1) F(t, T, T_1) dW_t^{Q^T}. \end{aligned} \qquad (21)$$

Options on zero-coupon bonds – 3

- An application of Ito's lemma gives:

$$d \log F(t, T, T_1) = -\frac{1}{2}\sigma_F^2(t, T, T_1) dt + \sigma_F(t, T, T_1) dW_t^{Q^T}. \qquad (22)$$

- For the extended Vasicek model:

$$\begin{aligned} \sigma_F(t, T, T_1) &= \frac{\sigma}{\kappa}\left(e^{-\kappa(T_1 - t)} - e^{-\kappa(T - t)}\right) \\ &= \frac{\sigma}{\kappa} e^{-\kappa(T - t)} \left(e^{-\kappa(T_1 - T)} - 1\right). \end{aligned} \qquad (23)$$

- It follows from (22) and (23) that $\log P(T, T_1) = \log F(T, T, T_1)$ is normally distributed.
- The mean of $\log P(T, T_1)$ is given by:

$$\begin{aligned} \mu_F(T, T_1) &= \log F(0, T, T_1) - \frac{1}{2}\int_0^T \sigma_F^2(t, T, T_1) dt \\ &= \log F(0, T, T_1) - \frac{1}{2}\omega_F^2(T, T_1). \end{aligned} \qquad (24)$$

Options on zero-coupon bonds – 4

- The variance of $\log P(T, T_1)$ is given by:

$$\begin{aligned}\omega_F^2(T, T_1) &= \int_0^T \sigma_F^2(t, T, T_1) dt \\ &= \left(\frac{e^{-\kappa(T_1 - T)} - 1}{\kappa} \right)^2 \times \left(\sigma^2 \frac{1 - e^{-2\kappa T}}{2\kappa} \right) \end{aligned} \quad (25)$$

- After some lengthy algebra, the price of the call follows:

$$C(T, T_1, K) = P(0, T_1) N(d_1) - P(0, T) K N(d_2) \quad (26)$$

$$d_1 = \left(\log \frac{P(0, T_1)}{P(0, T)} - \log K + \frac{1}{2} \omega_F^2 \right) / \omega_F \quad (27)$$

$$d_2 = d_1 - \omega_F. \quad (28)$$

- This is the Black-Scholes formula with a different variance . . .

Options on coupon bonds – 1

- Coupon bond with payments $\{a_j\}$ at times T_j, $1 \leq j \leq M$.
- The price is at time t is

$$P_a(t; r_t) = \sum_{j=1}^{M} a_j \cdot P(t, T_j; r_t) \quad (29)$$

- In the extended Vasicek model, all bond prices depend on r_t.
- The price of a call option expiring at time $T < T_1$ (date of the first payment) can be written as:

$$C_a(T, K) = P(0, T) \cdot E_0^{Q^T} \left(\sum_{j=1}^{M} a_j \cdot P(T, T_j; r_T) - K \right)^+ \quad (30)$$

- However, we are no longer taking the (truncated) expectation over a log-normal random variate.

Options on coupon bonds – 2

- We use the Jamshidian decomposition.
- Define r^* such that

$$\sum_{j=1}^{M} a_j \cdot P(T, T_j;\, r^*) - K = 0 \tag{31}$$

- If $P(T, T_j;\, r)$ is **monotonic** in r (for all maturities), we can show that:

$$\left(\sum_{j=1}^{M} a_j \cdot P(T, T_j;\, r_T) - K \right)^+ = \sum_{j=1}^{M} a_j \left(P(T, T_j;\, r_T) - K_j \right)^+ \tag{32}$$

where $K_j = P(T, T_j;\, r^*)$.
- This follows from monotonicity since $\sum_{j=1}^{M} a_j P(t, T_j;\, r) > K$, corresponding to $r < r^*$, implies $P(t, T_j;\, r) > K_j$ for each j.

Options on coupon bonds – 3

- Interpretation: an option on a portfolio of payments, $\{a_j\}$, is equivalent to a portfolio of options.
- This means that the price of an option on a coupon bond is given by the expression:

$$C_a(T, K) = \sum_{j=1}^{M} a_j \cdot C(T, T_j, K_j) \tag{33}$$

- This holds for the following models:
  - Vasicek model — derived by Jamshidian (1989)
  - CIR model — derived by Longstaff (1993)
- The result does **not** generalize to multi-factor models, such as the Gaussian double-decay model.

At-the-money interest-rate caps

- Consider a derivative (ATM cap) with payoff

$$V_T = \max(r_T - f(0,T), 0). \tag{34}$$

- Under the forward-risk adjusted measure:

$$r_T - f(0,T) \ = \ \int_0^T \sigma(t,T) dW_t^{Q^T}. \tag{35}$$

- Under the extended Vasicek model, the RHS of (35) is normally distributed with mean zero and variance

$$v^2(0,T) = \int_0^T \sigma^2(t,T)dt = \sigma^2 \frac{1 - e^{-2\kappa T}}{2\kappa}. \tag{36}$$

- The time $t = 0$ price of the ATM cap is:

$$C(T, f(0,T)) \ = \ P(0,T) E_0^{Q^T}(V_T) \ = \ P(0,T) \frac{v(0,T)}{\sqrt{2\pi}}. \tag{37}$$

Forward and futures contracts

- Agreement to deliver a financial asset on a future date t for a price which is fixed today (but paid upon delivery).
- There are no payments when entering into the contract — the **initial** value of the forward or futures contract is always zero.
- Difference between forward and futures contract: the latter is continuously marked to market to ensure zero value.
- **Pricing:** consider a forward and future on a T-maturity zero.

$$F_{\text{for}}(t,T) \ = \ P(0,T)/P(0,t) \tag{38}$$
$$F_{\text{fut}}(t,T) \ = \ E_0^Q[P(t,T)]. \tag{39}$$

- See chapter 14 in Tuckman for proofs and further discussion.
- When the underlying asset is a bond, futures prices are (generally) **below** forward prices.

Pricing a two-year callable bond − 1

- Binomial tree for the short rate with annual time steps, $\theta(n, s) = 0.5$, discrete compounding. The tree is calibrated to match a flat initial term-structure of 10 percent.

$$10.00 \begin{cases} 11.11 \\ 8.91 \end{cases}$$

- A two-year non-callable bullet with a 10 percent coupon is trading at par. To see this, calculate the bond price using the tree

$$100.00 \begin{cases} 109.00 = 10 + 110/1.1111 \quad (101.00) \\ 111.00 = 10 + 110/1.0891 \quad (99.00) \end{cases}$$

Pricing a two-year callable bond − 2

- Does the price change if the borrower is **allowed** (but not obligated) to pay back the entire principal after one year?
- In other words: what is the price of a two-year callable bond?
- After one year, the borrower can choose between continuing the fixed-rate loan (bullet) with a 10% interest rate, or calling the bullet and borrowing at the new short rate, $r(1, s)$.
- If $r(1, s)$ is below 10%, the **rational** borrower will call the bond.
- Price of the callable bond follows from the tree:

$$99.55 \begin{cases} 109.00 = 10 + 110/1.1111 \\ 110.00 = 10 + \mathbf{100.00} \end{cases}$$

- The price (premium) of the call feature is 0.45 cents.

Callable bonds in a multi-period setting

- The callable bond contains an **embedded** option to purchase the "otherwise identical" non-callable bond at par.
- Option payoff: $C(1,0) = 1$ and $C(1,1) = 0$, which means that the price is $C(0,0) = 0.5 \cdot [C(1,0) + C(1,1)]/1.10 = \mathbf{0.45}$.
- We always have: $P_C(n,s) = P_{NC}(n,s) - C(n,s)$.
- In a multi-period setting, the call feature is an **American option**.
- Description of the **optimal** call strategy:
  - Let $K(n,s)$ denote the call price (could be greater than par).
  - Let $P_C^H(n,s)$ denote the price of the callable bond if it is **not** called at the node (n,s).

    Note: $P_C^H(n,s)$ is calculated using the short-rate tree and the backward equation.
  - The optimal strategy is to call the bond if $P_C^H(n,s) > K(n,s)$.

Mortgage-backed bonds

- Most mortgage-backed bonds (MBBs) in Denmark are callable.
- The "otherwise identical" non-callable bond is annuity bond (sum of interest and principal payments are constant over time).
- Can we use the previous techniques to price the MBB?
- **No** – MBB's are different in several respects:
  - Empirical evidence shows that not all borrowers prepay at the same time.
  - Possible explanation: transaction costs which differ across different borrowers (borrower heterogeneity).
- Instead, we will price the MBB using the so-called **prepayment function** (with an more or less *ad hoc* specification).
- This function is defined as the fraction of remaining borrowers who prepay on a given node in the tree.