Motivation

- Consider a fixed-income derivative with payoff $V_T$ a time $T$.
- The price today ($t = 0$) is given by
  \[ V_0 = E_Q^0 \left[ e^{-\int_0^T r_s ds} V_T \right]. \]  
  (1)

- Problem: we are calculating the expectation of the product of two dependent random variables.
- In general, it is easier to calculate $V_0$ using the so-called \textbf{forward-risk adjusted measure} technique.
- This means that $V_0$ is given by
  \[ V_0 = P(0, T) \cdot E_Q^{Q_T} (V_T), \]  
  \[ \text{where } Q_T \text{ is a new probability measure.} \]  
- Basic idea: change probabilities of different events (again).
Model setup

- The general results are derived for a one-factor HJM model.
- Risk-neutral forward-rate dynamics:
  \[ df(t, T) = -\sigma(t, T)\sigma_P(t, T)dt + \sigma(t, T)dW^Q_t, \]  
  where
  \[ \sigma_P(t, T) = -\int_t^T \sigma(t, u)du. \]  
- Bond prices evolve according to the SDE
  \[ dP(t, T) = r_t P(t, T)dt + \sigma_P(t, T)P(t, T)dW^Q_t. \]  
- Thus, \( \sigma_P(t, T) \) is the time \( t \) volatility of the zero-coupon bond maturing at time \( T \).

Forward-risk adjusted measure \(-1\)

- The price of the derivative at time \( t \) is denoted \( V_t \).
- Under the risk-neutral distribution we have
  \[ dV_t = r_t V_t dt + \sigma_V(t)V_t dW^Q_t. \]  
- Note: we do not know \( V_t \) or \( \sigma_V(t) \), but the only important thing right now is the form of the SDE (6).
- Define the relative (deflated) price of the derivative
  \[ F_t \equiv V_t / P(t, T), \quad \text{for } t \in [0, T]. \]  
- SDE for \( F_t \) can be obtained from Ito’s lemma
  \[ dF_t = \sigma_P(\sigma_P - \sigma_V)F_t dt + (\sigma_V - \sigma_P)F_t dW^Q_t \]  
- Shorthand notation: \( \sigma_P \equiv \sigma_P(t, T) \) and \( \sigma_V \equiv \sigma_V(t) \).
Forward-risk adjusted measure – 2

- Define a new probability measure, denoted $Q^T_t$, such that
  \[ W^Q_{t} = W^Q_{0} - \int_{0}^{t} \sigma_p(u,T)du, \quad t \in [0,T], \]  
  is a Brownian motion under $Q^T_t$.
- The differential form of (9) is
  \[ dW^Q_{t} = dW^Q_{t} - \sigma_p(t,T)dt. \]  
- If we substitute (9) into the SDE for $F_t$, we obtain the process for $F_t$ under the forward-risk adjusted measure,
  \[ dF_t = -\sigma_p(\sigma_V - \sigma_p)F_t dt + (\sigma_V - \sigma_p)F_t \left( dW^Q_{t} + \sigma_p dt \right) \]
  \[ = (\sigma_V - \sigma_p)F_t dW^Q_{t}. \]  

Forward-risk adjusted measure – 3

- **Key result:** under the new measure (distribution) $Q^T$, the relative price $F_t$ is a martingale — since the drift is zero.
- Step 1: because of the martingale property, we have
  \[ F_t = E^{Q^T}_{t} (F_T), \quad \text{for } t \leq T. \]  
- Step 2: since $P(T,T) = 1$, we get $F_T = V_T$.
- Step 3: the time $t = 0$ price can be calculated as
  \[ V_0 = P(0,T)F_0 = P(0,T)E^{Q^T}_{0} (F_T) \]
  \[ = P(0,T)E^{Q^T}_{0} (V_T). \]  
- Only remaining problem: determine distribution of payoff under the forward-risk adjusted measure, $Q^T$.  

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Forward-risk adjusted measure – 4

- Forward-rate dynamics under the new measure $Q_T$,
  \[ df(t,T) = -\sigma(t,T) \sigma_p(t,T) dt + \sigma(t,T) \left( dW^Q_t + \sigma_p(t,T) dt \right) \]
  \[ = \sigma(t,T) dW^Q_t. \] (14)
- Thus, the $T$-maturity forward rate is a martingale under $Q_T$.
- Integrating (14) from $t = 0$ to $t = T$,
  \[ f(T,T) = f(0,T) + \int_0^T \sigma(t,T) dW^Q_t \] (15)
- Since the expectation of the second term in (15) is zero, and since $f(T,T) = r_T$, we get
  \[ f(0,T) = E_0^{Q_T} (r_T). \] (16)

Options on zero-coupon bonds – 1

- Notation:
  \begin{align*}
  K & \quad \text{exercise price of the call option.} \\
  T & \quad \text{maturity of the call option.} \\
  T_1 & \quad \text{maturity of the underlying zero-coupon bond.} \\
  C(T,T_1,K) & \quad \text{price of the call option at time } t = 0.
  \end{align*}
- For concreteness, we use the extended Vasicek model which corresponds to the HJM model with
  \[ \sigma(t,T) = \sigma e^{-\kappa(T-t)} \] (17)
  \[ \sigma_p(t,T) = \frac{\sigma e^{-\kappa(T-t)} - 1}{\kappa}. \] (18)
- The following results apply to any Gaussian HJM model, however.
Options on zero-coupon bonds – 2

- The price of the option is given by:
  \[ C(T, T_1, K) = P(0, T) E_0^{QT} \left[ \max \{ P(T, T_1) - K, 0 \} \right]. \] (19)

- In order to calculate this expectation, we must determine the distribution of \( P(T, T_1) \) under \( Q^T \).

- Since \( P(T, T) = 1 \), the distribution of \( P(T, T_1) \) can be obtained from the relative price
  \[ F(t, T, T_1) = P(t, T_1)/P(t, T). \] (20)

- Note that (20) is the forward price of the \( T_1 \)-maturity bond.

- SDE under \( Q^T \) for \( F(t, T, T_1) \):
  \[
  dF(t, T, T_1) = \{ \sigma_P(t, T_1) - \sigma_P(t, T) \} F(t, T, T_1) dW_t^{QT} \\
  = \sigma_F(t, T, T_1) F(t, T, T_1) dW_t^{QT}. \] (21)

Options on zero-coupon bonds – 3

- An application of Ito’s lemma gives:
  \[
  d \log F(t, T, T_1) = -\frac{1}{2} \sigma_F^2(t, T, T_1) dt + \sigma_F(t, T, T_1) dW_t^{Q}. \] (22)

- For the extended Vasicek model:
  \[
  \sigma_F(t, T, T_1) = \frac{\sigma}{\kappa} \left( e^{-\kappa(T_1 - t)} - e^{-\kappa(T - t)} \right) \\
  = \frac{\sigma}{\kappa} e^{-\kappa(T - t)} \left( e^{-\kappa(T_1 - T)} - 1 \right). \] (23)

- It follows from (22) and (23) that \( \log P(T, T_1) = \log F(T, T, T_1) \) is normally distributed.

- The mean of \( \log P(T, T_1) \) is given by:
  \[
  \mu_F(T, T_1) = \log F(0, T, T_1) - \frac{1}{2} \int_0^T \sigma_F^2(t, T, T_1) dt \\
  = \log F(0, T, T_1) - \frac{1}{2} \omega_F^2(T, T_1). \] (24)
Options on zero-coupon bonds – 4

- The variance of $\log P(T, T_1)$ is given by:

$$\omega_F^2(T, T_1) = \int_0^T \sigma_F^2(t, T, T_1) dt$$

$$= \left( \frac{e^{-\kappa(T_1 - T)} - 1}{\kappa} \right)^2 \times \left( \frac{\sigma^2 - e^{-2\kappa T}}{2\kappa} \right)$$ (25)

- After some lengthy algebra, the price of the call follows:

$$C(T, T_1, K) = P(0, T_1)N(d_1) - P(0, T)KN(d_2)$$ (26)

$$d_1 = \left( \log \frac{P(0, T_1)}{P(0, T)} - \log K + \frac{1}{2} \omega^2_F \right) / \omega_F$$ (27)

$$d_2 = d_1 - \omega_F.$$ (28)

- This is the Black-Scholes formula with a different variance . . .

Options on coupon bonds – 1

- Coupon bond with payments $\{a_j\}$ at times $T_j$, $1 \leq j \leq M$.

- The price is at time $t$ is

$$P_a(t; r_t) = \sum_{j=1}^M a_j \cdot P(t, T_j; r_t)$$ (29)

- In the extended Vasicek model, all bond prices depend on $r_t$.

- The price of a call option expiring at time $T < T_1$ (date of the first payment) can be written as:

$$C_a(T, K) = P(0, T) \cdot E_0^T \left( \sum_{j=1}^M a_j \cdot P(T, T_j; r_T) - K \right)^+$$ (30)

- However, we are no longer taking the (truncated) expectation over a log-normal random variate.
Options on coupon bonds – 2

- We use the Jamshidian decomposition.
- Define $r^*$ such that
  \[ \sum_{j=1}^{M} a_j \cdot P(T, T_j; r^*) - K = 0 \]  
  (31)
- If $P(T, T_j; r)$ is monotonic in $r$ (for all maturities), we can show that:
  \[ \left( \sum_{j=1}^{M} a_j \cdot P(T, T_j; r_T) - K \right)^+ = \sum_{j=1}^{M} a_j \left( P(T, T_j; r_T) - K_j \right)^+ \]  
  (32)
  where $K_j = P(T, T_j; r^*)$.
- This follows from monotonicity since $\sum_{j=1}^{M} a_j P(t, T_j; r) > K$, corresponding to $r < r^*$, implies $P(t, T_j; r) > K_j$ for each $j$.

Options on coupon bonds – 3

- Interpretation: an option on a portfolio of payments, $\{a_j\}$, is equivalent to a portfolio of options.
- This means that the price of an option on a coupon bond is given by the expression:
  \[ C_a(T, K) = \sum_{j=1}^{M} a_j \cdot C(T, T_j; K_j) \]  
  (33)
- This holds for the following models:
  - Vasicek model — derived by Jamshidian (1989)
  - CIR model — derived by Longstaff (1993)
- The result does not generalize to multi-factor models, such as the Gaussian double-decay model.
At-the-money interest-rate caps

- Consider a derivative (ATM cap) with payoff
  \[ V_T = \max(r_T - f(0,T), 0). \] (34)

- Under the forward-risk adjusted measure:
  \[ r_T - f(0,T) = \int_0^T \sigma(t,T) dW_t^Q. \] (35)

- Under the extended Vasicek model, the RHS of (35) is normally distributed with mean zero and variance
  \[ v^2(0,T) = \int_0^T \sigma^2(t,T) dt = \sigma^2 \frac{1 - e^{-2\kappa T}}{2\kappa}. \] (36)

- The time \( t = 0 \) price of the ATM cap is:
  \[ C(T, f(0,T)) = P(0,T) E_0^Q(V_T) = P(0,T) \frac{v(0,T)}{\sqrt{2\pi}}. \] (37)

Forward and futures contracts

- Agreement to deliver a financial asset on a future date \( t \) for a price which is fixed today (but paid upon delivery).

- There are no payments when entering into the contract — the initial value of the forward or futures contract is always zero.

- Difference between forward and futures contract: the latter is continuously marked to market to ensure zero value.

- Pricing: consider a forward and future on a \( T \)-maturity zero.
  \[
  F_{\text{for}}(t,T) = \frac{P(0,T)}{P(0,t)} \\
  F_{\text{fut}}(t,T) = E_0^Q[P(t,T)].
  \] (38) (39)

- See chapter 14 in Tuckman for proofs and further discussion.

- When the underlying asset is a bond, futures prices are (generally) below forward prices.
Pricing a two-year callable bond – 1

- Binomial tree for the short rate with annual time steps, \( \theta(n,s) = 0.5 \), discrete compounding. The tree is calibrated to match a flat initial term-structure of 10 percent.

\[
\begin{array}{c}
10.00 \\
11.11 \\
8.91 \\
\end{array}
\]

- A two-year non-callable bullet with a 10 percent coupon is trading at par. To see this, calculate the bond price using the tree

\[
\begin{array}{c}
100.00 \\
109.00 = 10 + 110/1.1111 \ (101.00) \\
111.00 = 10 + 110/1.0891 \ (99.00) \\
\end{array}
\]

Pricing a two-year callable bond – 2

- Does the price change if the borrower is allowed (but not obligated) to pay back the entire principal after one year?
- In other words: what is the price of a two-year callable bond?
- After one year, the borrower can choose between continuing the fixed-rate loan (bullet) with a 10% interest rate, or calling the bullet and borrowing at the new short rate, \( r(1,s) \).
- If \( r(1,s) \) is below 10%, the rational borrower will call the bond.
- Price of the callable bond follows from the tree:

\[
\begin{array}{c}
99.55 \\
109.00 = 10 + 110/1.1111 \\
110.00 = 10 + 100.00 \\
\end{array}
\]

- The price (premium) of the call feature is 0.45 cents.
Callable bonds in a multi-period setting

- The callable bond contains an **embedded** option to purchase the “otherwise identical” non-callable bond at par.
- Option payoff: $C(1,0) = 1$ and $C(1,1) = 0$, which means that the price is $C(0,0) = 0.5 \cdot \left[ C(1,0) + C(1,1) \right] / 1.10 = 0.45$.
- We always have: $P_C(n,s) = P_{NC}(n,s) - C(n,s)$.
- In a multi-period setting, the call feature is an **American option**.
- Description of the **optimal** call strategy:
  - Let $K(n,s)$ denote the call price (could be greater than par).
  - Let $P_C^H(n,s)$ denote the price of the callable bond if it is **not** called at the node $(n,s)$.
    - Note: $P_C^H(n,s)$ is calculated using the short-rate tree and the backward equation.
    - The optimal strategy is to call the bond if $P_C^H(n,s) > K(n,s)$.

Mortgage-backed bonds

- Most mortgage-backed bonds (MBBs) in Denmark are callable.
- The “otherwise identical” non-callable bond is annuity bond (sum of interest and principal payments are constant over time).
- Can we use the previous techniques to price the MBB?
  - **No** – MBB’s are different in several respects:
    - Empirical evidence shows that not all borrowers prepay at the same time.
    - Possible explanation: transaction costs which differ across different borrowers (borrower heterogeneity).
- Instead, we will price the MBB using the so-called **prepayment function** (with an more or less ad hoc specification).
- This function is defined as the fraction of remaining borrowers who prepay on a given node in the tree.