The fundamental PDE for bond prices – 1

- Model building blocks (assumptions):
  1. **Absence of arbitrage** opportunities (in a frictionless market).
  2. **One factor**: the bond price, \( P(t, T) \), depends only the short rate, \( r_t \).
  3. **Stochastic process**: \( r_t \) follows the SDE \( dr_t = \mu(r_t)dt + \sigma(r_t)dW_t \).

- Based on these assumptions, we first derive the APT-restriction

\[
\mu_P(t, T) = r_t + \lambda(r_t)\sigma_P(t, T) ; \quad \sigma_P(t, T) = \frac{\partial P}{\partial r}\sigma(r), \tag{1}
\]

where \( \mu_P(t, T) \) and \( \sigma_P(t, T) \) are the instantaneous **expected return** and **volatility** of the \( T \)-maturity bond,

\[
d P(t, T) / P(t, T) = \mu_P(t, T)dt + \sigma_P(t, T)dW_t, \tag{2}
\]

and \( \lambda(r) \) is the so-called **market price of risk**.
The fundamental PDE for bond prices – 2

• Using Ito’s lemma, \( \mu_P(t, T) \) may also be written as:

\[
\mu_P(t, T) P(t, T) = \frac{\partial P}{\partial r} \mu(r) + \frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r),
\]

(3)

• From the APT restriction (1) we have

\[
\mu_P(t, T) P(t, T) = r_t P(t, T) + \lambda(r_t) \frac{\partial P}{\partial r} \sigma(r_t)
\]

(4)

• By combining the two equations (3) and (4), we get the fundamental PDE which the bond price \( P(t, T) \) must satisfy:

\[
\frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r) + \frac{\partial P}{\partial r} [\mu(r) - \lambda(r) \sigma(r)] + \frac{\partial P}{\partial t} - r P = 0,
\]

(5)

with boundary condition \( P(T, T) = 1 \).

Risk-neutral valuation – basics

• Feynman-Kac representation of the solution to the PDE,

\[
P(t, T) = E_t^Q \left[ e^{-\int_t^T r_s \, ds} \right].
\]

(6)

• The expectation is taken under a new probability measure \( Q \) corresponding to the drift-adjusted SDE for the short rate

\[
dr_t = \{\mu(r_t) - \lambda(r_t) \sigma(r_t)\} \, dt + \sigma(r_t) \, dW_t^Q,
\]

(7)

where \( W_t^Q \) is a Brownian motion under the \( Q \)-measure.

• We refer to this as risk-neutral valuation.

• Risk-neutral valuation in two cases:
  
  – **SDE:** risk adjustment done by modifying the drift of the short-rate process.
  
  – **Binomial:** risk adjustment by modifying the probabilities of an up-move.
Risk-neutral valuation – extensions

• Consider a claim with the following payoff structure
  – For \( t \leq s \leq T \), there is a continuous annualized payment of \( c(r_s) \). That is, between \( s \) and \( s + ds \), the payment from the claim is \( c(r_s)ds \).
  – At maturity \( T \), there is a final lump-sum payment of \( C(r_T) \).

• Using risk-neutral valuation, the price can be expression as:
  \[
  V_t(r) = E^Q_t \left[ \int_t^T e^{-\int_t^s r_u du} c(r_s)ds \right] + E^Q_t \left[ e^{-\int_t^T r_s ds} C(r_T) \right].
  \] (8)

• Note how the future payoffs of \( c(r_s)ds \) and \( C(r_T) \) are discounted.

• By the Feynman–Kac duality, there is also a PDE representation:
  \[
  \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \frac{\sigma^2(r)}{\sigma^2(r)} + \frac{\partial V}{\partial r} \left[ \mu(r) - \lambda(r) \sigma(r) \right] + \frac{\partial V}{\partial t} + c(r) - rP = 0,
  \] (9)
  subject to the boundary condition \( V_T(r) = C(r) \).

The Vasicek model – 1

• The first paper about continuous-time term-structure models.

• Vasicek (1977) assumes that the short rate follows the Ornstein-Uhlenbeck process
  \[
  dr_t = \kappa(\mu - r_t)dt + \sigma dW_t.
  \] (10)

• The market price of risk is assumed to be a constant, \( \lambda(r) = \lambda \).

• Main features of the Vasicek model:
  – Mean reversion towards the unconditional mean \( \mu = E(r) \).
  – Speed of mean reversion determined by \( \kappa \) (a larger \( \kappa \) means faster mean reversion).
  – The short rate is normally distributed (Gaussian model).
  – Because of the normal distribution, we can obtain closed-form solutions for interest-rate derivatives in many important cases.
The Vasicek model – 2

- PDE for bond prices:

\[ \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2 + \frac{\partial P}{\partial r} [\kappa(\mu - r) - \lambda \sigma] + \frac{\partial P}{\partial t} - rP = 0, \quad (11) \]

with boundary condition \( P(T, T) = 1 \).

- We guess that the solution to (11) takes the following form:

\[ P(t, T) = \exp [A(\tau) + B(\tau) r], \quad \tau = T - t. \quad (12) \]

- In order to show that equation (12) is the solution to (11) and to determine \( A(\tau) \) and \( B(\tau) \), we do the following:

  - Calculate the requisite partial derivatives of (12), and substitute these expressions into the PDE (11).
  - If the PDE reduces to two ordinary differential equations (ODEs), we have verified that the solution is of the form (12).
  - Solve the ODEs, subject to the boundary condition \( A(0) = 0 \) and \( B(0) = 0 \).

The Vasicek model – 3

- Partial derivatives of (12),

\[ \frac{\partial P}{\partial r} = B(\tau) P(t, T), \quad \frac{\partial^2 P}{\partial r^2} = B(\tau)^2 P(t, T) \]

\[ \frac{\partial P}{\partial t} = - \frac{\partial P}{\partial \tau} = - [A'(\tau) + B'(\tau) r] \cdot P(t, T). \]

- Next, we substitute these expressions into the PDE:

\[ \left\{ \frac{1}{2} B^2(\tau) \sigma^2 + B(\tau) [\kappa(\mu - r) - \lambda \sigma] - \{A'(\tau) - B'(\tau) r - \} \right\} \cdot P = 0. \quad (13) \]

- After dividing by \( P \) and collecting terms with the factor \( r \), we get

\[ \left\{ \frac{1}{2} B^2(\tau) \sigma^2 + B(\tau) [\kappa \mu - \lambda \sigma] - A'(\tau) \right\} - \{ \kappa B(\tau) + B'(\tau) + 1 \} r = 0. \quad (14) \]

- Both terms in brackets must be zero (our two ODEs).
The Vasicek model – 4

- System of ODEs for the Vasicek model

\[
A'(\tau) = \frac{1}{2}\sigma^2(\tau)B^2(\tau) + \{\kappa\mu - \lambda\sigma\}B(\tau) \tag{15}
\]

\[
B'(\tau) = -\kappa B(\tau) - 1 \tag{16}
\]

- The PDE boundary condition

\[
P(T, T) = \exp[A(0) + B(0)r_T] = 1 \quad \text{for all } r_T, \tag{17}
\]

means that \(A(0) = 0\) and \(B(0) = 0\) — ODE initial conditions.

- The ODE system has a recursive structure — the ODE equation for \(B'(\tau)\), i.e. (16), does not involve \(A(\tau)\).

- This means that the function \(B(\tau)\) only depends on \(\kappa\) and \(\tau\).

The Vasicek model – 5

Four steps in finding the solution for \(B(\tau)\)

1. Multiply all terms by \(\exp(\kappa\tau)\) and rearrange,

\[
B'(\tau)e^{\kappa\tau} + \kappa B(\tau)e^{\kappa\tau} = -e^{\kappa\tau}. \tag{18}
\]

2. By the product rule for differentiation the LHS may be written as

\[
\frac{d}{d\tau} \{e^{\kappa\tau}B(\tau)\} = -e^{\kappa\tau}. \tag{19}
\]

3. Since, for any function, \(h(\tau) = h(0) + \int_0^\tau h'(s)ds\),

\[
e^{\kappa\tau}B(\tau) = B(0) + \int_0^\tau \frac{d}{ds} \{e^{\kappa s}B(s)\} \; ds = -\int_0^\tau e^{\kappa s} \; ds, \tag{20}
\]

4. Finally, multiply by \(\exp(-\kappa\tau)\), and calculate the integral

\[
B(\tau) = -e^{-\kappa\tau}\int_0^\tau e^{\kappa s} \; ds = \frac{e^{-\kappa\tau} - 1}{\kappa}. \tag{21}
\]
The Vasicek model – 6
Finding the solution for $A(\tau)$

- The ODE is $A'(\tau) = \frac{1}{2}\sigma^2(\tau)B^2(\tau) + \{\kappa\mu - \lambda\sigma\}B(\tau)$.
- No special “tricks” are needed here since $A(\tau)$ is not on the RHS.
- We calculate $A(\tau)$ by straightforward integration of the RHS

\[
A(\tau) = A(0) + \int_0^\tau A'(s)ds = \frac{1}{2}\sigma^2 \int_0^\tau B^2(s)ds + [\kappa\mu - \lambda\sigma] \int_0^\tau B(s)ds. \quad (22)
\]

- We know $B(\tau)$, and after a lot of calculations we get

\[
A(\tau) = -R(\infty)(\tau + B(\tau)) - \frac{\sigma^2}{4\kappa}B^2(\tau), \quad \text{where}
\]

\[
R(\infty) = \mu - \frac{\lambda\sigma}{\kappa} - \frac{1}{2} \left(\frac{\sigma}{\kappa}\right)^2. \quad (23)
\]

The CIR model – 1

- Similar to the Vasicek model — except that the short rate is restricted to be positive (non-negative).
- Stochastic process for the short rate:

\[
dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t \quad (24)
\]

- This process has a reflecting barrier at 0, hence $r_t \geq 0$.
- Market price of risk: $\lambda(r) = (\lambda/\sigma)\sqrt{r}$.
- PDE for bond prices:

\[
\frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2 r + \frac{\partial P}{\partial r} [\kappa(\mu - r) - \lambda r] + \frac{\partial P}{\partial t} - rP = 0, \quad (25)
\]

with boundary condition $P(T,T) = 1$. 

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The CIR model – 2

- Again we guess that $P(t, t + \tau) = \exp[A(\tau) + B(\tau)\tau]$.  
- We substitute the partial derivatives into the PDE (25),

$$\left\{ \frac{1}{2} B^2(\tau)\sigma^2 r + B(\tau)[\kappa(\mu - r) - \lambda r] - A'(\tau) - B'(\tau) - r \right\} \cdot P = 0. \quad (26)$$

- After dividing by $P$ and collecting terms with the factor $r$, we get

$$\left\{ \frac{1}{2} B^2(\tau)\sigma^2 - B(\tau)(\kappa + \lambda) - B'(\tau) - 1 \right\} r + \left\{ B(\tau)\kappa \mu - A'(\tau) \right\} = 0. \quad (27)$$

- From the two brackets, we get the ODE system:

$$A'(\tau) = \kappa \mu B(\tau) \quad (28)$$
$$B'(\tau) = \frac{1}{2} \sigma^2 B^2(\tau) - (\kappa + \lambda)B(\tau) - 1, \quad (29)$$

with initial conditions $A(0) = 0$ and $B(0) = 0$.

Does equation (12) always work?

- Q: Do we always get $P(t, t + \tau) = \exp[A(\tau) + B(\tau)\tau]$?

- A: No — as counter-example let $\lambda(r) = 0$ and

$$dr_t = \kappa(\mu - r_t)dt + \sigma r_t^\gamma dW_t. \quad (30)$$

- If the above guess is correct, the PDE becomes

$$\left\{ \frac{1}{2} B^2(\tau)\sigma^2 r^{2\gamma} + B(\tau)\kappa(\mu - r) - A'(\tau) - B'(\tau) - r \right\} \cdot P = 0. \quad (31)$$

- After dividing by $P$ and collecting terms we have,

$$\left\{ \frac{1}{2} B^2(\tau)\sigma^2 \right\} r^{2\gamma} - \left\{ B(\tau)\kappa + B'(\tau) + 1 \right\} r + \left\{ B(\tau)\kappa \mu - A'(\tau) \right\} = 0. \quad (32)$$

- The three expressions in brackets cannot be zero at the same time (unless $\gamma = 0$ or $\gamma = 1/2$), so our guess is wrong.