Fixed Income Analysis

Term-Structure Models in Continuous Time

Introduction and Mathematical Preliminaries
One-Factor Models (Equilibrium Models)

Jesper Lund

March 24, 1998

Outline

1. A brief survey of stochastic processes, SDEs and Ito's lemma
2. Motivation for continuous-time term-structure models
3. Equilibrium vs. arbitrage-free models
4. Yield (spot) and forward curves with continuous compounding
5. Basic idea of equilibrium term-structure models
6. The term-structure in a general one-factor model
7. Fundamental PDE and Feynman-Kac representation
8. A simple one-factor model (example).
9. Three one-factor models: Merton, Vasicek and CIR.

Stochastic processes — definitions

- A stochastic process can be defined as an ordered sequence of random variables $\{X_t\}$, indexed by time t. In general, X_{t_1} and X_{t_2} are dependent random variables.
- The AR(1) model (process) is an example of a stochastic process:

$$X_t = \phi X_{t-1} + u_t\,; \qquad u_t \;\sim\; N(0, \sigma^2) \tag{1}$$

- The words 'process' and 'model' are often used interchangeably.
- The AR(1) model is a discrete-time model. We observe X_t at $t = 0, 1, 2, 3, 4, \ldots$ — but not at $t = 1.5$. Formally, the time index is the set of natural numbers (integers).
- For **continuous-time** processes, the time index is the set of real numbers. In principle, we can observe the stochastic process X_t at all time points (that is, continuously).

The Brownian motion

- The **Brownian motion** $\{W_t\}$ is a continuous-time stochastic process with the following properties:
  1. $W_0 = 0$.
  2. For any times $s > t$, $W_s - W_t \;\sim\; N[0, (s-t)]$.
  3. For any times $t_1 < t_2 < t_3$, the non-overlapping increments $W(t_3) - W(t_2)$ and $W(t_2) - W(t_1)$ are independent.
  4. Sample path of W_t are continuous (the sample path can be drawn without lifting the pen).
- The third property of the Brownian motion implies that

$$\text{Cov}(W_t, W_s) = E(W_t W_s) = \min(t, s) \tag{2}$$

- By the third property, the Brownian motion is a **martingale**,

$$E\left[W_s \,|W_t\right] = W_t, \qquad \text{for all } s \geq t. \tag{3}$$

Stochastic differential equations

- Stochastic differential equations (SDEs) are constructed from the Brownian motion process.
- Sample paths of SDEs are continuous (like the Brownian motion).
- General form of a univariate (one-factor) stochastic differential equation (SDE):

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{4}$$

- This means that for a sufficiently small Δ

$$X_{t+\Delta} - X_t \;\sim\; N\left[\mu(X_t)\Delta, \sigma^2(X_t)\Delta\right], \tag{5}$$

- Strictly speaking, equation (5) is only an approximation of the SDE (known as the Euler discretization).

Ito's lemma

- Consider a function of X_t and time t, denoted $F_t = F(X_t, t)$.
- Ito's lemma gives us the stochastic process for F_t,

$$dF_t = \mu_F(X_t, t)dt + \sigma_F(X_t, t)dW_t \tag{6}$$

where

$$\mu_F(X,t) = \frac{\partial F(X,t)}{\partial X}\mu(X) + \frac{\partial F(X,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(X,t)}{\partial X^2}\sigma^2(X) \tag{7}$$

$$\sigma_F(X,t) = \frac{\partial F(X,t)}{\partial X}\sigma(X). \tag{8}$$

- Example: the logarithm of the GBM, $dS_t = \mu S_t dt + \sigma S_t dW_t$, satisfies the SDE

$$d \log S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \tag{9}$$

Why study continuous-time models?

- Arguments **against** continuous-time models
  - In the real world, price changes occur at discrete time intervals.
  - Binomial models are simpler to understand (or to learn, at least).
  - In some cases, we will use some discrete-time approximation (as a numerical solution procedure), even if we start with a continuous-time model.
- Arguments in **favor** of continuous-time models
  - In any discrete-time model (not just binomial), there is a great deal of ambiguity about the "right" time interval. The continuous-time specification may very well be the **least arbitrary** assumption!
  - In many important cases, we can find an analytical (closed-form) solution for bond prices and fixed-income derivatives.
  - Therefore, understanding the structure and **properties** of the model is easier in the continuous-time case.
  - The continuous-time specification generally makes it easier to find the **best** discrete-time approximation and numerical solution procedure.

Equilibrium vs. arbitrage-free models – 1

- We use the classification in Tuckman (1995, ch. 9) — but note that other (older) papers may use different definitions.
- Arbitrage-free models:
  - Per construction, arbitrage-free term-structure models fit the initial yield curve (i.e., today's yield curve) **exactly**.
  - Used for pricing fixed-income derivatives (not bonds).
  - The prices of these securities are often independent of investor **preferences**.
  - Model examples: HJM and Ho & Lee models, as well as equilibrium-style models with **time-dependent** parameters (calibrated models), e.g. the BDT model and the Hull-White extended Vasicek model.
  - In most cases, a single-factor model is used (with numerical solution).
  - Implementation issues: calibration to initial yield curve, and assumptions about the volatility structure.
  - The models are **not stable** — the time-dependent parameters must be re-calibrated over time (inconsistency).

Equilibrium vs. arbitrage-free models – 2

- Equilibrium (classical) models:
  - The original term-structure models belong to this group, hence the phrase "classical models".
  - Main building blocks: stochastic process for the short rate, and assumptions about investor **preferences** (risk premia, or market prices of risk).
  - The yield curve is determined **endogenously** in the model — it is not constrained to match the actual (market) yield curve.
  - Model parameters are constant over time (internal consistency), and typically there are at least two factors (multi-factor models).
  - Model examples: Vasicek, CIR and the Brennan-Schwartz model.
  - Used mainly for trading bonds (yield-curve strategies), less useful for fixed-income derivatives (where we have two bets).
  - Other applications: risk management, where single-factor models (with calibration) tend to be inappropriate.
  - Implementation issues: **statistical estimation** using historical data on the term structure (note: these methods are not covered in this course).

Definition of yield and forward curves

- Price at time t of a zero-coupon bond maturing at time T (maturity date) is denoted by $P(t,T)$.
- We always use **continuous compounding** when defining the yield curve and forward rates, since this simplifies many formulas.
- Yield-to-maturity, $R(t,T)$, and forward rate, $f(t,T)$:

$$R(t,T) = \frac{-\log P(t,T)}{T-t} \tag{10}$$

$$f(t,T) = \frac{-\partial \log P(t,T)}{\partial T} \tag{11}$$

- Inverse relationships:

$$P(t,T) = e^{-R(t,T)(T-t)} \tag{12}$$

$$P(t,T) = e^{-\int_t^T f(t,s)ds} \tag{13}$$

Basic idea of equilibrium models

- The purpose is deriving an expression for $P(t,T)$.
- We start by making assumptions about the number of factors (state variables) determining the yield curve, and the stochastic processes governing these factors.
- With these assumptions — and Ito's lemma — we find an expression for the expected bond return and risk exposure (volatility) for different maturity dates T_j.
- Suppose that we know the expected return at each time (instant) between t (today) and T (maturity) ...
- Then, using this knowledge and the terminal value of $P(T,T) = 1$, we can work backwards and calculate the price today, $P(t,T)$.
- We use the APT (arbitrage price theory) to determine the expected return as a function of some preference parameters.

A general one-factor model – 1

- Modeling assumptions:
  1. Frictionless bond market (no taxes, transactions costs, bid-ask spreads, divisibility problems, short-sale constraints, etc.).
  2. Investors prefers more wealth to less (implies absence of arbitrage opportunities in the bond market).
  3. All bond prices are a function of a single state variable, which we take as the short rate r_t (definition: continuously compounded interest rate on a money market account over a small horizon).
  4. The dynamics of the short rate are governed by the SDE:

$$dr_t = \mu(r_t)dt + \sigma(r_t)dW_t. \qquad (14)$$

- **Our problem:** determine the relationship between r_t and the price of the bond maturing at time T, $P(t,T)$.
- Limitation implicit in the third assumption: bond returns for different maturities are perfectly correlated.

A general one-factor model – 2

- The zero-coupon bond price, $P(t,T)$, is a function of r and t.
- By Ito's lemma, $P(t,T)$ evolves according to the SDE:

$$d\,P(t,T) = \mu_P(t,T)P(t,T)dt \;+\; \sigma_P(t,T)P(t,T)dW_t, \tag{15}$$

where

$$\mu_P(t,T)P(t,T) \;=\; \frac{\partial P}{\partial r}\mu(r) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r) \tag{16}$$

$$\sigma_P(t,T)P(t,T) \;=\; \frac{\partial P}{\partial r}\sigma(r). \tag{17}$$

- Consider a portfolio, consisting of w_1 bonds with maturity T_1 and w_2 bonds with maturity T_2 (where $T_1 \neq T_2$).
- Value of the portfolio: $\Pi_t = w_1 P(t,T_1) + w_2 P(t,T_2)$.

A general one-factor model – 3

- The instantaneous movement of Π_t, at time t, is given by:

$$d\Pi_t = w_1 \cdot d\,P(t,T_1) + w_2 \cdot d\,P(t,T_2) \tag{18}$$

- Using (15), this can also be written as:

$$\begin{aligned} d\Pi_t = &\{w_1\mu_P(t,T_1)P(t,T_1) + w_2\mu_P(t,T_2)P(t,T_2)\}\,dt \;+ \\ &\{w_1\sigma_P(t,T_1)P(t,T_1) + w_2\sigma_P(t,T_2)P(t,T_2)\}\,dW_t\,. \end{aligned} \tag{19}$$

- Since there are two bonds and only one source of risk, it must be possible to choose w_1 and w_2 such that the portfolio is riskless,

$$w_1\sigma_P(t,T_1)P(t,T_1) + w_2\sigma_P(t,T_2)P(t,T_2) = 0\,. \tag{20}$$

- Note: this requires continuous adjustment of w_1 and w_2.

A general one-factor model – 4

- By the "absence of arbitrage" assumption, the expected return of the portfolio must equal the riskless rate r_t:

$$\begin{aligned} d\Pi_t &= \{w_1\mu_P(t,T_1)P(t,T_1) \ + w_2\mu_P(t,T_2)P(t,T_2)\}\, dt \\ &= r_t\Pi_t dt, \end{aligned} \tag{21}$$

- Alternatively, the **excess return** must be zero:

$$w_1\{\mu_P(t,T_1) - r_t\}\, P(t,T_1) + w_2\{\mu_P(t,T_2) - r_t\}\, P(t,T_2) = 0. \tag{22}$$

- We will show (next slide) that this implies the APT restriction

$$\mu_P(t,T) = r_t + \lambda(r_t)\sigma_P(t,T), \quad \text{for all } T, \tag{23}$$

where $\lambda(r)$ is the market price of risk (risk premium).

A general one-factor model – 5

Proof of equation (23)

- We have shown that, if the vector $w = (w_1, w_2)$ solves the system of equations

$$\begin{bmatrix} \sigma_P(t,T_1)P(t,T_1) & \sigma_P(t,T_2)P(t,T_2) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \equiv A_1 w = 0\,, \tag{24}$$

the **same** vector w also solves the larger system

$$\begin{bmatrix} \sigma_P(t,T_1)P(t,T_1) & \sigma_P(t,T_2)P(t,T_2) \\ \{\mu_P(t,T_1) - r_t\}P(t,T_1) & \{\mu_P(t,T_2) - r_t\}P(t,T_2) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \equiv A_2 w = 0\,. \tag{25}$$

- Since $w \neq 0$, the 2×2 matrix A_2 must be singular (why?).
- Specifically, the rank of A_2 is 1, so the last row can be written as a linear combination of the first. This gives us (23).
- Note that $\lambda(r)$ cannot depend on the maturities T_1 and T_2.

A general one-factor model – 6

- The next step is combining the two different expressions for the expected bond return.
- First, from Ito's lemma and (16) we have

$$\mu_P(t,T)P(t,T) = \frac{\partial P}{\partial r}\mu(r) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r) \tag{26}$$

- Second, the APT restriction (23) can be written as

$$\begin{aligned} \mu_P(t,T)P(t,T) &= rP(t,T) + \lambda(r)\sigma_P(t,T)P(t,T) \\ &= rP(t,T) + \frac{\partial P}{\partial r}\lambda(r)\sigma(r) \end{aligned} \tag{27}$$

- Finally, we equate the right hand sides of (26) and (27) in order to obtain the **fundamental PDE** for $P(t,T)$.

Fundamental PDE for bond prices

- Fundamental PDE (partial differential equation)

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r) + \frac{\partial P}{\partial r}\left[\mu(r) - \lambda(r)\sigma(r)\right] + \frac{\partial P}{\partial t} - rP = 0, \tag{28}$$

with boundary condition $P(T,T) = 1$.

- Feynman-Kac representation:

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right], \tag{29}$$

where the expectation is taken under the probability measure corresponding to the risk-neutral short-rate process:

$$dr_t = \{\mu(r_t) - \lambda(r_t)\sigma(r_t)\}\,dt + \sigma(r_t)dW_t. \tag{30}$$

- Note how the drift and volatility of the SDE (30) are constructed from the coefficients of the PDE (28).

A simple one-factor model – 1

- The short-rate is governed by the random-walk process

$$dr_t = \sigma dW_t \tag{31}$$

- The market price of risk is zero (investors are risk-neutral).
- Fundamental PDE

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 + \frac{\partial P}{\partial t} - rP = 0. \tag{32}$$

- We **guess** that the solution is of the following form

$$P(t, Y) = \exp\left[A(\tau) + B(\tau)r_t\right], \quad \tau = T - t. \tag{33}$$

- In order to check whether equation (33) — our "educated" guess — is the solution of the PDE, we calculate the requisite partial derivatives of (33) and substitute them into (32).

A simple one-factor model – 2

- Partial derivatives:

$$\frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} = \left[A'(\tau) + B'(\tau)r\right] P(t, t+\tau) \tag{34}$$

$$\frac{\partial^2 P}{\partial r^2} = B^2(\tau)P(t, t+\tau). \tag{35}$$

- We substitute (34) and (35) into (32),

$$\frac{1}{2}B^2(\tau)\sigma^2 P - \left[A'(\tau) + B'(\tau)r\right] P - rP = 0 \tag{36}$$

- After dividing by $P > 0$ on both sides of (36), and collecting terms we get

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 - A'(\tau)\right\} - \left\{B'(\tau) + 1\right\} r = 0 \tag{37}$$

A simple one-factor model – 3

- Since (36) must hold for all values of r, both expression in braces must be zero.
- Hence, we obtain two ordinary differential equations (ODEs)

$$A'(\tau) = \frac{1}{2}\sigma^2 B^2(\tau) \tag{38}$$
$$B'(\tau) = -1 \tag{39}$$

- Boundary conditions: $B(0) = 0$ and $A(0) = 0$.
- The **final solution** is obtained by integration,

$$B(\tau) = B(0) + \int_0^\tau B'(s)ds = -\int_0^\tau ds = -\tau \tag{40}$$
$$A(\tau) = A(0) + \int_0^\tau A'(s)ds = \int_0^\tau \frac{1}{2}\sigma^2 s^2 ds = \frac{1}{6}\sigma^2\tau^3. \tag{41}$$

Three one-factor models

1. Merton (1973) model:
   - Short-rate process: $dr_t = \mu dt + \sigma dW_t$.
   - Market price of risk: $\lambda(r) = \lambda$.
   - Comments: negative interest rates possible, no mean reversion.
2. Vasicek (1977) model:
   - Short-rate process: $dr_t = \kappa(\mu - r_t)dt + \sigma dW_t$.
   - Market price of risk: $\lambda(r) = \lambda$.
   - Comments: mean reversion towards the unconditional mean μ, but still possibility of negative rates.
3. Cox, Ingersoll and Ross (CIR) (1985) model:
   - Short-rate process: $dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t$.
   - Market price of risk: $\lambda(r) = (\lambda/\sigma)\sqrt{r}$.
   - Comments: mean reversion as in the Vasicek model, and r_t is always positive (i.e., $r_t \geq 0$) — because of the continuity of SDE sample paths.