Fixed Income Analysis

Calibration in lattice models — Part II

Calibration to the initial volatility structure
Pitfalls in volatility calibrations
Mean-reverting log-normal models (Black-Karasinski)
Brownian-path independent (BPI) models
Trinomial lattice models (Hull-White)

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The volatility structure

- **Definition:** volatility of zero-coupon yields as a function of time to maturity.
- In a one-factor model, the zero-coupon rate is governed by

$$dR(t,T) = \mu_R(t,T)dt + \sigma_R(t,T)R(t,T)dW_t. \quad (1)$$

- Here, $\sigma_R(t,T)$ is the **proportional** volatility, and $\sigma_R(t,T)R(t,T)$ is the **basis-point** volatility.
- From Ito's lemma, the volatility structure is given by

$$\sigma_R(t,T)R(t,T) = \sigma(r_t)\frac{\partial R(t,T)}{\partial r} = \frac{-\sigma(r_t)}{(T-t)P(t,T)}\frac{\partial P(t,T)}{\partial r} \quad (2)$$

- Thus, the volatility structure depends on the effect of the short rate, r_t, on bond prices, $P(t,T)$.
- Mainly determined by the speed of **mean reversion**.

Calibration in the BDT model

- The BDT model is an approximation to the SDE

$$d \log r_t = \left\{ b(t) + \frac{\sigma'(t)}{\sigma(t)} \log r_t \right\} dt + \sigma(t) dW_t^Q. \tag{3}$$

- Note that the mean reversion coefficient is tied to $\sigma(t)$, the short-rate volatility at time t.
- Calibrating the BDT model to the initial yield **and** volatility curve:
  - The basic geometry of the the BDT binomial tree is unchanged.
  - The pair $\{b(n\Delta), \sigma(n\Delta)\}$ is chosen to match the yield and volatility of the $(n+1)$-period bond.
  - Alternative parameterization: $\{r(n,0), \delta_n\}$, where $r(n,0)$ is the bottom node and $\log \delta_n$ is the spacing for $\log r(n,s)$.
  - We have two equations in two unknowns, but they are easy to solve numerically (Newton-Raphson) when the forward-induction technique is used.

Pitfalls in volatility calibration – 1

- The **basic** problem can be illustrated for the extended Vasicek model (in continuous time)

$$dr_t = \kappa(t) \{\theta(t) - r_t\} dt + \sigma(t) dW_t^Q. \tag{4}$$

- The pair $\{\theta(t), \kappa(t)\}$ is chosen to fit the initial (time $t = 0$) yield and volatility curves.
- The initial forward-rate volatility structure is given by:

$$\sigma(0,T) = \sigma(0) e^{-\int_0^T \kappa(u) du}. \tag{5}$$

- Basis-point volatility structure for zero-coupon rates:

$$\sigma_R(0,T) = \frac{1}{T} \int_0^T \sigma(0,s) ds = \sigma(0) \int_0^T e^{-\int_0^s \kappa(u) du} ds. \tag{6}$$

- Note that the shape only depends on $\kappa(t)$, for $0 \le t \le T$.

Pitfalls in volatility calibration – 2

- At time t, we have a new volatility structure, and the payoffs from derivatives (e.g., call options) depend on the time t volatilities.
- **Caveat:** in a Markovian model, the new volatility structure is completely determined from the initial volatility structure.
- New forward-rate volatilities:

$$\sigma(t,T) = \sigma(t)e^{-\int_t^T \kappa(u)du} = \sigma(t)\frac{\sigma(0,T)}{\sigma(0,t)}. \tag{7}$$

- New volatility structure for zero-coupon rates:

$$\sigma_R(t,T) = \frac{\sigma(t)}{T-t} \cdot \frac{T\sigma_R(0,T) - t\sigma_R(0,t)}{\sigma(0,t)} \tag{8}$$

- Apart from $\sigma(t)$, which is common for all maturities, (7) and (8) only depend on the initial volatility structure.

Pitfalls in volatility calibration – 3

- Constraining the evolution of the volatility structure in this way **could** have undesirable effects on derivatives prices.
- If the current volatility curve is humped, the future curve will be steeply downward sloping, although flat eventually.
- Hull and White's recommendation: do **not** calibrate the model to the volatility structure (if anything, use cap prices instead).
- Additional problems for the BDT model:
  - Minor problem: no analytical solution for bond prices, so the dependencies are more difficult to analyze (and understand).
  - Major problem: mean reversion is tied to the future short-rate volatilities since $\kappa(t) = -\sigma'(t)/\sigma(t)$ in the BDT model.
- If the volatility structure is downward sloping (normal situation), we need BDT's $\sigma(t)$ to be decreasing in t — and this is an **unrealistic** property (in general).

Pitfalls in volatility calibration – 4

- Illustration of the BDT volatility problem, taken from a paper by Simon Schultz and Per Søgaard-Andersen, "Pricing Caps and Floors," Finans/Invest, 4/93 [in Danish].
- Market prices (bid-ask) and three model prices for interest-rate caps (all prices are relative to bid-ask midpoint).
- The BDT and Hull-White (extended) Vasicek models are calibrated to a downward-sloping volatility structure.
- Since $\sigma(t)$ in the BDT model is decreasing over time, the BDT cap prices generally are **too low** in the table below.

Maturity	Bid	Ask	BDT	Vasicek	Black-76
2	0.95	1.05	0.94	1.00	0.97
3	0.97	1.03	0.90	1.00	0.98
4	0.97	1.03	0.92	1.00	0.98
5	0.97	1.03	0.94	0.99	0.97

Mean reversion in log-normal models

- Black and Karasinski (1991) relax the BDT restriction on the mean reversion coefficient,

$$d \log r_t = \{b(t) - \phi(t) \log r_t\} \, dt + \sigma(t) dW_t^Q. \tag{9}$$

- The model (9) **cannot** be implemented in a recombining binomial tree with constant time steps and probabilities $\theta(n, s) = 0.5$.
- There are three possible modifications of the tree which allow for (arbitrary) mean reversion:

1. Non-constant time steps, Δ_n, with $\theta(n, s) = 0.5$. This is suggested by Black and Karasinski (1991). The main disadvantage is that the **spacing declines** over time, and we generally want the opposite (if anything).
2. Non-constant probabilities of an up-move, but with constant time steps. We match the expected change in r (mean reversion) at node (n, s) by adjusting $\theta(n, s)$. The disadvantage is **slower convergence**.
3. Trinomial trees (Hull-White) which have three branches at each node.

Brownian-path independence (BPI)

- Why is it possible to construct a **simple** binomial tree — with constant time steps and $\theta(n,s) = 0.5$ — in the BDT case?
- The solution to the BDT SDE (3) can be written as

$$\begin{aligned} \log r_t &= \frac{\sigma(t)}{\sigma(0)} \log r_0 + \sigma(t) \int_0^t \frac{b(s)}{\sigma(s)} ds + \sigma(t) W_t^Q \\ &\equiv B(t) + \sigma(t) W_t^Q \end{aligned} \quad (10)$$

- The BDT model modifies the Brownian motion tree only by scaling [through $\log(\delta_n) = c \cdot \sigma(t)$] and the bottom node, $r(n,0)$.
- In most **BPI models**, the short rate has the general form:

$$r_t = F\left(B(t) + \sigma(t) W_t^Q\right), \quad (11)$$

for some function $F(x)$. Note that BDT has $F(x) = \exp(x)$.

- Simple binomial trees requires a BPI model [Jamshidian (1991)].

Trinomial lattices – 1

- Introduced by Hull and White (1993, 1994).
- Extended Vasicek model with $\kappa(t) = \kappa$ and $\sigma(t) = \sigma$,

$$dr_t = \kappa \{\theta(t) - r_t\} dt + \sigma dW_t^Q. \quad (12)$$

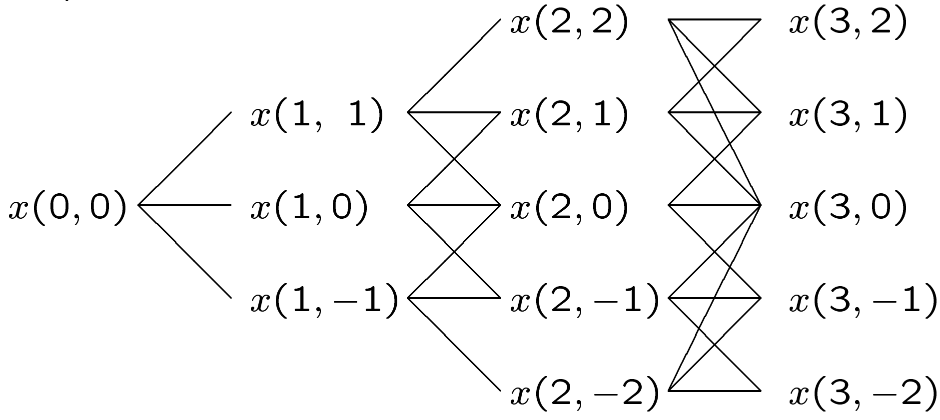
- We rewrite (12) as

$$\begin{aligned} r_t &= \alpha(t) + x_t && (13) \\ \alpha(t) &= e^{-\kappa t} r_0 + \int_0^t e^{-\kappa(t-s)} \kappa \theta(s) ds && (14) \\ dx_t &= -\kappa x_t dt + \sigma dW_t^Q, \quad \text{with } x_0 = 0. && (15) \end{aligned}$$

- First step: build a trinomial tree for x_t. This tree is symmetric around $x = 0$, and the geometry depends only on κ and σ.
- Second step: calibrate the time-dependent parameters, $\alpha_i = \alpha(t_i)$, to match the initial term structure.

Trinomial lattices – 2

- Three-period trinomial tree:



- **Comment 1:** the numbering scheme, $x(i,j)$, is different from binomial case. The center node has $j = 0$ for all times i.
- **Comment 2:** there is a different branching scheme for low and high values of j (node number). The purpose is accommodating mean reversion, while retaining positive probabilities.

Trinomial lattices – 3

- First and second moments of the SDE for x_t:

$$E_t\left[x_{t+\Delta} - x_t\right] \equiv Mx_t = \left(e^{-\kappa\Delta} - 1\right)x_t \approx -\kappa\Delta x_t \qquad (16)$$

$$\text{Var}_t\left[x_{t+\Delta} - x_t\right] \equiv V = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta}\right) \approx \sigma^2\Delta \qquad (17)$$

- The two approximations follow from $\exp(z) \approx 1 + z$.
- The trinomial model has a constant time step, denoted Δ.
- The spacing on the x-axis is specified as $\Delta x = \sqrt{3V}$.
- This means that $x(i,j) = j\Delta x$, for $-n_i \leq j \leq n_i$.
- From each node, (i,j), there are branches to three nodes with probabilities: p_u (top node), p_m (mid mode), and p_d (low node).

Trinomial lattices – 4

- We look at a "normal" node (i,j) — not special branching.
- The probabilities p_u, p_m and p_d are chosen in order to satisfy

$$p_u \Delta x - p_d \Delta = Mj\Delta x \quad (18)$$

$$p_u(\Delta x)^2 + p_d(\Delta x)^2 = V + M^2 j^2 (\Delta x)^2 \quad (19)$$

$$p_u + p_m + p_d = 1 \quad (20)$$

- That is, we match the moments of $(x_{t+\Delta} - x_t)$, see (16) and (17).
- Since $V = (\Delta x)^2/3$, the solution is easily found as

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2} \quad (21)$$

$$p_m = \frac{2}{3} - j^2M^2 \quad (22)$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2} \quad (23)$$

Trinomial lattices – 5

- Note that the probabilities are independent of the initial term structure. Apart from j, they only depend on κ and Δ.
- When branching out from the special top and bottom nodes, similar formulas apply — see Hull (1997) or Hull & White (1994).
- This completes the first step, setting up the nodes of the trinomial lattice.
- Second step: let $r(i,j) = \alpha_i + x(i,j)$, and **calibrate** α_i recursively so that the $(i+1)$-period bond price is matched exactly.
- As in the BDT model, the calibration is done with forward induction and Arrow-Debreu prices.
- We start with $\alpha_0 = r(0,0) = -\log P(1)/\Delta$.
- **Define** $p(i,j) = \exp[-r(i,j)\Delta] = \exp[-\alpha_i\Delta] \cdot \exp[-j(\Delta x)\Delta]$.

Trinomial lattices – 6

- Assume that we have computed α_{m-1}, where $m \geq 1$.
- First, we use the forward equation to compute $G(m, j)$,

$$G(m,j) = \sum_{k=-(n_{m-1})}^{n_{m-1}} q(k,j)p(m-1,k)G(m-1,k), \qquad (24)$$

where $q(k, j)$ is the probability of moving from the node $(m-1, k)$ to (m, j). **Note:** $q(k, j)$ is only non-zero for at most three k.

- Second, with $G(m, j)$ at hand, the $(m+1)$-period bond price is given by

$$\begin{aligned} P(m+1) &= \sum_{j=-n_m}^{n_m} G(m,j)p(m,j) \\ &= e^{-\alpha_m \Delta} \sum_{j=-n_m}^{n_m} G(m,j)e^{-j(\Delta x)\Delta} \qquad (25) \end{aligned}$$

Trinomial lattices – 7

- The solution to (25) is readily available in closed form:

$$\alpha_m = \frac{\log\left(\sum_{j=-n_m}^{n_m} G(m,j)e^{-j(\Delta x)\Delta}\right) - \log P(m+1)}{\Delta} \qquad (26)$$

- Having found α_m, we proceed to $m+1$ (next period) using the same recursions — forward equation (24) followed by (26).

- This completes the construction of the Hull-White trinomial tree for the extended Vasicek model.

- The parameters κ and σ can be calibrated to, e.g., cap prices by minimizing the squared pricing errors

$$S(\kappa,\sigma) = \sum_i \left(V_i^{\text{actual}} - V_i^{\text{model}}\right)^2. \qquad (27)$$