Fixed Income Analysis

Calibration in binomial models

Summary of binomial models
Arrow-Debreu securities
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Calibration using forward induction

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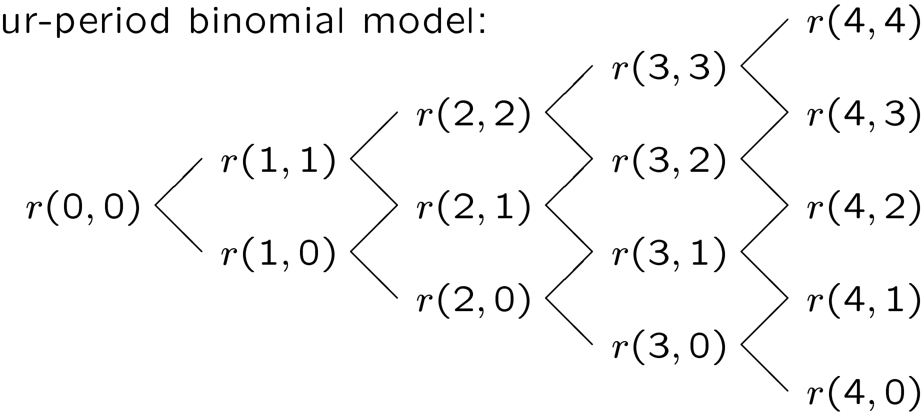
Binomial model — summary

- Four-period binomial model:



- The risk-neutral probability of an up-move at time n in state s is denoted $\theta(n,s)$.
- For a given tree (node values and probabilities), valuation is done using the backward equation

$$\begin{aligned} V(n,s) \;=\; & D(n,s) \;+\; p(n,s)\times \\ & [\theta(n,s)V(n+1,s+1) + (1-\theta(n,s))V(n+1,s)] \end{aligned} \tag{1}$$

Binomial distribution – summary

- A random variable, X, follows the binomial distribution if there are only two possible events (outcomes), denoted x_1 and x_2.
- We use the binomial distribution for modeling the dynamics of interest rates in a tree, typically a recombining tree.
- Let $\theta = \text{Pr}(X = x_1)$, which means that $\text{Pr}(X = x_2) = 1 - \theta$
- Mean and variance of the random variable X:

$$E(X) = \theta x_1 + (1-\theta)x_2 \tag{2}$$

$$\begin{aligned} \text{Var}(X) &= [x_1 - \theta x_1 - (1-\theta)x_2]^2 \cdot \theta \quad + \\ &\quad [x_2 - \theta x_1 - (1-\theta)x_2]^2 \cdot (1-\theta) \\ &= \theta(1-\theta)\Big(x_1 - x_2\Big)^2 \end{aligned} \tag{3}$$

- These formulae will be used later on when calibrating trees.

Arrow-Debreu securities

- Definition of **Arrow-Debreu** (AD) security: pays one dollar in state s at time n, and zero elsewhere.
- The price today is denoted $G(n, s)$. By construction, $G(0, 0) = 1$.
- AD securities as building block of the binomial model:
  - A n-period zero-coupon bond pays one dollar in all states at time n, so it is a portfolio of AD securities

$$P(n) = \sum_{s=0}^{n} G(n, s) \tag{4}$$

  - General fixed-income derivative with cash flows $D(n, s)$ can be priced in the following way:

$$V(0, 0) = \sum_{n=0}^{N} \sum_{s=0}^{n} G(n, s)D(n, s) \tag{5}$$

  - Equation (5) is an alternative to using the backward equation (1).
  - Of course, we must first determine $G(n, s)$ for all (n, s).

Arrow-Debreu prices – 1

- Insights from the geometry of the tree:
  - At time n there are only two nodes leading to $(n+1, s)$, an up-move from $(n, s-1)$ and down-move from (n, s).
  - Note: at the boundaries $s \in \{0, n+1\}$ there is only one node (down-move if $s = 0$, up-move if $s = n+1$).
- Basic idea for finding $G(n+1, s)$:
  - Determine the value of the AD security at time n in state u, denoted $F(n, u)$.
  - Note: $F(n, u) = 0$ if we cannot go to state s in the next period $(n+1)$.
  - Now, we can think of our $(n+1, s)$-AD security as a n-period security (derivative) with payoffs $F(n, u)$, for $0 \leq u \leq n$.
  - Hence, the value of the AD security today is given by:

$$G(n+1, s) = \sum_{u=0}^{n} G(n, u) F(n, u) \tag{6}$$

Arrow-Debreu prices – 2

- The time n prices, $F(n, u)$, for the $(n+1, s)$-AD security satisfy:
  - If $s \leq n$,
  $$F(n, s) = p(n, s)(1 - \theta(n, s)) \tag{7}$$
  since a down-move takes us to $(n+1, s)$.
  - If $s \geq 1$,
  $$F(n, s-1) = p(n, s-1)\theta(n, s-1) \tag{8}$$
  since an up-move takes us to $(n+1, s)$.
  - $F(n, u) = 0$ in all other cases.
- Equation (6) becomes the *forward equation*,

$$\begin{aligned} G(n+1, s) &= G(n, s)p(n, s)(1 - \theta(n, s)) + \\ &\quad G(n, s-1)p(n, s-1)\theta(n, s-1) \end{aligned} \tag{9}$$

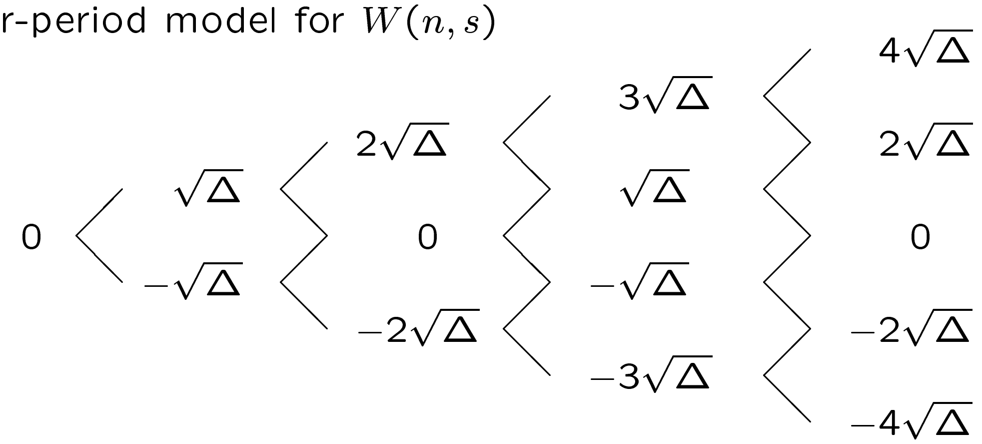
- If $G(n, s) \equiv 0$ for non-existing nodes, this holds for **all** (n, s).

Binomial approximation to the BM – 1

- There are five properties of the Brownian motion:
  - Conditional mean: $E_t(W_{t+\Delta}) = W_t$, the martingale property.
  - Conditional variance: $\text{Var}_t(W_{t+\Delta}) = \Delta$.
  - $W_{t+\Delta} - W_t$ independent of $W_t - W_{t-\Delta}$.
  - Increments in W_t are normally distributed.
  - Sample path of W_t is continuous (W_t does not jump).
- A Binomial model (approximation) with constant time steps, Δ, can match the first three properties.
- In the binomial approximation, we let
  - $\theta(n,s) = 1/2$ for all (n,s).
  - Up move: $W(n+1, s+1) = W(n,s) + \sqrt{\Delta}$
  - Down move: $W(n+1, s) = W(n,s) - \sqrt{\Delta}$

Binomial approximation to the BM – 2

- Four-period model for $W(n,s)$



- Local mean, $\mu(n,s)$, and variance, $\sigma^2(n,s)$, are given by

$$\mu(n,s) = 0.5\left\{\left(W(n,s)+\sqrt{\Delta}\right) + \left(W(n,s)-\sqrt{\Delta}\right)\right\} = W(n,s) \quad (10)$$

$$\sigma^2(n,s) = 0.5(1-0.5)\left\{\left(W(n,s)+\sqrt{\Delta}\right) - \left(W(n,s)-\sqrt{\Delta}\right)\right\}^2$$

$$= 0.25\left\{2\sqrt{\Delta}\right\}^2 = \Delta \quad (11)$$

BDT and Ho-Lee models – 1

- In their binomial version, both models are approximations to

$$dx_t = \left\{ b(t) + \frac{\sigma'(t)}{\sigma(t)} x_t \right\} dt + \sigma(t) dW_t^Q. \qquad (12)$$

- Ho-Lee: $x_t = r_t$ (normal) — BDT: $x_t = \log r_t$ (log-normal).
- Setup of the binomial model:
  - In the binomial model, we define $r(n,s)$ as the **one-period** interest rate (matter of scaling).
  - Risk-neutral probabilities: $\theta(n,s) = \theta = 0.5$ for all (n,s).
  - The time step is constant for all n — we denote it by Δ.
  - Additive relationship between states in the x-space:

$$x(n, s+1) = x(n,s) + h(n) \qquad (13)$$

  - Because of (13), we have $x(n,s) = x(n,0) + sh(n)$, so the only free parameter (for calibration) at time n is $x(n,0)$.

BDT and Ho-Lee models – 2

- The conditional variance of $x(n)$ in state $(n-1,s)$ is given by

$$\begin{aligned} \text{Var}(n-1,s) &= \theta(1-\theta) \left\{ x(n,s+1) - x(n,s) \right\}^2 \\ &= \theta(1-\theta) h^2(n) \end{aligned} \qquad (14)$$

- Note that the variance (14) is independent of the state s.
- From the SDE (12), we have $\text{Var}(n-1,s) = \sigma^2(n\Delta)\Delta$ (exactly), where Δ is the time step of the tree (measured in years).
- Hence, we determine the spacing parameter $h(n)$ as

$$h(n) = \frac{\sigma(n\Delta)\sqrt{\Delta}}{\sqrt{\theta(1-\theta)}} \qquad (15)$$

- Today, we **pre**-specify the volatility function $\sigma(t)$ and **calibrate** $r(n,0)$ (bottom node) to the current yield curve.

Calibration in the BDT model

- We focus on the BDT model where
$$r(n,s) = \delta_n^s r(n,0), \quad \text{with } \log \delta_n = h(n). \tag{16}$$
- We assume **discrete** compounding, so $p(n,s) = 1/\{1 + r(n,s)\}$.
- Assume that we have prices of zero-coupon bonds for **all** $N = T/\Delta$ time periods between $t = 0$ (today) and $t = T$ (last maturity).
- Normally, this requires some interpolation (curve fitting).
- We have N equations, $P(n+1)$, in N unknowns, but the equations can be solve recursively.
- We determine $r(0,0)$ from the first bond price,
$$P(1) = \frac{1}{1 + r(0,0)}. \tag{17}$$
- For $n > 0$ we use the *forward induction* method.

Forward induction in the BDT model – 1

- Assume that we have computed $r(n-1,0)$ in the calibration.
- At time n, the price of the zero maturing at time $n+1$ is
$$p(n,s) = \frac{1}{1 + r(n,s)} = \frac{1}{1 + \delta_n^s r(n,0)} \tag{18}$$
- The current bond price, $P(n+1)$, follows from the AD prices
$$\begin{aligned} P(n+1) &= \sum_{s=0}^{n} G(n,s) p(n,s) \\ &= \sum_{s=0}^{n} G(n,s) \frac{1}{1 + \delta_n^s r(n,0)} \end{aligned} \tag{19}$$
- Using the forward equation (9), we obtain $G(n,s)$ from $G(n-1,u)$ and $p(n-1,u)$ — both of which are known at this stage.
- Equation (19) is solved for $r(n,0)$, and we proceed to $n+1$.

Forward induction in the BDT model – 2

- Equation (19) can only be solved numerically.
- Let $z = r(n, 0)$ and

$$H(z) = P(n+1) - \sum_{s=0}^{n} G(n,s) \frac{1}{1+\delta_n^s z} \tag{20}$$

- We start by some guess for the solution, say z_0, and use the Newton-Raphson iteration scheme

$$z_{k+1} = z_k - \frac{H(z_k)}{H'(z_k)} \tag{21}$$

until $H(z_{k+1}) \approx 0$ (convergence).

- The first-order derivative of $H(z)$ is given by

$$H'(z) = \sum_{s=0}^{n} G(n,s) \frac{\delta_n^s}{(1+\delta_n^s z)^2}. \tag{22}$$

Calibration in the Ho-Lee model (briefly)

- Here, it is more convenient to assume continuous compounding,

$$p(n,s) = \exp[-r(n,s)] = \exp[-r(n,0) - sh(n)] \tag{23}$$

- Assume that we have computed $r(n-1,0)$ in the previous calibration steps, starting from $r(0,0) = -\log P(1)$.
- The bond price $P(n+1)$ can be written as [see eq. (19)].

$$\begin{aligned} P(n+1) &= \sum_{s=0}^{n} G(n,s) p(n,s) \\ &= \exp[-r(n,0)] \sum_{s=0}^{n} G(n,s) \exp[-sh(n)] \end{aligned} \tag{24}$$

- For the HL model, we can solve for $r(n,0)$ in **closed form**

$$r(n,0) = \log \left(\frac{\sum_{s=0}^{n} G(n,s) \exp[-sh(n)]}{P(n+1)} \right) \tag{25}$$