Fixed Income Analysis
Calibration in binomial models

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Binomial model — summary

• Four-period binomial model:

  $r(0,0) \leftarrow r(1,1) \leftarrow r(1,0) \leftarrow r(2,0) \leftarrow r(2,1) \leftarrow r(2,2) \leftarrow r(3,1) \leftarrow r(3,2) \leftarrow r(3,3) \leftarrow r(4,1) \leftarrow r(4,2) \leftarrow r(4,3) \leftarrow r(4,4)$

• The risk-neutral probability of an up-move at time $n$ in state $s$ is denoted $\theta(n,s)$.

• For a given tree (node values and probabilities), valuation is done using the backward equation

$$V(n,s) = D(n,s) + p(n,s) \times \left[ \theta(n,s)V(n+1,s+1) + (1 - \theta(n,s))V(n+1,s) \right]$$  \hspace{1cm} (1)
Binomial distribution – summary

- A random variable, \( X \), follows the binomial distribution if there are only two possible events (outcomes), denoted \( x_1 \) and \( x_2 \).
- We use the binomial distribution for modeling the dynamics of interest rates in a tree, typically a recombining tree.
- Let \( \theta = \Pr(X = x_1) \), which means that \( \Pr(X = x_2) = 1 - \theta \)
- Mean and variance of the random variable \( X \):
  \[
  E(X) = \theta x_1 + (1 - \theta) x_2 \\
  \text{Var}(X) = \left[ x_1 - \theta x_1 - (1 - \theta) x_2 \right]^2 \cdot \theta + \left[ x_2 - \theta x_1 - (1 - \theta) x_2 \right]^2 \cdot (1 - \theta)
  \]
  \[
  = \theta(1 - \theta) \left( x_1 - x_2 \right)^2 
  \]
- These formulae will be used later on when calibrating trees.

Arrow-Debreu securities

- Definition of Arrow-Debreu (AD) security: pays one dollar in state \( s \) at time \( n \), and zero elsewhere.
- The price today is denoted \( G(n, s) \). By construction, \( G(0, 0) = 1 \).
- AD securities as building block of the binomial model:
  - A \( n \)-period zero-coupon bond pays one dollar in all states at time \( n \), so it is a portfolio of AD securities
    \[
    P(n) = \sum_{s=0}^{n} G(n, s) 
    \]
  - General fixed-income derivative with cash flows \( D(n, s) \) can be priced in the following way:
    \[
    V(0, 0) = \sum_{n=0}^{N} \sum_{s=0}^{n} G(n, s) D(n, s)
    \]
    - Equation (5) is an alternative to using the backward equation (1).
    - Of course, we must first determine \( G(n, s) \) for all \( (n, s) \).
Arrow-Debreu prices – 1

• Insights from the geometry of the tree:
  – At time $n$ there are only two nodes leading to $(n + 1, s)$, an up-move from $(n, s - 1)$ and down-move from $(n, s)$.
  – Note: at the boundaries $s \in \{0, n + 1\}$ there is only one node (down-move if $s = 0$, up-move if $s = n + 1$).

• Basic idea for finding $G(n + 1, s)$:
  – Determine the value of the AD security at time $n$ in state $u$, denoted $F(n, u)$.
  – Note: $F(n, u) = 0$ if we cannot go to state $s$ in the next period $(n + 1)$.
  – Now, we can think of our $(n + 1, s)$-AD security as a $n$-period security (derivative) with payoffs $F(n, u)$, for $0 \leq u \leq n$.
  – Hence, the value of the AD security today is given by:

\[
G(n + 1, s) = \sum_{u=0}^{n} G(n, u) F(n, u)
\]  

(6)

Arrow-Debreu prices – 2

• The time $n$ prices, $F(n, u)$, for the $(n + 1, s)$-AD security satisfy:
  – If $s \leq n$,
    \[
    F(n, s) = p(n, s)(1 - \theta(n, s))
    \]
    since a down-move takes us to $(n + 1, s)$.
  – If $s \geq 1$,
    \[
    F(n, s - 1) = p(n, s - 1)\theta(n, s - 1)
    \]
    since an up-move takes us to $(n + 1, s)$.
  – $F(n, u) = 0$ in all other cases.

• Equation (6) becomes the *forward equation*,

\[
G(n + 1, s) = G(n, s)p(n, s)(1 - \theta(n, s)) + \\
G(n, s - 1)p(n, s - 1)\theta(n, s - 1)
\]

(9)

• If $G(n, s) \equiv 0$ for non-existing nodes, this holds for all $(n, s)$. 
Binomial approximation to the BM – 1

- There are five properties of the Brownian motion:
  - Conditional mean: \( E_t(W_{t+\Delta}) = W_t \), the martingale property.
  - Conditional variance: \( \text{Var}_t(W_{t+\Delta}) = \Delta \).
  - \( W_{t+\Delta} - W_t \) independent of \( W_t - W_{t-\Delta} \).
  - Increments in \( W_t \) are normally distributed.
  - Sample path of \( W_t \) is continuous (\( W_t \) does not jump).

- A Binomial model (approximation) with constant time steps, \( \Delta \), can match the first three properties.

- In the binomial approximation, we let
  - \( \theta(n,s) = 1/2 \) for all \( (n,s) \).
  - Up move: \( W(n+1,s+1) = W(n,s) + \sqrt{\Delta} \)
  - Down move: \( W(n+1,s) = W(n,s) - \sqrt{\Delta} \)

Binomial approximation to the BM – 2

- Four-period model for \( W(n,s) \)

\[
\begin{array}{cccccc}
0 & \sqrt{\Delta} & 2\sqrt{\Delta} & 3\sqrt{\Delta} & 4\sqrt{\Delta} \\
-\sqrt{\Delta} & 0 & \sqrt{\Delta} & 2\sqrt{\Delta} \\
-2\sqrt{\Delta} & -\sqrt{\Delta} & 0 & \sqrt{\Delta} \\
-3\sqrt{\Delta} & -2\sqrt{\Delta} & -\sqrt{\Delta} & 0 \\
-4\sqrt{\Delta} & -3\sqrt{\Delta} & -2\sqrt{\Delta} & -\sqrt{\Delta}
\end{array}
\]

- Local mean, \( \mu(n,s) \), and variance, \( \sigma^2(n,s) \), are given by
  \[ \mu(n,s) = 0.5 \left\{ \left( W(n,s) + \sqrt{\Delta} \right) + \left( W(n,s) - \sqrt{\Delta} \right) \right\} = W(n,s) \]  
  \[ \sigma^2(n,s) = 0.5(1-0.5) \left\{ \left( W(n,s) + \sqrt{\Delta} \right) - \left( W(n,s) - \sqrt{\Delta} \right) \right\}^2 
  = 0.25 \left\{ 2\sqrt{\Delta} \right\}^2 = \Delta \]
BDT and Ho-Lee models – 1

• In their binomial version, both models are approximations to
  \[ dx_t = \left\{ b(t) + \frac{\sigma'(t)}{\sigma(t)} x_t \right\} dt + \sigma(t) dW_t^Q. \]  
  (12)

• Ho-Lee: \( x_t = r_t \) (normal) — BDT: \( x_t = \log r_t \) (log-normal).

• Setup of the binomial model:
  – In the binomial model, we define \( r(n,s) \) as the one-period interest rate (matter of scaling).
  – Risk-neutral probabilities: \( \theta(n,s) = \theta = 0.5 \) for all \( (n,s) \).
  – The time step is constant for all \( n \) — we denote it by \( \Delta \).
  – Additive relationship between states in the \( x \)-space:
    \[ x(n,s + 1) = x(n,s) + h(n) \]  
    (13)
  – Because of (13), we have \( x(n,s) = x(n,0) + sh(n) \), so the only free parameter (for calibration) at time \( n \) is \( x(n,0) \).

BDT and Ho-Lee models – 2

• The conditional variance of \( x(n) \) in state \( (n-1,s) \) is given by
  \[ \text{Var}(n-1,s) = \theta(1-\theta) \left\{ x(n,s + 1) - x(n,s) \right\}^2 \]
  \[ = \theta(1-\theta)h^2(n) \]  
  (14)

• Note that the variance (14) is independent of the state \( s \).

• From the SDE (12), we have \( \text{Var}(n-1,s) = \sigma^2(n\Delta)\Delta \) (exactly), where \( \Delta \) is the time step of the tree (measured in years).

• Hence, we determine the spacing parameter \( h(n) \) as
  \[ h(n) = \frac{\sigma(n\Delta)\sqrt{\Delta}}{\sqrt{\theta(1-\theta)}} \]  
  (15)

• Today, we pre-specify the volatility function \( \sigma(t) \) and calibrate \( r(n,0) \) (bottom node) to the current yield curve.
Calibration in the BDT model

- We focus on the BDT model where
  \[ r(n,s) = \delta_n^s r(n,0), \] with \( \log \delta_n = h(n) \). \hfill (16)

- We assume discrete compounding, so \( p(n,s) = 1/(1 + r(n,s)) \).
- Assume that we have prices of zero-coupon bonds for all \( N = T/\Delta \) time periods between \( t = 0 \) (today) and \( t = T \) (last maturity).
- Normally, this requires some interpolation (curve fitting).
- We have \( N \) equations, \( P(n+1) \), in \( N \) unknowns, but the equations can be solve recursively.
- We determine \( r(0,0) \) from the first bond price,
  \[ P(1) = \frac{1}{1 + r(0,0)}. \] \hfill (17)
- For \( n > 0 \) we use the forward induction method.

Forward induction in the BDT model \(-1\)

- Assume that we have computed \( r(n-1,0) \) in the calibration.
- At time \( n \), the price of the zero maturing at time \( n+1 \) is
  \[ p(n,s) = \frac{1}{1 + r(n,s)} = \frac{1}{1 + \delta_n^s r(n,0)} \] \hfill (18)
- The current bond price, \( P(n+1) \), follows from the AD prices
  \[ P(n+1) = \sum_{s=0}^{n} G(n,s) p(n,s) \]
  \[ \quad = \sum_{s=0}^{n} G(n,s) \frac{1}{1 + \delta_n^s r(n,0)} \] \hfill (19)
- Using the forward equation (9), we obtain \( G(n,s) \) from \( G(n-1,u) \) and \( p(n-1,u) \) — both of which are known at this stage.
- Equation (19) is solved for \( r(n,0) \), and we proceed to \( n+1 \).
Forward induction in the BDT model

- Equation (19) can only be solved numerically.
- Let \( z = r(n, 0) \) and
  \[
  H(z) = P(n + 1) - \sum_{s=0}^{n} G(n, s) \frac{1}{1 + \delta^{s}_{n}z} 
  \]
  \hspace{1cm} (20)
- We start by some guess for the solution, say \( z_0 \), and use the Newton-Raphson iteration scheme
  \[
  z_{k+1} = z_k - \frac{H(z_k)}{H'(z_k)} 
  \]
  \hspace{1cm} (21)
  until \( H(z_{k+1}) \approx 0 \) (convergence).
- The first-order derivative of \( H(z) \) is given by
  \[
  H'(z) = \sum_{s=0}^{n} G(n, s) \frac{\delta^{s}_{n}}{(1 + \delta^{s}_{n}z)^2}. 
  \]
  \hspace{1cm} (22)

Calibration in the Ho-Lee model (briefly)

- Here, it is more convenient to assume continuous compounding,
  \[
  p(n, s) = \exp[-r(n, s)] = \exp[-r(n, 0) - sh(n)] 
  \]
  \hspace{1cm} (23)
- Assume that we have computed \( r(n - 1, 0) \) in the previous calibration steps, starting from \( r(0, 0) = -\log P(1) \).
- The bond price \( P(n + 1) \) can be written as [see eq. (19)].
  \[
  P(n + 1) = \sum_{s=0}^{n} G(n, s)p(n, s) 
  = \exp[-r(n, 0)] \sum_{s=0}^{n} G(n, s) \exp[-sh(n)] 
  \]
  \hspace{1cm} (24)
- For the HL model, we can solve for \( r(n, 0) \) in \textbf{closed form}
  \[
  r(n, 0) = \log \left( \frac{\sum_{s=0}^{n} G(n, s) \exp[-sh(n)]}{P(n + 1)} \right) 
  \]
  \hspace{1cm} (25)