Summary from last week

- We price fixed-income derivatives by constructing a replicating portfolio, typically from bonds with different maturities.
- The replicating portfolio has the same payoff as the derivative security in all states in the future.
- The **arbitrage-free price** of the derivative equals the price of the replicating portfolio.
- If a security can be priced by arbitrage, there exists a risk-neutral distribution, such that the price of the security equals the expected, discounted payoff (under the Q-distribution).
- That is, we can “pretend” that investors are risk-neutral — once the up-down probabilities are modified.
- A more general version of this result is known as the **Equivalent Martingale Theory**, formulated by Harrison and Kreps (1979).
Multi-period binomial models – 1

- The two-date examples are sufficient to explain all aspects of the theory (and intuition) of risk-neutral valuation.
- In practice, multi-period models are needed
  - Some derivative securities make payments at more than one day, e.g., interest-rate caps. All distinct payment dates should be represented in the binomial model (tree).
  - The real world does not exactly evolve according to a simple binomial model. Instead, the binomial model is an approximation, usually to a continuous distribution such as the normal distribution.
  - Reducing the step size (and thereby increasing the number of periods) results in a better approximation, see Figures 7.1–7.4 in Tuckman (1995).

- To keep the computational work manageable, we must use a re-combining tree (lattice), that is an 'up-down' move takes us to the same node as a 'down-up' move.

Multi-period binomial models – 2

- To account for the nodes in a lattice, we use the following notation $(n, s)$, where $n = 0, 1, \ldots, N$ is the date, and $s = 0, \ldots, n$ denotes the state, numbered from below.
- Four-period example — short-rate tree:

```
       r(4,4)
      /  \  \\
   r(3,3)  r(4,3)
  /  \    /  \  \\
 r(2,2) r(3,2) r(4,2)
 /  \    /  \    /  \ \\
r(2,1) r(3,1) r(4,1)
 /  \    /  \    /  \ \\
r(2,0) r(3,0) r(4,0)
 r(1,1)
 /  \\
r(1,0)
 r(0,0)
```

- Constructing the tree — such that the prices of all $N$ zeros are matched exactly — is an exercise known as calibration.
Backward equation − 1

• Assume we have a binomial tree with risk-neutral probabilities.
• Define the following notation:

\[(n, s)\] indicates that we are at time \(n\), in state \(s\).

\(r(n, s)\) is the short rate at time \(n\) in state \(s\).

\(p(n, s)\) is the discount factor for one period (in the tree).

If \(r(n, s)\) is quoted as the short rate for \(m\) periods with simple interest, we have 
\[p(n, s) = 1/(1 + r(n, s)/m).\] In chapters 5–7, \(m = 2\).

\(\theta(n, s)\) is the risk-neutral probability of an up-move, that is to state \(s + 1\) at
time \(n + 1\), from the current state \(s\) at time \(n\).

\(D(n, s)\) is the payment in state \(s\) at time \(n\). If the payment is made in the
next period, it must be discounted using \(p(n, s)\).

\(V(n, s)\) The value (price) of the security in state \(s\) at time \(n\).

• The basic idea of the backward equation is calculating \(V(0, 0)\) —
the price of the security today.

Backward equation − 2

• Suppose that the tree covers \(N\) dates, that is \(n = 0, 1, \ldots, N\).
• At the final maturity \(n = N\), the value of the claim is \(D(N, s)\) if
we are in state \(s\), where \(s \in \{0, 1, \ldots, N\}\).
• If we are in state \(s\) at time \(N − 1\), we can only move to states
\(s\) (down) and \(s + 1\) (up). Therefore, the (present) value of the
payments received in the next period is

\[p(N − 1, s) [\theta(N − 1, s)D(N, s + 1) + (1 − \theta(N − 1, s))D(N, s)]\]

• If we add the additional payments received in state \(s\), we obtain
the total value of the security in \((N − 1, s)\),

\[V(N − 1, s) = D(N − 1, s) + p(N − 1, s) \times\]

\[\theta(N − 1, s) V(N, s + 1) + (1 − \theta(N − 1, s))V(N, s)\] (1)

since \(V(N, s) = D(N, s)\).
Equation (1) is an example of the **backward equation**.

In general, for any \((n,s)\), the no-arbitrage condition implies that

\[
V(n,s) = D(n,s) + p(n,s) \times \\
[\theta(n,s)V(n + 1, s + 1) + (1 - \theta(n,s))V(n + 1, s)]
\]  

(2)

The **first** step in pricing a fixed-income derivative is specifying the payments in all possible states, that is \(D(n,s)\) for all \((n,s)\).

Many fixed-income securities (derivatives) only make payments at maturity (expiration). This means that \(D(n,s) = 0\) for \(n < N\). Examples: European bond options and zero-coupon bonds.

**Second**, the backward equation is used **recursively**. We start from the last date \(n = N\) and work backwards using equation (2) until we get to \(V(0,0)\).

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**Numerical example – 1**

The examples in chapter 7 use the following tree (below, the short rate is for one period):

```
1.995 0.661 0.339
1.953 2.450
2.250 0.047 0.953
2.150
```

Note that there is an error in chapter 7 of Tuckman (1995).

The first up probability (0.661) was calculated last week, and the second calculation (0.953) will be explained later.
Numerical example – 2

- The first example is a 1.5Y zero-coupon bonds, where \( D(3,s) = 1000 \) for all \( s \in \{0,1,2,3\} \), and \( D(n,s) = 0 \) for \( n < 3 \).
- Price tree for the zero:

\[
\begin{array}{c}
937.7641 \\
954.7405 \\
959.8469 \\
978.9525 \\
980.8730 \\
1000.00
\end{array}
\]

- Sample calculations:
  - (2,1) \( 978.9525 = 1000.0 / 1.0215 \)
  - (1,0) \( 959.8469 = (0.953 \times 978.9525 + 0.047 \times 980.8730) / 1.02 \)
  - (0,0) \( 937.7641 = (0.661 \times 954.7405 + 0.339 \times 959.8469) / 1.01995 \)

Numerical example – 3

- The second example is a security with the following payments at date 2: \( D(2,0) = -10 \), \( D(2,1) = 100 \) and \( D(2,2) = 500 \).
- There are no payments at time 1 or 0, so \( D(n,s) = 0 \) for \( n < 2 \).
- Price tree for the security:

\[
\begin{array}{c}
335.6695 \\
470.3970 \\
92.9115 \\
100.00 \\
500.00
\end{array}
\]

- Calculations in the tree:
  - (1,0) \( 92.9115 = (0.9525 \times 100.00 - 0.0475 \times 10.00) / 1.02 \)
  - (1,1) \( 470.3970 = (0.9525 \times 500.00 + 0.0475 \times 100.00) / 1.0225 \)
  - (0,0) \( 335.6695 = (0.661 \times 470.3970 + 0.339 \times 92.9115) / 1.01995 \)
Introduction to calibration – 1

- Typical situation: we want to price fixed-income derivatives relative to the current yield curve.
- That is, the zero-coupon bond prices are taken as given — therefore the binomial tree should price the zeros correctly.
- Adjusting the tree such that the current (initial) yield curve is matched exactly is known as calibration.
- Calibration means determining the short rate at the nodes of the tree, \( r(n, s) \), and the risk-neutral probabilities, \( \theta(n, s) \).
- Data input: if the time step is 3 months, we need zeros in maturity intervals of 3 months up to the final horizon, and so on.
- In practice, a certain amount of interpolation between missing maturities is needed.

Introduction to calibration – 2

- In the simplest case, we have one degree of freedom per time period, namely the zero price for that maturity.
- There are two possibilities:
  1. At time \( n \), we fix \( \theta(n, s) = 0.5 \), and adjust the short rate \( r(n, s) \) in the \( n + 1 \) nodes. Note that the \( n + 1 \) values of \( r(n, s) \) are not uniquely determined since there is only one degree of freedom.
  2. Specify \( r(n, s) \) freely, and adjust the risk-neutral probabilities \( \theta(n, s) \). Since there is only one degree of freedom, we need to assume (for example) that \( \theta(n, s) \) is the same for all \( s \).
- The second method is used in chapters 5–7, and we focus on this method today — because it is simpler (at first).
- However, it is widely recognized that fixing \( \theta(n, s) = 0.5 \) and adjusting \( r(n, s) \) is preferable — due to faster convergence.
Introduction to calibration — 3

- How is the risk-neutral (up) probability for \( n = 1, \theta(1, s) = 0.9525 \) calculated in the earlier example?

- The node values \( r(n, s) \) are specified more or less arbitrarily, and we calculated \( \theta(0, 0) = 0.661 \) last week.

- Tree with unknown \( q = \theta(1, s) \)

![Tree diagram]

Introduction to calibration — 4

- Price tree for the 1.5 Y zero, with a yield of 4.33%

![Price tree]

- We know that

\[
P_{1.5}(1, 0) = \{q \cdot 978.9525 + (1 - q) \cdot 980.8730\} / 1.02
\]

\[
P_{1.5}(1, 1) = \{q \cdot 976.0859 + (1 - q) \cdot 978.9525\} / 1.0225
\]
Introduction to calibration — 5

- Moreover we have

\[
937.7641 = \frac{1}{1.01995} \{0.661P_{1.5}(1, 1) + 0.339P_{1.5}(1, 0)\} \tag{3}
\]

- If we substitute the expressions for \( P_{1.5}(1, i) \), \( i = 1, 2 \), into (3), we get one equation in one unknown, \( q \).

- The solution for \( q \) is

\[
q = \frac{0.661p(1, 1) [P_{1.5}(2, 2) - P_{1.5}(2, 1)] + 0.339p(1, 0) [P_{1.5}(2, 1) - P_{1.5}(2, 0)]}{937.7641 \times 1.01995 - (0.661p(1, 1)P_{1.5}(2, 1) + 0.339p(1, 0)P_{1.5}(2, 0))}
\]

- Note that \( P_{1.5}(2, 0) = 980.8730 \), \( p(1, 0) = 1/1.02 \), and so forth.

- The solution is \( q = 0.9525 \) — there is an error in the book.