Fixed Income Analysis

Risk-Neutral Pricing and Binomial Models

Multi-Period Models and the Backward Equation
Introduction to Calibration of Binomial Models

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Summary from last week

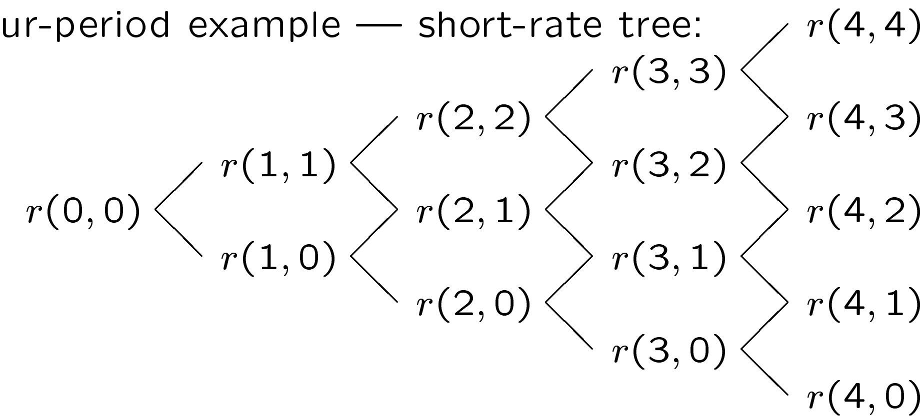
- We price fixed-income derivatives by constructing a replicating portfolio, typically from bonds with different maturities.
- The replicating portfolio has the same payoff as the derivative security in **all** states in the future.
- The **arbitrage-free price** of the derivative equals the price of the replicating portfolio.
- If a security can be priced by arbitrage, there exists a risk-neutral distribution, such that the price of the security equals the expected, discounted payoff (under the Q-distribution).
- That is, we can "pretend" that investors are risk-neutral — once the up-down probabilities are modified.
- A more general version of this result is known as the **Equivalent Martingale Theory**, formulated by Harrison and Kreps (1979).

Multi-period binomial models – 1

- The two-date examples are sufficient to explain **all** aspects of the theory (and intuition) of risk-neutral valuation.
- In practice, multi-period models are needed
  - Some derivative securities make payments at more than one day, e.g., interest-rate caps. All distinct payment dates should be represented in the binomial model (tree).
  - The real world does not exactly evolve according to a simple binomial model. Instead, the binomial model is an approximation, usually to a continuous distribution such as the normal distribution.
  - Reducing the step size (and thereby increasing the number of periods) results in a better approximation, see Figures 7.1–7.4 in Tuckman (1995).
- To keep the computational work manageable, we must use a **re-combining** tree (lattice), that is an 'up-down' move takes us to the same node as a 'down-up' move.

Multi-period binomial models – 2

- To account for the nodes in a lattice, we use the following notation (n, s), where $n = 0, 1, \ldots, N$ is the date, and $s = 0, \ldots, n$ denotes the state, numbered from below.
- Four-period example — short-rate tree:



- Constructing the tree — such that the prices of all N zeros are matched exactly — is an exercise known as **calibration**.

Backward equation – 1

- Assume we have a binomial tree with risk-neutral probabilities.
- Define the following notation:

(n, s)	indicates that we are at time n, in state s.
$r(n, s)$	is the short rate at time n in state s.
$p(n, s)$	is the discount factor for one period (in the tree). If $r(n, s)$ is quoted as the short rate for m periods with simple interest, we have $p(n, s) = 1/\{1 + r(n, s)/m\}$. In chapters 5–7, $m = 2$.
$\theta(n, s)$	is the risk-neutral probability of an up-move, that is to state $s + 1$ at time $n + 1$, from the current state s at time n.
$D(n, s)$	is the payment in state s at time n. If the payment is made in the next period, it must be discounted using $p(n, s)$.
$V(n, s)$	The value (price) of the security in state s at time n.

- The basic idea of the backward equation is calculating $V(0, 0)$ — the price of the security today.

Backward equation – 2

- Suppose that the tree covers N dates, that is $n = 0, 1, \ldots, N$.
- At the final maturity $n = N$, the value of the claim is $D(N, s)$ if we are in state s, where $s \in \{0, 1, \ldots, N\}$.
- If we are in state s at time $N - 1$, we can only move to states s (down) and $s + 1$ (up). Therefore, the (present) value of the payments received in the next period is

$$p(N-1, s)\,[\theta(N-1, s)D(N, s+1) + (1 - \theta(N-1, s))D(N, s)]$$

- If we add the additional payments received in state s, we obtain the total value of the security in $(N - 1, s)$,

$$\begin{aligned} V(N-1, s) \;=\; & D(N-1, s) \;+\; p(N-1, s)\times \\ & [\theta(N-1, s)V(N, s+1) + (1 - \theta(N-1, s))V(N, s)] \end{aligned} \quad (1)$$

since $V(N, s) = D(N, s)$.

Backward equation – 3

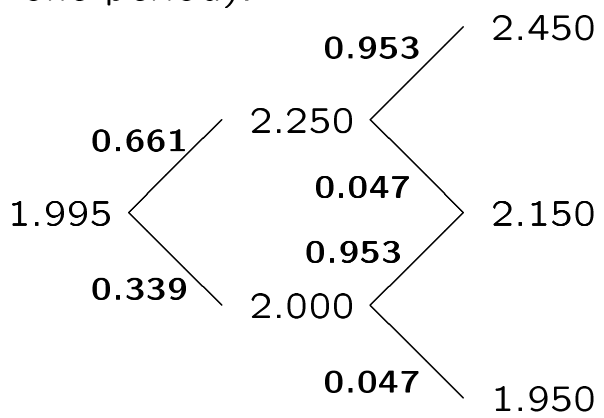
- Equation (1) is an example of the **backward equation**.
- In general, for any (n,s), the no-arbitrage condition implies that

$$\begin{aligned} V(n,s) &= D(n,s) + p(n,s)\times \\ &\quad [\theta(n,s)V(n+1,s+1) + (1-\theta(n,s))V(n+1,s)] \end{aligned} \qquad (2)$$

- The **first** step in pricing a fixed-income derivative is specifying the payments in all possible states, that is $D(n,s)$ for all (n,s).
- Many fixed-income securities (derivatives) only make payments at maturity (expiration). This means that $D(n,s) = 0$ for $n < N$. Examples: European bond options and zero-coupon bonds.
- **Second**, the backward equation is used **recursively**. We start from the last date $n = N$ and work backwards using equation (2) until we get to $V(0,0)$.

Numerical example – 1

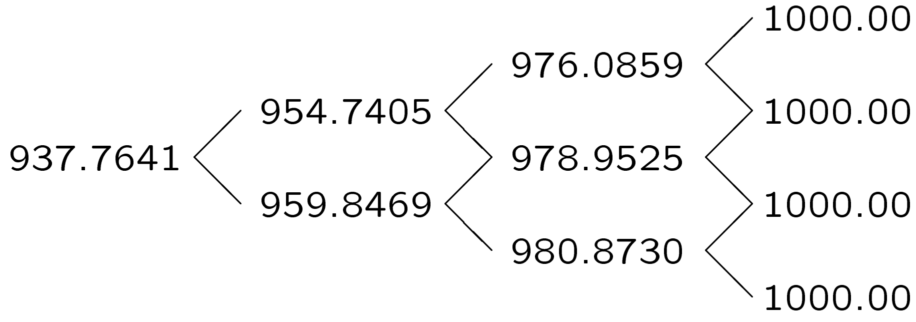
- The examples in chapter 7 use the following tree (below, the short rate is for one period):



- Note that there is an error in chapter 7 of Tuckman (1995).
- The first up probability (0.661) was calculated last week, and the second calculation (0.953) will be explained later.

Numerical example – 2

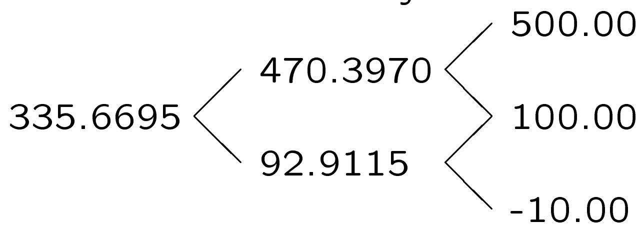
- The first example is a 1.5Y zero-coupon bonds, where $D(3, s) = 1000$ for all $s \in \{0, 1, 2, 3\}$, and $D(n, s) = 0$ for $n < 3$.
- Price tree for the zero:



- Sample calculations:
  - (2,1) 978.9525 = 1000.0 / 1.0215
  - (1,0) 959.8469 = (0.953 × 978.9525 + 0.047 × 980.8730) / 1.02
  - (0,0) 937.7641 = (0.661 × 954.7405 + 0.339 × 959.8469) / 1.01995

Numerical example – 3

- The second example is a security with the following payments at date 2: $D(2, 0) = -10$, $D(2, 1) = 100$ and $D(2, 2) = 500$.
- There are no payments at time 1 or 0, so $D(n, s) = 0$ for $n < 2$.
- Price tree for the security:



- Calculations in the tree:
  - (1,0) 92.9115 = (0.9525 × 100.00 – 0.0475 × 10.00) / 1.02
  - (1,1) 470.3970 = (0.9525 × 500.00 + 0.0475 × 100.00) / 1.0225
  - (0,0) 335.6695 = (0.661 × 470.3970 + 0.339 × 92.9115) / 1.01995

Introduction to calibration – 1

- Typical situation: we want to price fixed-income derivatives relative to the current yield curve.
- That is, the zero-coupon bond prices are taken as given — therefore the binomial tree should price the zeros correctly.
- Adjusting the tree such that the current (initial) yield curve is matched exactly is known as **calibration**.
- Calibration means determining the short rate at the nodes of the tree, $r(n,s)$, and the risk-neutral probabilities, $\theta(n,s)$.
- Data input: if the time step is 3 months, we need zeros in maturity intervals of 3 months up to the final horizon, and so on.
- In practice, a certain amount of interpolation between missing maturities is needed.

Introduction to calibration – 2

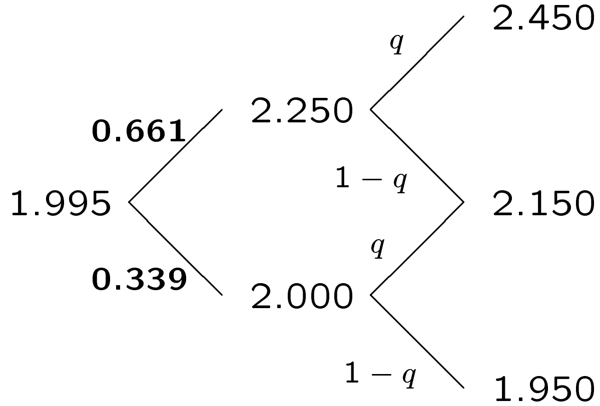
- In the simplest case, we have one degree of freedom per time period, namely the zero price for that maturity.
- There are two possibilities:

  1. At time n, we fix $\theta(n,s) = 0.5$, and adjust the short rate $r(n,s)$ in the $n+1$ nodes. Note that the $n+1$ values of $r(n,s)$ are not uniquely determined since there is only one degree of freedom.
  2. Specify $r(n,s)$ freely, and adjust the risk-neutral probabilities $\theta(n,s)$. Since there is only one degree of freedom, we need to assume (for example) that $\theta(n,s)$ is the same for all s,

- The second method is used in chapters 5–7, and we focus on this method today — because it is simpler (at first).
- However, it is widely recognized that fixing $\theta(n,s) = 0.5$ and adjusting $r(n,s)$ is **preferable** — due to faster convergence.

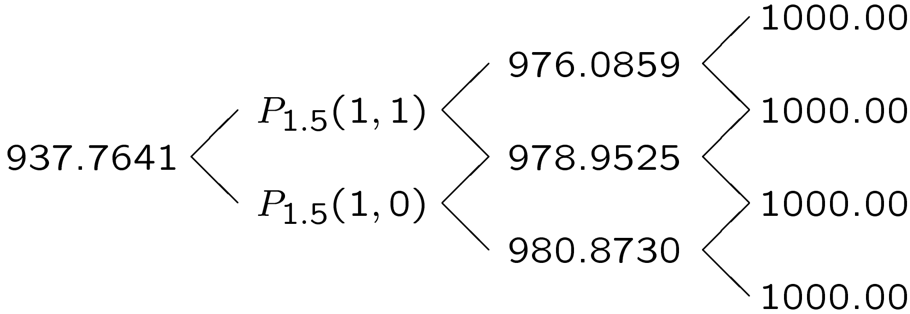
Introduction to calibration — 3

- How is the risk-neutral (up) probability for $n = 1$, $\theta(1, s) = 0.9525$ calculated in the earlier example?
- The node values $r(n, s)$ are specified more or less arbitrarily, and we calculated $\theta(0, 0) = 0.661$ last week.
- Tree with unknown $q = \theta(1, s)$



Introduction to calibration — 4

- Price tree for the 1.5 Y zero, with a yield of 4.33%



- We know that

$$P_{1.5}(1,0) = \{q\,978.9525 + (1-q)\,980.8730\}\,/\,1.02$$
$$P_{1.5}(1,1) = \{q\,976.0859 + (1-q)\,978.9525\}\,/\,1.0225$$

Introduction to calibration — 5

- Moreover we have

$$937.7641 = \frac{1}{1.01995}\{0.661P_{1.5}(1,1) + 0.339P_{1.5}(1,0)\} \qquad (3)$$

- If we substitute the expressions for $P_{1.5}(1,i)$, $i = 1,2$, into (3), we get one equation in one unknown, q.
- The solution for q is

$$q = \frac{0.661p(1,1)\,[P_{1.5}(2,2) - P_{1.5}(2,1)] + 0.339p(1,0)\,[P_{1.5}(2,1) - P_{1.5}(2,0)]}{937.7641 \times 1.01995 - \{0.661p(1,1)P_{1.5}(2,1) + 0.339p(1,0)P_{1.5}(2,0)\}}$$

- Note that $P_{1.5}(2,0) = 980.8730$, $p(1,0) = 1/1.02$, and so forth.
- The solution is $q = 0.9525$ — there is an error in the book.