Pricing term-structure derivatives

- Contrary to stock options, the distinction between “real” assets and derivatives is not clearcut (and not important, either).
- Generally, we define a term-structure derivative (contingent claim) as an asset with uncertain cash flows, which are linked to the level of interest rates (or bond prices).
- Hence, a bullet is not a derivative (cash flows are fixed), but a floating-rate note is, as cash flows are linked to the short rate.
- In the Black-Scholes model — and its binomial counterpart — we assume constant interest rates and constant volatility.
- This makes the model useless for most (all) term-structure derivatives. The volatility of the bond return is gradually reduced over time, and just before maturity the value is equal to par.
Binomial models – an introduction

- A random variable, $X$, follows the binomial distribution if there are only two possible events (outcomes), denoted $x_1$ and $x_2$.
- We use the binomial distribution for modeling the evolution (over time) of stock prices or interest rates in a tree, typically a recombining tree (sometimes called a lattice).
- Let $p = \Pr(X = x_1)$, which means that $\Pr(X = x_2) = 1 - p$
- Mean and variance of the random variable $X$:
  \[ E(X) = px_1 + (1 - p)x_2 \]  
  \[ \text{Var}(X) = [x_1 - px_1 - (1 - p)x_2]^2 \cdot p + [x_2 - px_1 - (1 - p)x_2]^2 \cdot (1 - p) \]
  \[ = p(1 - p)(x_1 - x_2)^2 \]  
- These formulae will be used later on when calibrating trees.

Stock options – 1

- The current stock price is $S$. In the next period the price can increase to $Su$ or drop to $Sd$, with (true) probabilities $p$ and $1 - p$, respectively. The one-period risk-free interest rate is $r$.
- **Question**: How do we price a one-period call option on the stock price?
- Evolution of the prices of the stock and the call option:

\[ S \xrightarrow{p} Su \quad \xleftarrow{1-p} Sd \]

\[ C =? \xrightarrow{p} Cu = \max(Su - K, 0) \quad \xleftarrow{1-p} Cd = \max(Sd - K, 0) \]
Stock options – 2

- There are two assets (stock, bond) and two future states (up, down). This means that we can construct a replicating portfolio.
- Let $w_1$ denote the number of bonds, and $w_2$ the number of stocks. The weights $w_1$ and $w_2$ must satisfy
  \[ C_u = w_1 + w_2 Su \]  \[ C_d = w_1 + w_2 Sd \]  
  (3)  
  (4)
- The solution to equations (3) and (4) is
  \[ w_1 = \frac{uC_d - dC_u}{u - d} \] \text{and} \[ w_2 = \frac{C_u - C_d}{S(u - d)} \]  
  (5)
- Price of the call option
  \[ C = \frac{w_1}{1 + r} + w_2 S = \frac{uC_d - dC_u}{u - d} \frac{1}{1 + r} + \frac{C_u - C_d}{u - d} \]  
  (6)

Stock options – 3

- Next, we collect terms with $C_u$ and $C_d$ in (6),
  \[ C = \frac{1}{1 + r} \left\{ C_u \frac{1 + r - d}{u - d} + C_d \frac{u - (1 + r)}{u - d} \right\} \]
  \[ = \frac{1}{1 + r} \left\{ qC_u + (1 - q)C_d \right\}, \]  
  (7)
  where
  \[ q = \frac{1 + r - d}{u - d}, \quad \text{with} \quad 0 < q < 1 \quad \text{(why?)} \]  
  (8)
- This is the expected (discounted) payoff if the probability of an up-move is $q$ — called the risk-neutral probability.
- The true probability, $p$, of an up-move does not matter.
- The risk-neutral probability, $q$, does not depend on $C_u$ and $C_d$. 

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Stock options – 4

• Expected stock return under the risk-neutral distribution

\[
\frac{[qS_u + (1-q)S_d] - S}{S} = qu + (1-q)d - 1 = r
\]  

(9)

• In the risk-neutral world, all assets have the same expected return, namely the risk-free rate \( r \). No need to distinguish between different asset types — only their payoffs in different states.

• We are **not** assuming that investors are risk-neutral, but the risk adjustment done by modifying the up probability from \( p \) to \( q \).

• The payoffs in the states are not modified, only the probabilities.

• Note that we can determine \( q \) from the relationship:

\[
S = \frac{1}{1+r} \{qS_u + (1-q)S_d\} \quad \Rightarrow \quad q = \frac{1+r-d}{u-d}
\]  

(10)

Using binomial models for term-structure derivatives — how?

• The same basic idea (risk-neutral valuation) can still be used, but some modifications are needed.

• Suppose we need to price a one-year option on a five-year zero.

• Should we set up a binomial tree for the 5Y zero?

• **No** — in one year, the bond is no longer a 5Y bond, and the option prices also depend on short-term rates for discounting the option payoff.

• Instead, we need to describe the evolution of the entire yield curve.

• This is done by modeling the dynamics of the short rate (maturity equal to time interval of the binomial tree).

• As a consequence of this assumption, the entire yield curve is a function of the short rate (will be relaxed later).
Interest-rate binomial models – 1

- We focus on the example from chapters 5–6 in Tuckman (1995).
- The time interval is 6 months, so the short rate is the 6M rate.
- Two-date example: current short rate is 3.99%, which can change to either 4.00% or 4.50%, with equal probabilities. Note these are objective or true probabilities.
- Evolution of the short rate and the price of a 6 month zero, which at time 0 (today) is 980.4402 = 1000/(1 + 0.0399/2).

\[
\begin{align*}
3.99\% & \quad \begin{cases} 0.5 & \quad 4.00\% \\ 0.5 & \quad 4.50\% \end{cases} \\
980.4402$ & \quad \begin{cases} 0.5 & \quad 1000.0$ \\ 0.5 & \quad 1000.0$ \end{cases}
\end{align*}
\]

Note that the current price differs from the price under risk-neutrality, which is \( \frac{1}{2}(977.9951 + 980.3922)/1.01995 = 960.04 \)

Interest-rate binomial models – 2

- The one-year rate is 4.16%, that is the 1Y zero is priced at 959.6628 = 1000/(1 + 0.0208)² per 1000$ face value.
- In 6 months the bond is a 6M zero, and the price is either 1000/(1 + 0.0225) = 977.9951 or 1000/1.02 = 980.3922.
- Price tree for the one-year zero

\[
\begin{align*}
959.6628 & \quad \begin{cases} 0.5 & \quad 980.3922 \\ 0.5 & \quad 977.9951 \end{cases} \\
1000.0 & \quad \begin{cases} 0.5 & \quad 1000.0 \\ 0.5 & \quad 1000.0 \end{cases}
\end{align*}
\]

- Note that the current price differs from the price under risk-neutrality, which is \( \frac{1}{2}(977.9951 + 980.3922)/1.01995 = 960.04 \)
Interest-rate binomial models – 3

- We want to price a 6-month call option on the bond with an exercise price of 978.50.
- Payoff from the call option in 6 months

\[
\begin{align*}
0.0000 &= \max(977.9951 - 978.50, 0) \\
? &= 1.8922 = \max(980.3922 - 978.50, 0)
\end{align*}
\]

- There are two equivalent ways to price the option (by arbitrage)
  1. Construct a portfolio of the 1Y and 6M bonds which replicates the payoff from the option in both states (up, down).
  2. Determine the risk-neutral probability \( q \) of an up move, and discount the expected payoff under the risk-neutral distribution.

Interest-rate binomial models – 4

Replicating portfolio

- Replicating portfolio \( F_5 \) and \( F_1 \) must satisfy

\[
\begin{align*}
F_5 + 0.9779951F_1 &= 0 \\
F_5 + 0.9803922F_1 &= 1.8922
\end{align*}
\]

- The solution is

\[
F_1 = \frac{1.8922}{0.9803922 - 0.9779951} = 789.3705
\]

\[
F_5 = -0.9779951F_1 = -772.0005
\]

- The price of the option is

\[
C = \frac{F_5}{1 + 0.0399/2} + \frac{F_1}{(1 + 0.0416/2)^2} = 0.6292 \quad (11)
\]
Interest-rate binomial models — 5
Risk-neutral valuation

- We determine \( q \), such that the 1Y bond is priced correctly by risk-neutral valuation.
- In 6 months the value of the bond is either 977.9951 (up) or 980.3922 (down), and the current price is 959.6628.
- Hence, the risk-neutral probability should satisfy

\[
q \times 977.9951 + (1 - q) \times 980.3922 = 959.6628.
\]  

(12)

- Solving for \( q \) yields \( q = 0.66085 \), and \( 1 - q = 0.33915 \).
- The price of the option is

\[
C = \left[ 0.66085 \times 0.0 + 0.33915 \times 1.8922 \right] = 0.6292
\]  

(13)

Pricing derivatives — Summary

- The call option can be priced by arbitrage, which means that the price is independent of investor preferences.
- One way to see this is that the objective probabilities (0.5) do not enter the option-pricing formula.
- This statement is conditional: given the prices of the 6M and 1Y zeros, the price of the call option does not depend further on investor preferences.
- Unconditionally, the price of the option does depend on investor preferences, but only through the two bond prices.
- Basic idea of risk-neutral valuation: adjust the probabilities of the tree, and assume that the investor is risk-neutral, i.e., all prices are computed as the expected discounted payoff (under \( Q \)).
- The risk-neutral probabilities do not depend on the payoffs.
Multi-period binomial models – 1

- The above two-date example is sufficient to explain all aspects of the theory (and intuition) of risk-neutral valuation.
- In practice, multi-period models are needed
  - Some derivative securities have payoffs at more than one day, e.g., interest-rate caps. All distinct payoff dates should be represented in the binomial model (tree).
  - The real world does not exactly evolve according to a simple binomial model. Instead, the binomial model is an approximation, usually to a continuous distribution such as the normal distribution.
  - Reducing the step size (and thereby increasing the number of periods) results in a better approximation, see Figures 7.1–7.4 in Tuckman (1995).
- To keep the computational work manageable, we must use a recombining tree (lattice), that is an 'up-down' move takes us to the same node as a 'down-up' move.

Multi-period binomial models – 2

- To account for the nodes in a lattice, we use the following notation \((n, s)\), where \(n = 0, 1, \ldots, N\) is the date, and \(s = 0, \ldots, n\) denotes the state, numbered from below.
- Four-period example:

  \[
  \begin{array}{c}
  (0,0) \\
  (1,0) \\
  (1,1) \\
  (2,0) \\
  (2,1) \\
  (2,2) \\
  (3,0) \\
  (3,1) \\
  (3,2) \\
  (4,1) \\
  (4,2) \\
  (4,3) \\
  (4,4)
  \end{array}
  \]

- Constructing the tree — such that the prices of all \(N\) zeros are matched exactly — is an exercise known as calibration.