Fixed Income Analysis

Introduction to Risk-Neutral Pricing and Binomial Models

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Pricing term-structure derivatives

- Contrary to stock options, the distinction between "real" assets and derivatives is not clearcut (and not important, either).
- Generally, we define a term-structure derivative (contingent claim) as an asset with uncertain cash flows, which are linked to the level of interest rates (or bond prices).
- Hence, a bullet is not a derivative (cash flows are fixed), but a floating-rate note is, as cash flows are linked to the short rate.
- In the Black-Scholes model — and its **binomial** counterpart — we assume constant interest rates and constant volatility.
- This makes the model useless for most (all) term-structure derivatives. The volatility of the bond return is gradually reduced over time, and just before maturity the value is equal to par.

Binomial models – an introduction

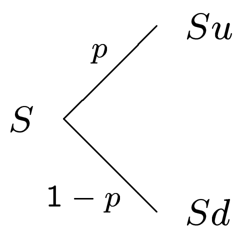
- A random variable, X, follows the binomial distribution if there are only two possible events (outcomes), denoted x_1 and x_2.
- We use the binomial distribution for modeling the evolution (over time) of stock prices or interest rates in a tree, typically a recombining tree (sometimes called a lattice).
- Let $p = \Pr(X = x_1)$, which means that $\Pr(X = x_2) = 1 - p$
- Mean and variance of the random variable X:

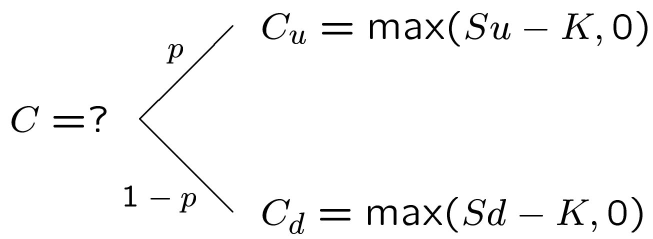
$$
\begin{aligned}
E(X) &= px_1 + (1-p)x_2 && (1)\\
\mathrm{Var}(X) &= [x_1 - px_1 - (1-p)x_2]^2 \cdot p \quad + \\
&\quad\; [x_2 - px_1 - (1-p)x_2]^2 \cdot (1-p) \\
&= p(1-p)\,(x_1 - x_2)^2 && (2)
\end{aligned}
$$

- These formulae will be used later on when calibrating trees.

Stock options – 1

- The current stock price is S. In the next period the price can increase to Su or drop to Sd, with (true) probabilities p and $1-p$, respectively. The one-period risk-free interest rate is r.
- **Question:** How do we price a one-period call option on the stock price?
- Evolution of the prices of the stock and the call option:





Stock options – 2

- There are two assets (stock, bond) and two future states (up, down). This means that we can construct a **replicating** portfolio.
- Let w_1 denote the number of bonds, and w_2 the number of stocks. The weights w_1 and w_2 must satisfy

$$C_u = w_1 + w_2 S u \quad (3)$$
$$C_d = w_1 + w_2 S d \quad (4)$$

- The solution to equations (3) and (4) is

$$w_1 = \frac{uC_d - dC_u}{u-d} \quad \text{and} \quad w_2 = \frac{C_u - C_d}{S(u-d)} \quad (5)$$

- Price of the call option

$$C = \frac{w_1}{1+r} + w_2 S = \frac{uC_d - dC_u}{u-d}\frac{1}{1+r} + \frac{C_u - C_d}{u-d} \quad (6)$$

Stock options – 3

- Next, we collect terms with C_u and C_d in (6),

$$\begin{aligned} C &= \frac{1}{1+r}\left\{C_u\frac{1+r-d}{u-d} + C_d\frac{u-(1+r)}{u-d}\right\} \\ &= \frac{1}{1+r}\left\{qC_u + (1-q)C_d\right\}, \end{aligned} \quad (7)$$

where

$$q = \frac{1+r-d}{u-d}, \quad \text{with } 0 < q < 1 \quad \text{(why?)} \quad (8)$$

- This is the expected (discounted) payoff if the probability of an up-move is q — called the **risk-neutral** probability.
- The true probability, p, of an up-move does **not** matter.
- The risk-neutral probability, q, does **not** depend on C_u and C_d.

Stock options – 4

- Expected stock return under the risk-neutral distribution

$$\frac{[\,qSu + (1-q)Sd\,] - S}{S} \;=\; qu + (1-q)d - 1 \;=\; r \qquad (9)$$

- In the risk-neutral world, all assets have the same expected return, namely the risk-free rate r. No need to distinguish between different asset types — only their payoffs in different states.
- We are **not** assuming that investors are risk-neutral, but the risk adjustment done by modifying the up probability from p to q.
- The payoffs in the states are not modified, only the probabilities.
- Note that we can determine q from the relationship:

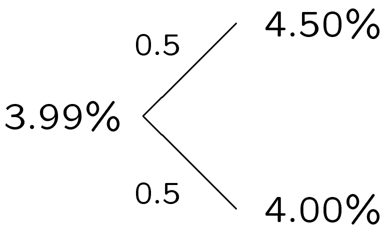
$$S = \frac{1}{1+r}\{qSu + (1-q)Sd\} \quad \Rightarrow \quad q = \frac{1+r-d}{u-d} \qquad (10)$$

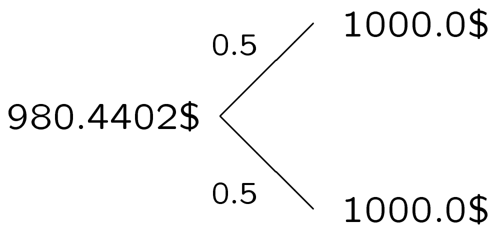
Using binomial models for term-structure derivatives — how?

- The same basic idea (risk-neutral valuation) can still be used, but some modifications are needed.
- Suppose we need to price a one-year option on a five-year zero.
- Should we set up a binomial tree for the 5Y zero?
- **No** — in one year, the bond is no longer a 5Y bond, and the option prices also depend on short-term rates for discounting the option payoff.
- Instead, we need to describe the evolution of the entire yield curve.
- This is done by modeling the dynamics of the short rate (maturity equal to time interval of the binomial tree).
- As a consequence of this assumption, the entire yield curve is a function of the short rate (will be relaxed later).

Interest-rate binomial models − 1

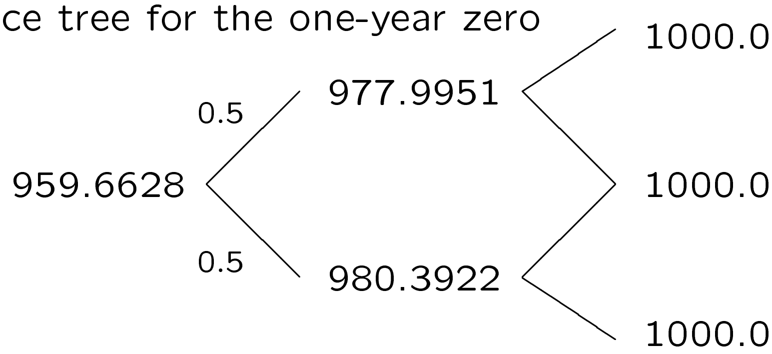
- We focus on the example from chapters 5–6 in Tuckman (1995).
- The time interval is 6 months, so the **short rate** is the 6M rate.
- Two-date example: current short rate is 3.99%, which can change to either 4.00% or 4.50%, with equal probabilities. Note these are **objective** or **true** probabilities.
- Evolution of the short rate and the price of a 6 month zero, which at time 0 (today) is $980.4402 = 1000/(1 + 0.0399/2)$.





Interest-rate binomial models − 2

- The one-year rate is 4.16%, that is the 1Y zero is priced at $959.6628 = 1000/(1 + 0.0208)^2$ per 1000$ face value.
- In 6 months the bond is a 6M zero, and the price is either $1000/(1 + 0.0225) = 977.9951$ or $1000/1.02 = 980.3922$.
- Price tree for the one-year zero



- Note that the current price **differs** from the price under risk-neutrality, which is $\frac{1}{2}(977.9951 + 980.3922)/1.01995 = 960.04$

Interest-rate binomial models – 3

- We want to price a 6-month call option on the bond with an exercise price of 978.50.
- Payoff from the call option in 6 months

$$
\begin{array}{cccl}
 & 0.5 \nearrow & 0.0000 & = \max(977.9951 - 978.50,\ 0) \\
? & & & \\
 & 0.5 \searrow & 1.8922 & = \max(980.3922 - 978.50,\ 0)
\end{array}
$$

- There are two **equivalent** ways to price the option (by arbitrage)
  1. Construct a portfolio of the 1Y and 6M bonds which replicates the payoff from the option in both states (up, down).
  2. Determine the risk-neutral probability q of an up move, and discount the expected payoff under the risk-neutral distribution.

Interest-rate binomial models – 4
Replicating portfolio

- Replicating portfolio $F_{.5}$ and F_1 must satisfy

$$
\begin{aligned}
F_{.5} + 0.9779951 F_1 &= 0 \\
F_{.5} + 0.9803922 F_1 &= 1.8922
\end{aligned}
$$

- The solution is

$$
F_1 = \frac{1.8922}{0.9803922 - 0.9779951} = \mathbf{789.3705}
$$

$$
F_{.5} = -0.9779951 F_1 = \mathbf{-772.0005}
$$

- The price of the option is

$$
C = \frac{F_{.5}}{1 + 0.0399/2} + \frac{F_1}{(1 + 0.0416/2)^2} = \mathbf{0.6292} \qquad (11)
$$

Interest-rate binomial models – 5
Risk-neutral valuation

- We determine q, such that the 1Y bond is priced correctly by risk-neutral valuation.
- In 6 months the value of the bond is either 977.9951 (up) or 980.3922 (down), and the current price is 959.6628.
- Hence, the risk-neutral probability should satisfy

$$\frac{q \times 977.9951 + (1-q) \times 980.3922}{1 + 0.0399/2} = 959.6628. \quad (12)$$

- Solving for q yields $q = \mathbf{0.66085}$, and $1 - q = 0.33915$.
- The price of the option is

$$C = \frac{[0.66085 \times 0.0 + 0.33915 \times 1.8922]}{1 + 0.0399/2} = \mathbf{0.6292} \quad (13)$$

Pricing derivatives — Summary

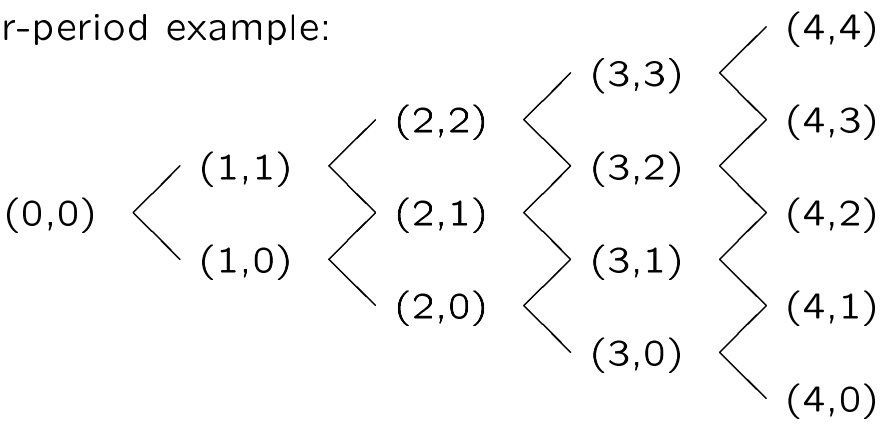
- The call option can be **priced by arbitrage**, which means that the price is independent of investor preferences.
- One way to see this is that the objective probabilities (=0.5) do **not** enter the option-pricing formula.
- This statement is conditional: **given** the prices of the 6M and 1Y zeros, the price of the call option does **not** depend (further) on investor preferences.
- Unconditionally, the price of the option **does** depend on investor preferences, but **only** through the two bond prices.
- Basic idea of risk-neutral valuation: adjust the probabilities of the tree, and assume that the investor is risk-neutral, i.e., **all** prices are computed as the expected discounted payoff (under Q).
- The risk-neutral probabilities do not depend on the payoffs.

Multi-period binomial models – 1

- The above two-date example is sufficient to explain **all** aspects of the theory (and intuition) of risk-neutral valuation.
- In practice, multi-period models are needed
  - Some derivative securities have payoffs at more than one day, e.g., interest-rate caps. All distinct payoff dates should be represented in the binomial model (tree).
  - The real world does not exactly evolve according to a simple binomial model. Instead, the binomial model is an approximation, usually to a continuous distribution such as the normal distribution.
  - Reducing the step size (and thereby increasing the number of periods) results in a better approximation, see Figures 7.1–7.4 in Tuckman (1995).
- To keep the computational work manageable, we must use a **re-combining** tree (lattice), that is an 'up-down' move takes us to the same node as a 'down-up' move.

Multi-period binomial models – 2

- To account for the nodes in a lattice, we use the following notation (n, s), where $n = 0, 1, \ldots, N$ is the date, and $s = 0, \ldots, n$ denotes the state, numbered from below.
- Four-period example:



- Constructing the tree — such that the prices of all N zeros are matched exactly — is an exercise known as **calibration**.