Introduction to the course — 1

- Rapid growth in all fixed income markets recently (government debt, mortgage financing, corporate borrowing, and fixed-income derivatives).

- Many new instruments differ significantly from traditional, plain-vanilla bonds (default-free, non-callable bonds).

- For these securities, traditional measures of interest-rate risk (duration) are either inappropriate, or difficult to compute.

- Black-Scholes model cannot be used for fixed income derivatives (e.g., bond options) because of volatility assumptions.

- **Pricing** and **hedging** requires a stochastic term-structure model describing the future evolution of the entire term structure (not just the underlying asset/bond).
Introduction to the course — 2

- Stochastic term-structure models are the main emphasis of the course.
- Most of these models are relative pricing model: we take the prices of some assets as given (correct), and compute arbitrage-free prices of remaining assets.
- We need to study different models and understand their strengths and weaknesses. No single model can be used for all purposes.
- Institutional details of fixed income markets (and taxes) are only briefly discussed in this course. The securities we study are often simplified as some real-world features are ignored.
- On the other hand, the models we are going to study are quite representative of current “industry practice.”

Zero-coupon bonds

- The basic building blocks of all term-structure models are the so-called zero-coupon bonds.
- Definition: a single payment (e.g., one Dollar) at maturity.
- The price of a zero maturing in \( t \) years is sometimes called the discount factor for time \( t \), denoted \( d(t) \).
- Coupon bonds (bonds with more than one payment) are portfolios (packages) of zero-coupon bonds.
- Alternative representation of \( d(t) \) in terms of spot rates, \( R(t) \):
  \[
  d(t) = \frac{1}{[1 + R(t)]^t} \quad \text{or} \quad d(t) = \frac{1}{[1 + R(t)/2]^{2t}}
  \]
- Tuckman uses the latter formula because of semi-annual coupon payments in the US. We generally prefer the former.
Coupon bonds and absence of arbitrage

- Bond characteristics:
  - Payments at times \( t_1, t_2, \ldots, t_m \). Typical payment frequency for government bonds: annually or semi-annually. Danish mortgage bonds: quarterly.
  - Coupon rate, \( C \), and the repayment scheme of principal (bullets or annuities) determine the \( m \) payments \( b_i \).
  - Bullet bond: principal paid at maturity of the bond.
  - Annuity bond: sum of interest and principal payments is constant.

- Absence of arbitrage requires that the bond price \( P \) satisfies
  \[
  P + A = \sum_{i=1}^{m} b_i d(t_i) = \sum_{i=1}^{m} b_i [1 + R(t_i)]^{-t_i}
  \]
  (1)

- The bond price is quoted without accrued interest (clean price).
- Note that we use different (spot) rates for the \( m \) payment dates.

Day-counting rules

- Rules for computing accrued interest, interest payments on money-market deposits/loans, and payments for interest-rate swaps.
- Typical rules are ACT/360 and ACT/365 for money market deposits, and 30/360 for the bond market (accrued interest).
- Payments are computed as
  \[
  N r \left( \frac{d_1}{d_2} \right),
  \]
  (2)

where

- \( N \) is the nominal amount (deposit/loan amount, or face value of bond).
- \( r \) is the interest rate (or coupon rate) per year, on the given basis.
- \( d_1 \) is the (calculated) number of days until the payment date. For ACT rules, \( d_1 \) is the actual number of days. For the 30/360 rule, each month is assumed to consist of 30 days (bond market convention).
- \( d_2 \) is the number of days in a year (360 or 365), corresponding to the denominator in the rule.
Yield-to-Maturity (YTM) for coupon bonds

- Yield-to-maturity is defined as the $Y$ that solves the equation
  \[ P + A = \sum_{i=1}^{m} b_i (1 + Y)^{-t_i} \] (3)

- Compared to (1), all payments in (3) are discounted using the same interest rate.

- For a zero-coupon bond, YTM is equivalent to the spot rate $R(t)$.

- In the Danish bond market, the official YTM is computed on a 30/360 basis (assumptions about the payment dates $t_i$).

- Interpretation of YTM: another way of quoting the price $P$.

- Note that: YTM is not a bond return, and two bullets with the same maturity can have different YTM’s.

Forward rates

- Consider the definition of the spot rate for maturity $t_2$,
  \[ [1 + R(t_2)]^{t_2} = 1/d(t_2) \] (4)

- If $R(t_1)$ is the spot rate for another maturity, and $t_1 < t_2$, we can (re)write equation (4) as
  \[ [1 + R(t_2)]^{t_2} = [1 + R(t_1)]^{t_1} \times [1 + F(t_1, t_2)]^{t_2-t_1} \] (5)
  where $F(t_1, t_2)$ is defined as the forward rate between $t_1$ and $t_2$.

- $F(t_1, t_2)$ can be interpreted as the promised interest rate today ($t = 0$) for a future deposit/loan between times $t_1$ and $t_2$.

- There are three ways to represent the term structure: discount factors, spot (zero-coupon) rates, and forward rates.
Duration

- Duration (Macauley) is defined as
  \[ D = \frac{\sum_{i=1}^{m} t_i b_i (1 + Y)^{-t_i}}{P + A}. \]  

(6)

- For a zero-coupon bond, duration equals the maturity, \( D = t \). For coupon bonds, we always have \( D < t_m \).

- Duration is a measure of interest-rate risk, since
  \[ D = -\frac{dP}{dY} \left( \frac{1 + Y}{P + A} \right) \]  

(7)

- Useful approximation to bond return (here \( K = P + A \)) if there is a small change in the yield \( Y \),
  \[ \frac{\Delta K}{K} \approx \frac{-D}{1 + Y} \Delta Y \]  

(8)

Continuous compounding − 1

- If the interest on a bank account is compounded \( n \) times in a year, at the annual rate \( r \), investing one Dollar for \( t \) years gives
  \[ \left(1 + \frac{r}{n}\right)^{nt} \]  

(9)

- If \( n \to \infty \), we get \( \exp(rt) \), and \( r \) is called an interest rate with continuous compounding.

- In many cases, assuming continuous compounding simplifies the mathematical formulas, especially for continuous-time models.

- The spot rate for maturity \( t \), with continuous compounding, is defined as
  \[ R(t) = -\frac{\log d(t)}{t} \]  

(10)
Continuous compounding

- If $Y$ is an annual, discretely compounded rate, the continuously compounded rate for the same maturity is $\log(1 + Y)$.

- In Appendix 4A of Tuckman (p. 57-58), the continuously compounded forward rate for time (maturity) $t$ is derived:

$$f(t) = - \frac{d(t)/dt}{d(t)} = -\frac{d}{dt}\log d(t)$$  \hspace{1cm} (11)

- Hence, the forward rate $f(t)$ is the derivative of the logarithm of the discount function.

- The inverse relationship between $f(t)$ and $d(t)$ is given by:

$$d(t) = \exp \left( - \int_0^t f(s) ds \right).$$  \hspace{1cm} (12)