Fixed Income Analysis

Introduction to the course
Bond market basics

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Introduction to the course — 1

- Rapid growth in all fixed income markets recently (government debt, mortgage financing, corporate borrowing, and fixed-income derivatives).
- Many new instruments differ significantly from traditional, plain-vanilla bonds (default-free, non-callable bonds).
- For these securities, traditional measures of interest-rate risk (duration) are either inappropriate, or difficult to compute.
- Black-Scholes model cannot be used for fixed income derivatives (e.g., bond options) because of volatility assumptions.
- **Pricing** and **hedging** requires a stochastic term-structure model describing the future evolution of the entire term structure (not just the underlying asset/bond).

Introduction to the course — 2

- Stochastic term-structure models are the main emphasis of the course.
- Most of these models are **relative** pricing model: we take the prices of some assets as given (correct), and compute arbitrage-free prices of remaining assets.
- We need to study different models and understand their strengths and weaknesses. No single model can be used for all purposes.
- Institutional details of fixed income markets (and taxes) are only briefly discussed in this course. The **securities** we study are often simplified as some real-world features are ignored.
- On the other hand, the **models** we are going to study are quite representative of current "industry practice."

Zero-coupon bonds

- The basic building blocks of all term-structure models are the so-called **zero-coupon bonds**.
- Definition: a single payment (e.g., one Dollar) at maturity.
- The price of a zero maturing in t years is sometimes called the discount factor for time t, denoted $d(t)$.
- Coupon bonds (bonds with more than one payment) are portfolios (packages) of zero-coupon bonds.
- Alternative representation of $d(t)$ in terms of **spot rates**, $R(t)$:

$$d(t) = \frac{1}{[1+R(t)]^t} \qquad \text{or} \qquad d(t) = \frac{1}{[1+R(t)/2]^{2t}}$$

- Tuckman uses the latter formula because of semi-annual coupon payments in the US. We generally prefer the former.

Coupon bonds and absence of arbitrage

- Bond characteristics:
  - Payments at times $t_1, t_2, \ldots, t_m$. Typical payment frequency for government bonds: annually or semi-annually. Danish mortgage bonds: quarterly.
  - Coupon rate, C, and the repayment scheme of principal (bullets or annuities) determine the m payments b_i.
  - Bullet bond: principal paid at maturity of the bond.
  - Annuity bond: sum of interest and principal payments is constant.
- Absence of arbitrage requires that the bond price P satisfies

$$P + A = \sum_{i=1}^{m} b_i \, d(t_i) = \sum_{i=1}^{m} b_i \, [1 + R(t_i)]^{-t_i} \tag{1}$$

- The bond price is quoted without accrued interest (clean price).
- Note that we use different (spot) rates for the m payment dates.

Day-counting rules

- Rules for computing accrued interest, interest payments on money-market deposits/loans, and payments for interest-rate swaps.
- Typical rules are ACT/360 and ACT/365 for money market deposits, and 30/360 for the bond market (accrued interest).
- Payments are computed as

$$N \, r \, (d_1/d_2), \tag{2}$$

where

- N is the nominal amount (deposit/loan amount, or face value of bond).
- r is the interest rate (or coupon rate) per year, on the given basis.
- d_1 is the (calculated) number of days until the payment date. For ACT rules, d_1 is the actual number of days. For the 30/360 rule, each month is assumed to consist of 30 days (bond market convention).
- d_2 is the number of days in a year (360 or 365), corresponding to the denominator in the rule.

Yield-to-Maturity (YTM) for coupon bonds

- Yield-to-maturity is defined as the Y that solves the equation

$$P + A = \sum_{i=1}^{m} b_i (1+Y)^{-t_i} \tag{3}$$

- Compared to (1), all payments in (3) are discounted using the same interest rate.
- For a zero-coupon bond, YTM is equivalent to the spot rate $R(t)$.
- In the Danish bond market, the official YTM is computed on a 30/360 basis (assumptions about the payment dates t_i).
- Interpretation of YTM: another way of quoting the price P.
- Note that: YTM is **not** a bond return, and two bullets with the same maturity can have **different** YTM's.

Forward rates

- Consider the definition of the spot rate for maturity t_2,

$$[1 + R(t_2)]^{t_2} = 1/d(t_2) \tag{4}$$

- If $R(t_1)$ is the spot rate for another maturity, and $t_1 < t_2$, we can (re)write equation (4) as

$$[1 + R(t_2)]^{t_2} = [1 + R(t_1)]^{t_1} \times [1 + F(t_1, t_2)]^{t_2 - t_1} \tag{5}$$

  where $F(t_1, t_2)$ is defined as the **forward rate** between t_1 and t_2.
- $F(t_1, t_2)$ can be interpreted as the promised interest rate today ($t = 0$) for a future deposit/loan between times t_1 and t_2.
- There are three ways to represent the term structure: discount factors, spot (zero-coupon) rates, and forward rates.

Duration

- Duration (Macauley) is defined as

$$D = \frac{\sum_{i=1}^{m} t_i b_i (1+Y)^{-t_i}}{P+A}. \tag{6}$$

- For a zero-coupon bond, duration equals the maturity, $D = t$. For coupon bonds, we always have $D < t_m$.
- Duration is a measure of interest-rate risk, since

$$D = -\frac{dP}{dY}\left(\frac{1+Y}{P+A}\right) \tag{7}$$

- Useful approximation to bond return (here $K = P + A$) if there is a small change in the yield Y,

$$\frac{\Delta K}{K} \approx \frac{-D}{1+Y}\Delta Y \tag{8}$$

Continuous compounding − 1

- If the interest on a bank account is compounded n times in a year, at the annual rate r, investing one Dollar for t years gives

$$\left(1+\frac{r}{n}\right)^{nt} \tag{9}$$

- If $n \to \infty$, we get $\exp(rt)$, and r is called an interest rate with **continuous compounding**.
- In many cases, assuming continuous compounding simplifies the mathematical formulas, especially for continuous-**time** models.
- The spot rate for maturity t, with continuous compounding, is defined as

$$R(t) = \frac{-\log d(t)}{t} \tag{10}$$

Continuous compounding – 2

- If Y is an annual, discretely compounded rate, the continuously compounded rate for the same maturity is $\log(1+Y)$.
- In Appendix 4A of Tuckman (p. 57-58), the continuously compounded forward rate for time (maturity) t is derived:

$$f(t) = -\frac{d(t)/d\,t}{d(t)} = -\frac{d}{dt}\log d(t) \tag{11}$$

- Hence, the forward rate $f(t)$ is the derivative of the logarithm of the discount function.
- The inverse relationship between $f(t)$ and $d(t)$ is given by:

$$d(t) = \exp\left(-\int_0^t f(s)ds\right). \tag{12}$$