Uncertain Interest Rate Modelling

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Abstract

In this thesis, we introduce a non-probabilistic model for the short-term interest rate. The key concepts involved in this new approach are the non-diffusive nature of the short rate process and the uncertainty in the model parameters. The model assumes the worst possible outcome for the short rate path when pricing a fixed-income product (from the point of view of the holder) and differs in many important ways from the traditional approaches of fully deterministic or stochastic rates. In this new model, delta hedging and unique pricing play no role, nor does any market price of risk term appear. We present the model and explore the analytical and numerical solutions of the associated partial differential equation. We show how to optimally hedge the interest rate risk of a fixed-income portfolio and price and hedge common and exotic fixed-income products. Finally, we consider extensions to the model and present conclusions and areas for further research.

Acknowledgements

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Statement of originality

Originality is claimed for the entirety of the thesis, excluding Chapter 1 which comprises a review of current practise.

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Chapter 1

Introduction

In contrast to the asset price world, there is no commonly accepted model for the movement of the underlying in the interest rate world. Consequently, there are a number of different approaches to the pricing of fixed-income products.

The simplest approach is to price a product off a yield curve. This method is effective for simple contracts, bonds for instance. However, for more complex products, where optionality or convexity play a role, the precise nature of the interest rate movements is significant and so the method does not give accurate results.

The 'traditional' approach to pricing these more complicated products is to introduce stochastic variables to model a number of 'unknown' factors, on which we believe the interest rate movements depend. These models can be single- or multi-factor models for the movement of the short-term interest rate, or models for the movement of the whole yield curve (the Heath, Jarrow & Morton approach). All of these methods rely on the estimation of parameters. Not only are these parameters (for instance, volatility) difficult to estimate, but they can also be unstable [2].

The single- and multi-factor models have the additional disadvantages that they can require fitting to the current yield curve, again in an unstable fashion, and that they can assume an equally difficult to estimate and unstable correlation between yields of different maturities.

In the following work, we present an alternative approach to the pricing of fixed-income products. We introduce a non-stochastic non-probabilistic model for the short-term interest rate. This work has, in part, been inspired by the work on uncertain volatility in equity derivatives by Avellaneda, Levy and Paras [4] and Lyons [48]. However, the ideas cannot be directly translated into the interest rate world, because the underlying that we consider is not a traded quantity.

Rather than specify how the short rate evolves, we will just constrain the possible movements. We will make no probabilistic statements whatsoever, solely stating what is possible and what is not. Clearly, there are therefore going to be a number of possible paths that the short rate could take. For each path, when we use the short rate as a discount rate, the contract in question could have a different value (where we consider the position of the holder of the contract). We will consequently find a range of possible values for the price of a contract. We identify the lowest of these as the 'worst-case scenario value'.

The analysis of this worst-case valuation problem leads to a nonlinear, first-order, hyperbolic partial differential equation. We can solve this equation either analytically or via numerical methods. The results motivate us to investigate whether there is any role for hedging. Rather than dynamic hedging, we find that there is an optimal static hedge for a product [15], [21], [26], [27]. This form of hedging mirrors the yield curve fitting that is often applied to stochastic interest rate models, but has none of the associated problems with inconsistency.

There are a number of practical applications for this model. Clearly, it can be used to find price ranges for instruments and spot potential arbitrage opportunities in the market [30], [32], [33]. If we have an over-the-counter (OTC) contract - one that is not listed in the market - then we can use the uncertain interest rate model to construct an optimal static hedge of market-traded products and reduce the inherent interest rate risk [34], [35]. Finally, we can use the model as a risk management tool. With a sensible choice of parameters, it is possible to show that the model is completely consistent with past interest rate history. In this case, the worst-case scenario value is a definitive lower bound for the value of a portfolio. The same consistency cannot easily be shown for any other model [31], [36].

In the next section, we present a review of the common fixed-income products available in the market and of the traditional approaches and techniques used to price them, some of which we will use for comparison later on. We then summarise the theory of uncertain volatility in the equity world, which is analogous to our uncertain interest rate theory. Finally we present an outline of the contents of the thesis.

1.1 Common fixed-income contracts

The following products are all contracts between two parties. The writer is, in general, paid a premium at the origination of the contract, for the obligation to pay specified cashflows to the holder, at specified dates over the life of the contract. (Further details of the specification of these and many other contracts can be found in Fabozzi [37]).

1.1.1 Bonds

A bond is a borrowing arrangement in which a borrower (the writer) issues an IOU to an investor (the holder). In its simplest form, it only requires the writer to pay a specified amount, the principal, to the holder, at a specified date in the future, the maturity. This contract is called a zero-coupon bond.

The more general contract, the coupon bond, also requires the writer to make interim payments, or coupons, of a specified proportion of the principal, the coupon rate, at specified dates, up to and including the maturity of the bond, as well as paying the principal at maturity.

Figure 1.1 shows a diagramatic representation of a zero-coupon bond and a coupon bond. Horizontal distance represents time, with maturity at the right hand end. Each arrow represents a cashflow. Arrows above the horizontal axis represent positive cashflow payments from the writer to the holder, and those below the axis represent negative payments. An arrow with a straight shaft is indicative of a payment of a known quantity at the origination of the contract, whereas an arrow with a wavy shaft (as we will see for the next contract) is indicative of a cashflow which is dependent on some quantity that is not known at the origination of the contract (for instance, an interest rate).

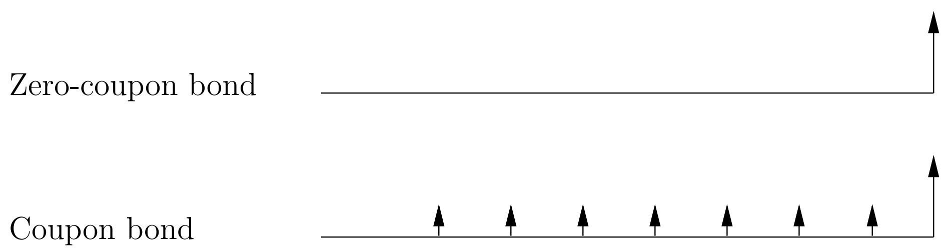


Figure 1.1: Diagramatic representation of a coupon and zero-coupon bond

1.1.2 Swaps

A swap is an agreement whereby two parties exchange interest payments on a principal. At specified payment dates, one party pays the the interest that would be due on the principal due to a predetermined fixed rate. The other party pays the interest that would be due on the principal due to some designated interest rate (the reference rate). Further details of the particulars of swap specification can be found in numerous sources [24], [50], [60]. The individual set of cashflows for a particular payment date is called a swaplet. We can express these cashflows mathematically as

$$P(r - r_f), \tag{1.1}$$

where P is the principal, r is the reference rate and r_f is the fixed rate.

The contract must specify which interest rate is to be used, and at what time it is to be measured, since this may be prior to the payment date [25]. The fixed rate is usually chosen so that there is no premium payable to either party at the origination of the contract (in this case, the contract is called a par swap). Figure 1.2 shows a diagramatic representation of a swap where the holder receives the floating payments and pays the fixed.

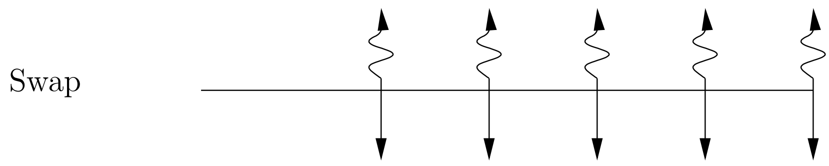


Figure 1.2: Diagramatic representation of a swap

In certain circumstances, it is possible to decompose a swap into a portfolio of zero-coupon bonds. Consider a single floating rate payment, as shown in Figure 1.3.

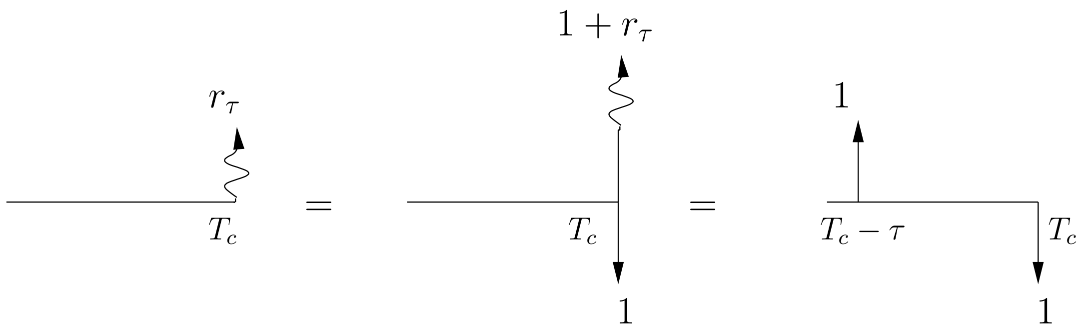


Figure 1.3: Decomposition of a single floating rate payment

If the floating rate is the interest rate for a period of τ and is measured at a time τ before the payment date, T_c, then a cashflow of 1 at the date $T_c - \tau$ is equivalent to a cashflow of $1 + r_\tau$ at the date T_c, since r_τ is the τ period interest rate. We can therefore decompose the single floating rate payment into two zero-coupon bonds.

If we now consider the whole floating rate side of the swap, we can see that this also decomposes into two zero-coupon bond payments, but only in the case when the payment dates are also τ apart. This is shown in Figure 1.4.

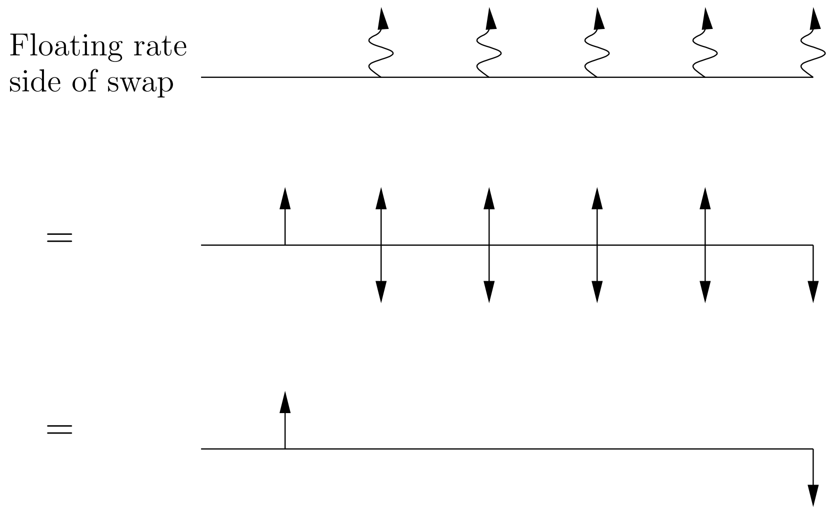


Figure 1.4: Decomposition of the floating rate side of a swap

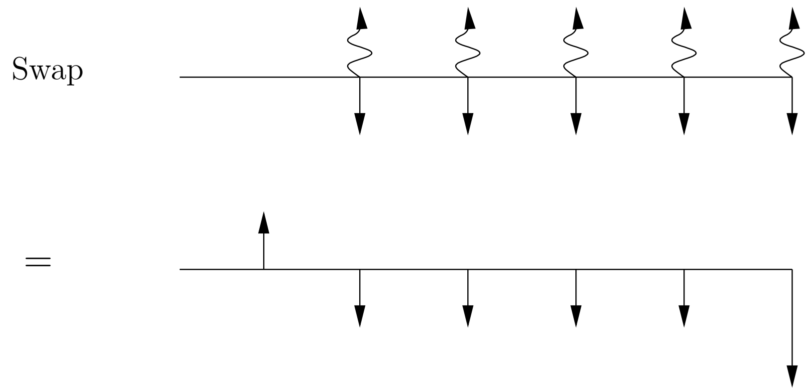


Figure 1.5: Decomposition of a swap into zero-coupon bonds

The swap, as a whole, can therefore be expressed as a sum of zero-coupon bonds, as shown in Figure 1.5. If this swap has N payment dates, at times $T_1(=\tau), T_2, \ldots, T_N$, then we can write the value of the swap in terms of the fixed interest rate, r_f, and zero-coupon bonds, as

$$P\left(1 - Z(t;T_N) - r_f \sum_{i=1}^{N} Z(t;T_i)\right),$$

where $Z(t;T)$ is the value at time t of a zero-coupon bond with principal 1 and maturity at time T. For a par swap, we must choose r_f so that the swap initially has no value, i.e.

$$r_f = \frac{1 - Z(t;T_N)}{\sum_{i=1}^{N} Z(t;T_i)}.$$

This swap decomposition is model-independent. It therefore holds regardless of which interest rate model we use to value the contracts. We note that the rate r_f

is a τ-period rate. To annualise the rate, we must divide by τ (assuming that τ is measured in years).

1.1.3 Caps and floors

Caps and floors are interest rate agreements whereby one party (the writer), for an upfront premium, agrees to compensate the other (the holder) at specific dates if a designated interest rate, the reference rate, differs from a predetermined level [19]. The agreement is called a cap if payment occurs when the reference rate exceeds a predetermined level. The agreement is called a floor if payment occurs when the reference rate falls below a predetermined level. The predetermined level is called the strike rate.

The individual set of cashflows for a particular cap payment date is called a caplet. We can express these cashflows mathematically as

$$P \max(r - r_s, 0), \tag{1.2}$$

where P is the principal, r is the reference rate and r_s is the strike rate.

The individual set of cashflows for a particular floor payment date is called a floorlet. We can express these cashflows mathematically as

$$P \max(r_s - r, 0). \tag{1.3}$$

As with the swap contract, a cap or floor contract must specify which interest rate is to be used, and at what time it is to be measured. It is again possible to decompose the contract. In this case, into a portfolio of bond options. First of all, we define a bond option.

1.1.4 Bond options

A vanilla bond option is a contract that gives the holder the right, but not the obligation, to buy or sell a bond to the writer at, or between prescribed times, for a specified price. A European option gives the holder this right at a specified date in the future (the expiry). An American option gives the holder the right at all times until expiry.

A call option gives the holder the right to buy the prescribed bond (the underlying) for a prescribed amount (the exercise price). This payoff can be expressed mathematically as

$$\max(B - E, 0), \tag{1.4}$$

where B is the value of the bond at expiry of the option and E is the exercise price (since we only exercise the option if $B > E$ at expiry).

A put option gives the holder the right to sell the prescribed bond for a prescribed amount. This payoff can be expressed mathematically as

$$\max(E - B, 0). \tag{1.5}$$

1.1.5 Decomposition of caps/floors into bond options

We consider a single caplet, with a floating rate that is the interest rate for a period of τ and is measured at a time τ before the payment date, T_c. In this case, a cashflow of 1 at the date $T_c - \tau$ is equivalent to a cashflow of $1 + r_\tau$ at the date T_c, since r_τ is the τ period interest rate. (We assume, without loss of generality, that the principal is 1).

The caplet has cashflow

$$\max(r_\tau - r_s, 0),$$

received at time T_c. This is equivalent to a cashflow of

$$\frac{1}{1 + r_\tau} \max(r_\tau - r_s, 0),$$

received at time $T_c - \tau$. We can rewrite this as

$$\max\left(1 - \frac{1 + r_s}{1 + r_\tau}, 0\right),$$

where we can think of

$$\frac{1 + r_s}{1 + r_\tau},$$

as being the price at time $T_c - \tau$ of a bond that pays out $1 + r_s$ at time T_c. We can therefore consider the caplet to be equivalent to a put option, with this bond as the underlying, with exercise price 1 and expiry at time $T_c - \tau$. Hence, we can decompose a cap into a portfolio of put options.

Similarly, we can decompose a floor into a portfolio of the corresponding call options.

1.2 Traditional approaches to interest rate modelling

This section is *not* intended to be a complete review of the current state of affairs in the world of interest rate modelling. It is instead a review of some of the key definitions and methods and, in detail, only those which we will refer to during this thesis. Further details of all these approaches can be found in a number of sources, for instance, Hull [44] or Wilmott [63].

1.2.1 Arbitrage

An arbitrage opportunity is the opportunity to make a risk-free profit. We shall assume that there is an absence of such arbitrage opportunities throughout this work.

1.2.2 Present value

The present value (at time t) of an amount of cash E to be received at time T is the amount we would pay now for this future cash flow. To find the present value of the cash flow, we must discount it using a specified interest rate. If we have a continuously-compounded short-term interest rate, r, then money invested in the bank, $M(t)$, grows exponentially according to

$$dM = rMdt.$$

When this short interest rate is a known function of time, $r(t)$, and $M(T) = E$ then we can solve the resulting ordinary differential equation to find

$$M(t) = Ee^{-\int_t^T r(\tau)d\tau}. \tag{1.6}$$

(Note that throughout this work, we shall assume that the short-term interest rate is continuously-compounded).

1.2.3 Yield to maturity

The yield to maturity is a measure of the rate of return of a bond held until maturity. It is the constant interest rate that we would have to use to discount all of the bond's cashflows to value the bond at its current market price.

For a zero-coupon bond, with principal P, expiry at time T and market price Z_M, the yield to maturity at time t is

$$Y = -\frac{\log(Z_M/P)}{T-t}. \tag{1.7}$$

If we have a number of traded instruments then we can calculate the yield for each of their maturities. We can then interpolate between these points to construct the yield curve. This curve provides us with rates of return for all maturities.

Figure 1.6 gives an example of a yield curve constructed with linear interpolation and one constructed with spline interpolation [1] between the points where the yields have been calculated (the x's).

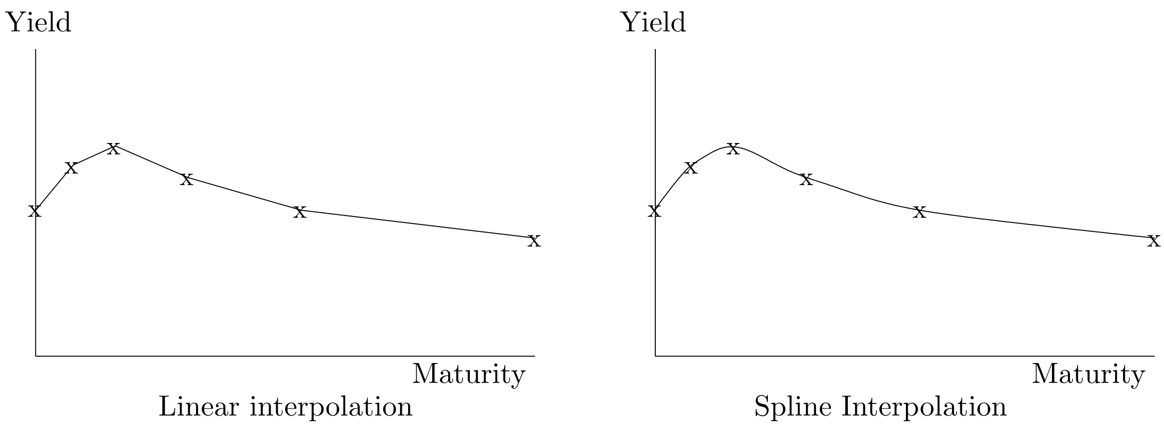


Figure 1.6: An interpolated yield curve

1.2.4 Pricing off the yield curve

We can price a product off the yield curve as long as all of its cashflows are known quantities at the origination of the contract. To do this, we just add up the present values of all the cashflows. To find the present value of a cashflow, we read off the rate of return for the payment date of the cashflow from the yield curve and discount the cashflow at that rate.

However, it is possible to find two instruments with the same maturity but different yields, for example, two coupon bonds with the same maturity but different coupon structures. It is not possible to construct a yield curve consistent with both of these instruments.

In addition, we assume that a yield is constant from now until maturity, so we cannot use this rate to evaluate any cashflow that depends on an interest rate of shorter term, for instance, a swap.

We require a method of construction that produces an interest rate curve both consistent with our yield data and also suitable for the evaluation of rate-dependent cashflows. For this, we introduce the concept of forward rates.

1.2.5 Forward rates

Forward rates are interest rates that apply over given periods of time and are consistent with all our yield data. If we have a continuous set of zero-coupon bond prices, for all maturities, $Z(t;T)$, then the implied forward rate is the short rate curve that is consistent with all of these prices, $F(t;T)$, and satisfies

$$Z(t;T) = e^{-\int_t^T F(t;\tau)d\tau}. \tag{1.8}$$

We can differentiate this equation to find

$$F(t;T) = -\frac{\partial}{\partial T}(\log Z(t;T)). \tag{1.9}$$

If we have a finite set of zero-coupon bonds from which we want to generate our forward rate curve, then we use the following methodology:

- Rank the bonds in order of increasing maturity, $T_1, T_2, \ldots, T_N$.
- Find the constant interest rate that must apply between now and T_1, implied by the market value of the first bond. This is the forward rate that holds between now and T_1.
- Find the constant interest rate that must apply between T_1 and T_2, implied by the market value of the second bond, when we apply the first forward rate between now and T_1. This is the forward rate that holds between T_1 and T_2.
- For the ith forward rate, find the constant interest rate that must apply between T_{i-1} and T_i, implied by the market value of the ith bond, when we apply the previous forward rates between the appropriate times. This is the forward rate that holds between T_{i-1} and T_i.
- Repeat the previous step as necessary.

This method is called bootstrapping. Figure 1.7 shows the forward rate curve generated from the yield data used to construct the yield curves of Figure 1.6. We include the yields of all the instruments for comparison.

This method still applies if there are two contracts with the same maturity, but different cashflow structures, which may occur if we include coupon bonds, for example. In this case, we will have only one market value for a number of cashflow dates. This means that we will have fewer equations than unknowns. To solve this problem,

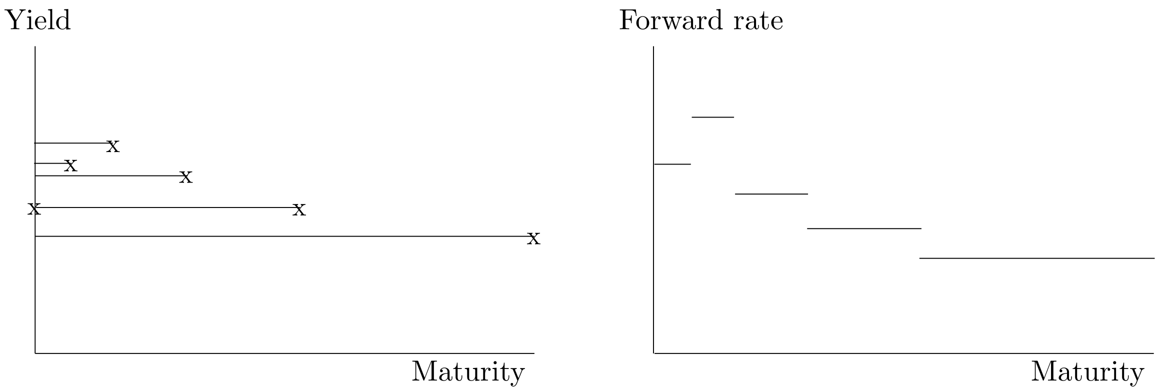


Figure 1.7: The yield to maturity and forward rate

we must make some additional assumptions, grouping some of the cashflow dates, for instance.

We also note that rather than a piecewise constant forward rate curve, we could instead construct a continuous curve, using some form of interpolation.

1.2.6 Pricing off the forward rate curve

We can use the forward rate curve to price any simple fixed-income contract. As with yield curve pricing, we again just add up the present values of the cashflows. If we have a forward rate curve, $F(t;T)$, and a cashflow $C(r)$ at time T_c, then the present value, at time t, of the cashflow is

$$C(F(t;T_c))e^{-\int_t^{T_c} F(t;\tau)d\tau}. \tag{1.10}$$

However, this method is still inappropriate for more complex products, such as caps, floors or bond options, whose values depend more strongly on the exact nature of the underlying interest rate movements. To price these contracts, we must first construct a model for the interest rate.

1.2.7 Stochastic models

A popular approach to interest rate modelling is to construct a stochastic model for the movement of the short-term interest rate. We can then price a contract as the expected value of its cashflows, where we discount at this short rate, and also use the rate to value any rate-dependent cash flows, i.e.

$$V = \sum_i \left(\mathbf{E}^*_{\mathbf{t}} \left[C_i(r) e^{-\int_t^{T_i} r_\tau d\tau} \right] \right), \tag{1.11}$$

where the contract has cashflows $C_i(r)$ at times T_i and we take the risk-neutral expectation, $\mathbf{E}^*_\mathbf{t}$ [53], [56].

1.2.8 One-factor models

The simplest of these stochastic models are one-factor models. Many such models have been proposed [11], [23], [28], [39], [58]. They assume that interest rate movements are driven by a single random factor. They have the general form

$$dr = u(r,t)dt + v(r,t)dX, \tag{1.12}$$

where u and v are some specified functions of r and t and dX is a Wiener process (that is, a random variable drawn from a Normal distribution with mean 0 and variance dt).

We can derive a pricing equation for the value of a fixed income product under this model. We find that the price of a contract, $V(r,t)$, satisfies

$$V_t + \tfrac{1}{2}v^2 V_{rr} + (u - \lambda v)V_r - rV = 0, \tag{1.13}$$

where λ is the market price of risk [65].

This is a second-order, parabolic, partial differential equation. It has final condition $V(r,T)$ given by the value of the contract at maturity and boundary conditions which depend on the specification of the contract. We include a cashflow at time T_c as a jump condition (due to an absence of arbitrage opportunities) of the form

$$V(r, T_c^-) = V(r, T_c^+) + \Lambda(r), \tag{1.14}$$

when there is a cashflow $\Lambda(r)$ at time T_c and where the superscript '$-$' denotes just before the cashflow date and '$+$' just after.

The market price of risk is the ratio of the excess return above the risk-free rate to the level of risk inherent in a portfolio. The increase in the value of the portfolio over a time step dt is an extra λdt for each unit of risk, dX. It is necessary to introduce such a measure because the underlying process, the short rate, is not a traded quantity.

There are many possible choices for the functions u and v. Below, we consider two particular examples that we will refer to later in the thesis.

1.2.8.1 Vasicek

In the Vasicek model, we set

$$u(r,t) = a - br \text{ and } v(r,t) = \nu,$$

(where a, b and ν are constants), so that the short rate is mean-reverting to the level a/b at a rate b [61]. In this case, the short rate process satisfies

$$dr = (a - br)dt + \nu dX.$$

We can generalise this model by allowing a and b to be functions of r and t. In the extended Vasicek model of Hull & White, a is time-dependent, so that

$$dr = (a(t) - br)dt + \nu dX. \tag{1.15}$$

If we estimate b and ν, then we can choose $a(t)$ to fit the current yield curve (i.e. so that the theoretical and actual market bond prices coincide). To fit the yield curve at time t^*, we find that $a(t)$ must satisfy

$$a(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - b\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \frac{c^2}{2b}\left(1 - e^{-2b(t-t^*)}\right), \tag{1.16}$$

where $Z_M(t^*, T)$ is the market price of the T-maturity zero-coupon bond at time t^* [42], [43].

1.2.8.2 ACKW

The ACKW model is an empirical model of the short rate, proposed by Apabhai, Choe, Khennach and Wilmott [2]. Rather than choosing a model that has tractable solutions, they consider a general form for the short rate model and perform an empirical analysis of short rate data to choose the precise parameters. They assume that

$$u(r,t) = \nu^2 r^{2\beta-1}\left(\beta - \tfrac{1}{2} - \frac{1}{2a^2}\log(r/\bar{r})\right)$$

and

$$v(r,t) = \nu r^\beta,$$

so that the short rate process is

$$dr = \nu^2 r^{2\beta-1}\left(\beta - \tfrac{1}{2} - \frac{1}{2a^2}\log(r/\bar{r})\right)dt + \nu r^\beta dX. \tag{1.17}$$

They then perform a statistical analysis of US short rate data to choose the model parameters, and find that

$$\beta = 1.13 \text{ and } \nu = 0.126, \tag{1.18}$$

from an examination of the expected average value of $(\delta r)^2$, and that

$$a = 0.4 \text{ and } \bar{r} = 0.08, \tag{1.19}$$

by considering the steady-state probability density function for the short rate. The model is therefore approximately lognormal and mean-reverts to 8%.

1.2.9 Multi-factor models

The simplest generalisation of the one-factor stochastic model is the multi-factor model. This assumes that movements in the yield curve depend on more than one random factor. If an instrument depends on the difference between different sections of the yield curve, rather than just its level, then we need at least a second source of randomness to model this movement effectively [18], [46], [47].

Generally, we model the short-term interest rate, r, along with another independent variable, l, where

$$dr = udt + vdX_1, \tag{1.20}$$

and

$$dl = pdt + qdX_2. \tag{1.21}$$

u, v, p and q are some specified functions of r, l and t and dX_1 and dX_2 are random variables drawn from Normal distributions with mean 0 and variance dt, with correlation ρ.

We can derive a pricing equation for the value of a fixed income product under this model. We find that the price of a contract, $V(r, l, t)$, satisfies

$$V_t + \tfrac{1}{2}v^2 V_{rr} + \rho v q V_{rl} + \tfrac{1}{2}q^2 V_{ll} + (u - \lambda_r v)V_r + (p - \lambda_l q)V_l - rV = 0, \tag{1.22}$$

where $\lambda_r(r, l, t)$ and $\lambda_l(r, l, t)$ are the market prices of risk for r and l respectively.

1.2.10 Heath, Jarrow & Morton

All of the previous stochastic models have been models of the movement of one or more interest rate factors. However, Heath, Jarrow & Morton suggest a more general approach, by modelling the movement of the whole forward rate curve [38]. The method consistently reproduces the current yield curve, since this information is contained in the initial forward rate curve.

We assume that zero-coupon bonds evolve, in a risk-neutral world, according to

$$dZ(t;T) = r(t)Z(t;T)dt + \sigma(t,T)Z(t;T)dX. \tag{1.23}$$

We can then determine the stochastic differential equation for the evolution of the risk-neutral forward rate curve

$$dF(t;T) = \left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX, \tag{1.24}$$

where

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T). \tag{1.25}$$

Since

$$r(t) = F(t;t) = F(t^*;t) + \int_{t^*}^t dF(\tau;t), \tag{1.26}$$

we can also find the short rate for any time t in the future of today, t^*.

Using this information, we can price a contract by calculating the present value of all of its cashflows. To do this, we must first choose a specific form for σ and then either proceed analytically, or implement a Monte Carlo method to simulate the various forward and short rate paths [22].

We remark that more recently, Brace, Gatarek and Musiela have proposed a similar model for a non-infinitessimal short rate, BGM, which can be used to model discrete (and observable) forward rates directly [16].

1.3 The uncertain volatility model

In the final section of this review, we present the uncertain volatility model for equity derivatives. Our motivation for including this model is that it was one of the original inspirations for the form of our uncertain interest rate model.

The uncertain volatility model is an extension of the original Black-Scholes model for equity derivatives [7], [12]. In the Black–Scholes model, we consider an asset price random walk of the form,

$$dS = \mu S dt + \sigma S dX, \tag{1.27}$$

where μ and σ are known parameters, and dX is drawn from a Normal distribution with mean 0 and variance dt.

It is then possible to derive a pricing equation for the value of an equity derivative, $V(S,t)$, of the form

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \tag{1.28}$$

with appropriate final and boundary conditions for the specific contract.

In the uncertain volatility model, we generalise the assumption that σ is known, to allow σ to lie anywhere within a given range,

$$\sigma^- \le \sigma \le \sigma^+, \tag{1.29}$$

Since there is a range of possible values for σ, we find that there is a range of possible values for the equity derivative. We identify the lowest of these as the worst-case scenario value and find that it is possible to derive the following nonlinear, second-order, parabolic differential equation for this value,

$$V_t + \tfrac{1}{2}\sigma^2\left(V_{SS}\right)S^2 V_{SS} + rSV_S - rV = 0, \tag{1.30}$$

where

$$\sigma(X) = \begin{cases} \sigma^+ & \text{if } X < 0 \\ \sigma^- & \text{if } X > 0. \end{cases} \tag{1.31}$$

Since this is a nonlinear problem, the value of a portfolio of contracts is not necessarily the same as the sum of their individual values. The price of a product therefore depends on what it is hedged with. Consequently, we can statically-hedge an OTC contract with market-traded contracts and find that there is an optimal static hedge which gives the OTC contract the highest possible worst-case scenario value [3], [6].

We remark that Avellaneda and Lewicki have also applied this approach to interest rate modelling, where they propose a Heath, Jarrow & Morton model with an uncertain volatility [5]. In the following work, we will eliminate the consideration of volatility altogether and propose a more general form of uncertain model for the interest rate.

1.4 Overview

In Chapter 2, we discuss the concept of a worst-case scenario valuation and illustrate the idea with a simple uncertain model for the short-term interest rate. We then describe our general uncertain, non-probabilistic model for the short rate. In this model, we prescribe bounds on both the short-term interest rate and its growth rate. We derive the partial differential equation for the worst-case scenario value of a contract under this model and find that it is first-order, nonlinear and hyperbolic. We then examine the solution of this equation via the method of characteristics.

We illustrate the solution by the method of characteristics in Chapter 3. We first discuss the general methodology and then considered various examples of final data. In each case, we consider all of the possible characteristic pictures that could occur and find the solution of the equation for each of these situations.

We begin Chapter 4 with a discussion of the consequences of the nonlinearity of our pricing equation and consider the problem of the contract value in a best-case scenario. Using the zero-coupon bond to illustrate the procedure, we then consider in detail the pricing and hedging of a contract. We show that there is an optimal static hedge for which the worst-case scenario value of the contract reaches a maximum level. Similarly, there is another optimal static hedge for which the best-case scenario value of the contract reaches a minimum level. Associated with these results is the Yield Envelope. This is similar in form to the yield curve, however, at a maturity where no traded contract exists, there is a yield spread. We then apply the model to the pricing and hedging of swaps, caps and floors, describing the appropriate jump and final conditions for the pricing equation in each case. Finally, in the light of these results, we discuss possible applications for the model and price and hedge a real-world leasing portfolio.

In Chapter 5, we apply our model to the pricing and hedging of more exotic fixed-income contracts. We begin with the European bond option and derive two different methodologies for the pricing of the option, dependent on the type of the option and the hedging strategy to be followed. Second, we price the multi-choice swap, a contract with embedded decisions. This swap allows the holder to choose on which m of the M possible cashflow dates to exchange interest rate payments. To price this contract, we introduce a set of functions for the contract value, dependent on how many cashflows there are left to take.

We then consider the index amortising rate swap. In this contract, the principal amortises on cashflow dates, at a rate determined by an amortising schedule. We derive the pricing equation for the worst-case scenario value of the contract and determine a similarity reduction to reduce the problem from three independent variables to two. Finally, we examine the convertible bond. This contract has coupon payments, of the same form as a vanilla coupon bond, but has the additional property that the holder can choose to exchange the bond for a specified number of an underlying asset.

We describe the partial differential equation for the worst-case value of the bond and compare the results of the pricing process to a number of more traditional approaches to interest rate modelling.

In Chapter 6, we present extensions to our uncertain model. These allow for interest rate paths that are indistinguishable from those seen in practice. We consider the concept of the uncertainty band, in which our model for the short-term interest rate becomes an estimate of the real short-term rate. We derive the new partial differential equation for the worst-case value of a contract under this assumption and describe how to relate the short-term interest rate to rates of a longer period. Using this concept, we can examine past interest rate data to choose a sensible width for the band. As a further extension, we include the possibility for crashes in the short-term interest rate. These crashes can take one of two forms. There can either be a maximum possible total number or a maximum possible frequency for the crashes. We describe the pricing equation framework for each case and then re-examine the data to find an adjusted uncertainty bandwidth, along with sensible parameters for these events. To close the chapter, we study the effect of illiquidity of the hedging instruments on the worst-case scenario valuation of a contract and present an illiquid version of the Yield Envelope.

Finally, in Chapter 7, we summarise the key ideas and results of the thesis. We then consider areas for further research and draw our conclusions.

Chapter 2

An uncertain interest rate model

In this chapter, we present a non-probabilistic model for the short-term interest rate. We use this as a discount rate to price fixed-income products. If a contract has cashflows $C_i(r)$ at times T_i then the present value of the contract under this model is

$$\sum_i C_i(r(T_i))e^{-\int_t^{T_i} r(\tau)d\tau}, \tag{2.1}$$

where $r(t)$ is the evolution of the short-term interest rate over the maturity of the contract.

We will place bounds on the possible movements of the short rate. Consequently, there will be a range of possible prices for a fixed-income contract, since there will be a number of possible paths for the short rate and each of these could give the contract a different present value. We identify the lowest of these prices as the worst-case scenario value (for the holder of the contract). Under our short rate model, the contract must be worth at least this much, regardless of which path the short rate actually takes. We remark that the worst-case scenario for the holder of the contract will be the best-case scenario for the writer of the contract (the best-case scenario is discussed in Section 4.2). In this thesis, we will always consider the case of the holder of the contract. In the next section, we motivate this concept using the simplest possible model.

2.1 A simple example of worst-case scenario valuation

Consider the simple case where the only constraint on the movement of r is that it is bounded above and below, i.e.

$$r^- \leq r \leq r^+, \tag{2.2}$$

where r^+ and r^- are constants. This means that it is possible for r to jump instantaneously from any value to any other value within the range $[r^-, r^+]$.

2.1.1 Pricing a single cashflow

Consider a zero-coupon bond with principal P that matures at time T. We want to find the value of this bond in a worst-case scenario, at time t. This value will depend on the evolution of the short rate, r, in the intervening period.

Since there is only a single cashflow to consider, it is simple to identify the worst-case path for the interest rate. If the cashflow is positive, then the worst-case scenario will occur when the interest rate is always as high as possible, so that the present value of the cashflow is as low as possible. However, if the cashflow is negative, then the worst-case scenario will occur when the interest rate is always as low as possible.

Therefore, if $P > 0$ then the worst-case scenario occurs when r immediately jumps from its initial value to r^+ and remains at this rate until maturity. In this case, the value of the bond is then

$$Pe^{-r^+(T-t)}.$$

If $P < 0$ then the worst-case scenario occurs when r instantaneously falls to r^- and stays there until maturity. The value of the bond is then

$$Pe^{-r^-(T-t)}.$$

Figure 2.1 shows the evolution of the interest rate in a worst-case scenario for these two cases of zero-coupon bond valuation.

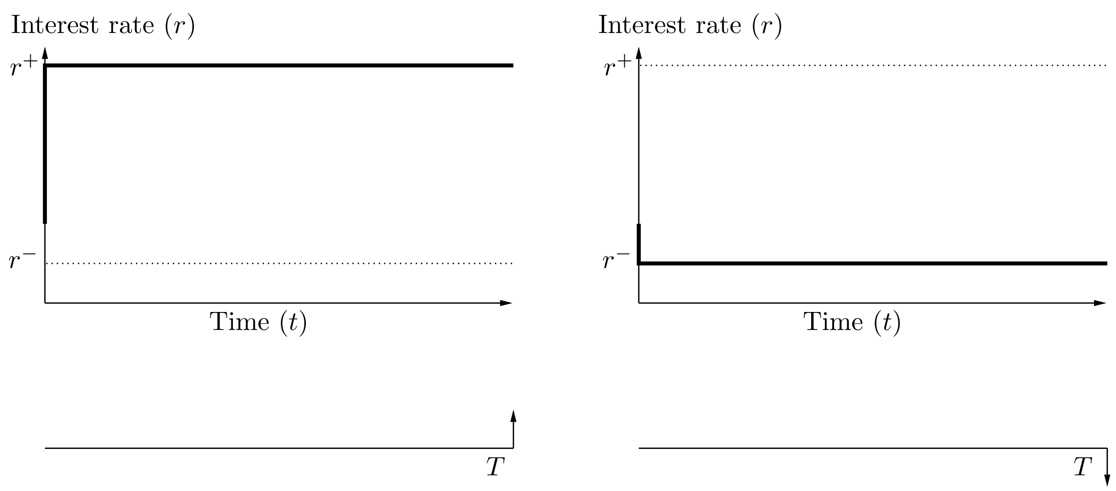


Figure 2.1: The interest rate paths for a zero-coupon bond under our simple model

2.1.2 Pricing two cashflows

Now, let us make the problem slightly harder. We consider a coupon bond that has a cashflow of C at time T_1 and P at maturity, T_2 ($T_1 < T_2$). This is equivalent to two zero-coupon bonds, one with a principal of C and maturity at time T_1 and the other with a principal of P and maturity at time T_2.

If both cashflows are of the same sign, then the problem reduces to that of the previous section for a single cashflow. If the cashflows are positive, then the worst-case scenario will occur when the interest rate is always as high as possible, and if the cashflows are negative, then the worst-case scenario will occur when the interest rate is always as low as possible.

Consequently, If $C > 0$ and $P > 0$ then the worst-case scenario occurs when r immediately jumps to r^+ and stays there until maturity. The value of the bond at time t is then

$$Ce^{-r^+(T_1-t)} + Pe^{-r^+(T_2-t)}.$$

If $C < 0$ and $P < 0$ then the worst-case scenario occurs when r instantaneously falls to r^- and stays there until maturity. The value of the bond in a worst-case scenario is then

$$Ce^{-r^-(T_1-t)} + Pe^{-r^-(T_2-t)}.$$

But what if the cashflows are of opposite sign? By way of illustration, we consider the case where $C < 0$ and $P > 0$. Since the latter cashflow is positive, the worst-case scenario will occur when the interest rate is always as high as possible between the two cashflow dates, T_1 and T_2. We must then examine the present value of the sum of the two cashflows at the first cashflow date. If this value is still positive, then the worst-case scenario will occur when the interest rate is always as high as possible from now until the first cashflow date. However, if this value is negative, then the worst-case scenario will occur when the interest rate is always as low as possible from now until the first cashflow date.

Under our simple model, r can instantaneously jump from one value to another. Therefore, in the worst-case scenario, the interest rate will be r^+ between T_1 and T_2. This is because it will be possible for the interest rate to jump to r^+ at time T_1, regardless of what value it takes between t and T_1. The present value of the second cashflow at time T_1 is then

$$Pe^{-r^+(T_2-T_1)}.$$

The present value of the two cashflows at time T_1 is therefore

$$C + Pe^{-r^+(T_2-T_1)}.$$

If $C + Pe^{-r^+(T_2-T_1)} > 0$, then the worst case-scenario occurs when the interest rate immediately jumps to r^+ (at time t) and stays there until T_2. The value of the bond at time t is therefore

$$Ce^{-r^+(T_1-t)} + Pe^{-r^+(T_2-t)}.$$

However, if $C + Pe^{-r^+(T_2-T_1)} < 0$, then the worst case-scenario occurs when the interest rate immediately jumps to r^- (at time t) and stays there until T_1 when it jumps to r^+ and remains there until T_2. The value of the bond at time t is then

$$Ce^{-r^-(T_1-t)} + Pe^{-r^-(T_1-t)-r^+(T_2-T_1)}.$$

This interest rate evolution is shown in Figure 2.2.

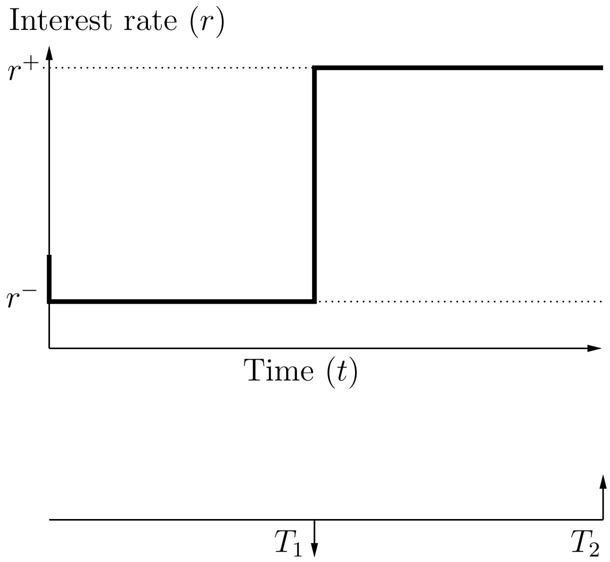


Figure 2.2: The interest rate path for a coupon bond under our simple model.

(We note that if $C + Pe^{-r^+(T_2-T_1)} = 0$, then the value of the bond today will be zero regardless of which path the interest rate takes between t and T_1. This is because it is the present value of zero at time T_1).

In this simple case, where r can instantaneously jump between values, the worst-case scenario interest rate only changes on a cashflow date and it is clear when this change should occur. Unfortunately, this 'bounded r' model is too broad to be useful. In the next section, we fine tune our model for the interest rate.

2.2 A model for the evolution of the interest rate

We now propose our non-probabilistic model for the evolution of the short-term interest rate. We assume that r is continuous and has a given initial value, $r(t)$. We do not describe a model for the actual behaviour of the short rate. Instead, we bound the possibilities by placing the following constraints on its movement:

$$r^- \leq r \leq r^+, \tag{2.3}$$

and

$$c^- \leq \frac{dr}{dt} \leq c^+. \tag{2.4}$$

Equation (2.3) states that the short rate is bounded above and below. For instance, we could say that the short rate must be at least 3% and no more than 20%. However, the two bounds, r^+ and r^-, can be time dependent.

Equation (2.4) places similar constraints on the change in the short rate. For instance, we could say that the short rate cannot increase or fall by more than 4% per annum. These two bounds, c^+ and c^-, can be dependent on both r and t. However, we assume that $c^+ > 0$ and $c^- < 0$.

We remark that this model does not appear to replicate the locally unbounded growth seen in the traditional stochastic models for the short rate and, to some extent, observed in practice. Since we are trying to model a long-term behaviour, we are not so concerned with these short-term movements as they will not significantly affect the worst-case price. However, in Section 6.1, we address the problem by considering certain modifications to our model. When we perform a statistical analysis of short-term interest rate data, we find that we can make this extended model indistinguishable from the actual underlying process.

2.2.1 Pricing a single cashflow

For the zero-coupon bond, with principal P at maturity T, there is still an obvious solution to the worst-case scenario valuation problem.

Since there is only a single cashflow to consider, it is again simple to identify the worst-case path for the interest rate. If the cashflow is positive, then the worst-case scenario will occur when the interest rate is always as high as possible, and if the cashflow is negative, then it will occur when the interest rate is always as low as possible.

Therefore, if $P > 0$ then the worst-case scenario occurs when r increases from its initial value as quickly as possible (i.e. $\frac{dr}{dt} = c^+$) until it reaches r^+. It then remains at this rate until maturity.

Similarly, if $P < 0$ then the worst-case scenario occurs when r decreases from its initial value as quickly as possible (i.e. $\frac{dr}{dt} = c^-$) until it reaches r^- and then stays there until maturity.

The zero-coupon bond value is then given by

$$Pe^{-\int_t^T r(\tau)d\tau},$$

where $r(t)$ is the realised short rate path for the particular case under consideration.

Figure 2.3 shows the evolution of the interest rate in a worst-case scenario for these two cases of zero-coupon bond valuation. In this and the following figure, we assume that r^-, r^+, c^- and c^+ are constants.

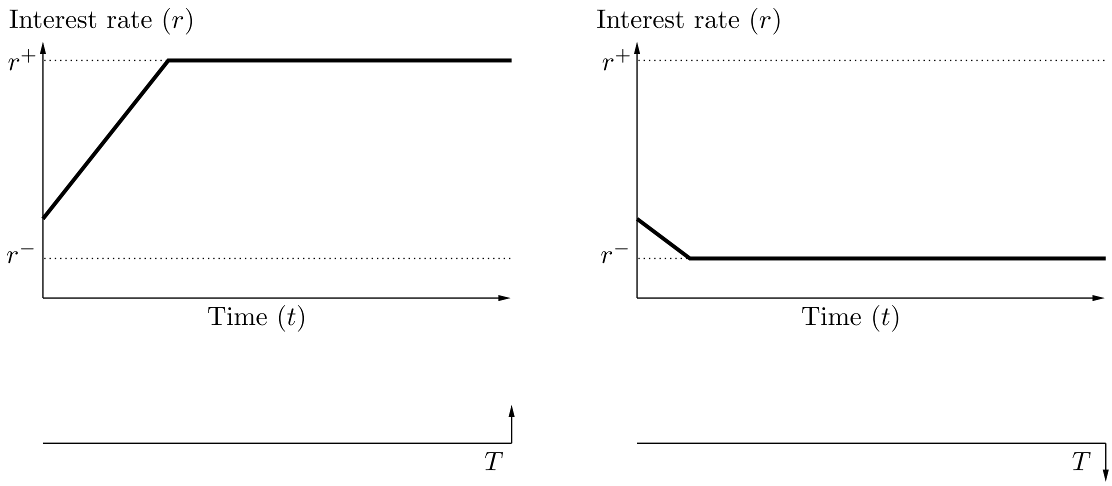


Figure 2.3: The interest rate paths for a zero-coupon bond under our non-probabilistic model

2.2.2 Pricing two cashflows

Now we consider the coupon bond, with a cashflow of C at time T_1 and P at maturity, T_2.

If both cashflows are of the same sign, then the problem again reduces to that of the previous section for a single cashflow. If the cashflows are positive, then the worst-case scenario will occur when the interest rate is always as high as possible, and if the cashflows are negative, then it will occur when the interest rate is always as low as possible.

Therefore, if $C > 0$ and $P > 0$ then the worst-case scenario occurs when r increases from its initial value as quickly as possible (i.e. $\frac{dr}{dt} = c^+$) until it reaches r^+ and then remains at this rate until maturity.

Similarly, if $C < 0$ and $P < 0$ then the worst-case scenario occurs when r decreases from its initial value as quickly as possible (i.e. $\frac{dr}{dt} = c^-$) until it reaches r^- and then stays there until maturity.

The coupon bond value is then given by

$$Ce^{-\int_t^{T_1} r(\tau)d\tau} + Pe^{-\int_t^{T_2} r(\tau)d\tau},$$

where $r(t)$ is the realised short rate path for the particular case under consideration.

But if the cashflows are of different sign, then the solution to the problem is far less obvious. By way of illustration, we again consider the case where $C < 0$ and $P > 0$. Since the latter cashflow is positive, it will have least value when the interest rate is always as high as possible (between t and T_2). On the other hand, the former cashflow is negative and will have least value when the interest rate is as low as possible between t and T_1.

If the interest rate were to remain at r^+ for the entire period T_1 to T_2 then the present value of the second cashflow at time T_1 would be

$$Pe^{-r^+(T_2-T_1)}.$$

If we include the first cashflow, then we find that the present value of the bond at time T_1 would be

$$C + Pe^{-r^+(T_2-T_1)}.$$

If this value is positive, then the worst-case scenario occurs when r is always as high as possible over the entire period until maturity. The evolution of r will consequently be the same as for the case with solely positive cashflows. (Note that we have assumed a growth rate such that r can grow from its original value to r^+ before T_1). However, if the value is negative, then in the worst-case scenario, we need r to be as low as possible for all times preceding T_1 and as high as possible for all times after T_1.

Unfortunately, r cannot instantly jump from one value to another under this model and it is not clear at what time r should start increasing from r^- to reach r^+. Figure 2.4 shows three plausible short rate evolutions. The outer paths are in some sense 'bounding paths'. These paths either begin or end the change from r^- to r^+ at time T_1. Since we want the interest rate to be as low as possible before T_1 and as high as possible afterwards, the worst-case scenario short rate path will lie within this region.

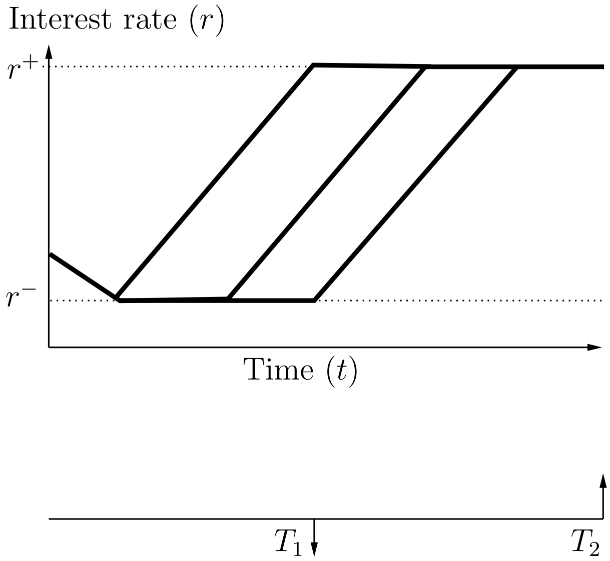


Figure 2.4: Possible interest rate paths for a coupon bond under our non-probabilistic model

However, there are still infinitely many possible paths, and only one may give the correct worst-case scenario value.

We must develop a method for establishing the realised interest rate over the period for which we want to perform a worst-case scenario valuation, or for directly calculating the value of a contract in this scenario. In the next section, we approach the problem from the perspective of the change in the contract's value over a small time step dt and obtain a differential equation for the worst-case price.

2.3 The differential equation for $V(r, t)$

2.3.1 The formulation of the differential equation

Let $V(r, t)$ be the value of our contract, when the short-term interest rate is r at time t. We consider the movement in the value of the contract over a time step dt.

Using Taylor's theorem to expand the value of the contract over a small time step dt and space step dr:

$$V(r + dr, t + dt) = V(r, t) + V_r(r, t)dr + V_t(r, t)dt + O(dr^2) + O(dt^2).$$

We note that, under our model, dr is bounded (from Equation (2.4)) in the form

$$c^- dt \leq dr \leq c^+ dt, \tag{2.5}$$

and so $dr = O(dt)$.

Hence, we approximate (to $O(dt)$)

$$dV = V(r + dr, t + dt) - V(r, t) = V_r(r, t)dr + V_t(r, t)dt.$$

We want to find the worst-case scenario value of this contract. This is the value of the contract when the short rate evolves, consist with the bounds of Equations (2.3) and (2.4), such that no other possible evolution would give the contract a lower value. Over a time step dt, this translates to the choice of dr such that the value of the contract increases by the minimum possible amount.

This worst-case increase must be equal to the risk-free increase. Otherwise, we could make an arbitrage profit on our belief that it is the worst-case scenario. We illustrate this point with the following example:

Consider the contract shown in Figure 2.5. Today, it is worth 1. There are five possible paths for the interest rate. Depending on which path the interest rate takes, the contract can have a final value ranging between 1.01 and 1.03. Over the time step, the risk-free increase is rdt.

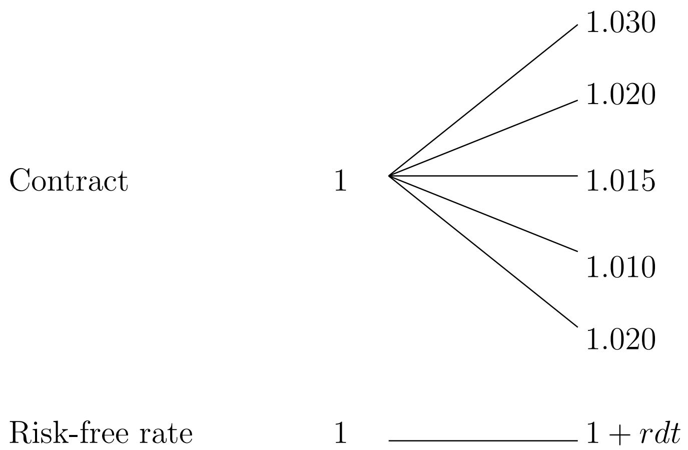


Figure 2.5: The worst-case increase and risk-free increase for a contract

The worst-case scenario path for the contract is the one that results in a final value of 1.01. This is an increase of 0.01.

If the risk-free increase were lower than this, then we could make a risk-free profit by borrowing from the bank and buying the contract. The contract would be guaranteed to increase by at least the worst-case amount and this would be greater than the interest owed to the bank.

On the other hand, if the risk-free increase were higher than 0.01, then we would own a contract that was going to increase, in our belief (that the worst-case scenario will occur), by less than the risk-free rate. This would lead to the contradiction that we would rather sell and invest the money in the bank than hold the contract in the first place.

We therefore have

$$\min(dV) = dV_{\text{worst case}} = rV\,dt.$$

Hence, we find

$$\min_{dr}(dV) = \min_{dr}\left(V_r dr + V_t dt\right) = rV\,dt.$$

Thus,

$$\min_{dr}\left(V_r dr + V_t dt\right) = rV\,dt.$$

Since dr is bounded by Equation (2.5), we can take the minimisation inside the brackets, to give

$$V_t + c\left(r, V_r\right) V_r - rV = 0, \tag{2.6}$$

where

$$c(r, X) = \begin{cases} c^+ & \text{if } X < 0 \\ c^- & \text{if } X > 0. \end{cases} \tag{2.7}$$

This is a first-order, nonlinear, hyperbolic partial differential equation (pde) for the contract value. We can solve this equation to value a contract with cashflows $C_i(r)$ at times T_i, for $i = 1, 2, \ldots, N$. We apply the last cashflow as final data for the pde,

$$V(r, T_N) = C_N(r), \tag{2.8}$$

and solve backwards in time from maturity, T_N, to the present day, t. Since the initial short rate is known, this solution contains the current worst-case price for the contract, $V(r, t)$. We remark that with this form for $c(r, V_r)$, our problem is similar in nature to a bang-bang optimal control problem [8], [40].

In the absence of arbitrage opportunities, V is everywhere continuous except at cash flow dates. If there is a cash flow $C_i(r)$ at time T_i, then a no-arbitrage assumption gives us that over the cash flow date,

$$V(r, T_i^-) = V(r, T_i^+) + C_i(r). \tag{2.9}$$

We first show how to solve the equation for an unbounded interval using the method of characteristics, when V_r is nonzero everywhere. We then consider the various cases in which $V_r = 0$ can occur on the unbounded interval, before finally examining the bounded problem.

2.4 The method of characteristics

To solve a hyperbolic partial differential equation analytically, we use the method of characteristics. Essentially, this method allows us to construct the solution surface as a family of characteristic curves which pass through a given curve of Cauchy data [62]. To illustrate the method, we consider the linear problem,

$$V_t + cV_r - rV = 0,$$

where c is some positive constant. We will solve the problem on the unbounded interval, $(-\infty, \infty)$, with final data $V(r,T) = \Lambda(r)$. We can rewrite this as Cauchy data for the problem, in the form

$$\Gamma(r,t,V) = (p, T, \Lambda(p)) \text{ for } -\infty < p < \infty, \tag{2.10}$$

where p measures distance along the data curve.

The characteristics for the problem are defined to be

$$\frac{dt}{1} = \frac{dr}{c} = \frac{dV}{rV} = ds, \tag{2.11}$$

where we have introduced a second parameter, s, to measure distance along the characteristic. The characteristic projections in the (r,t) plane are then

$$\frac{dr}{dt} = c. \tag{2.12}$$

It is possible to find a unique solution to this problem as long as the Cauchy data is not tangent to the characteristics. We can then construct a solution surface which is made up of characteristics which pass through the Cauchy data curve [55].

Since all of the jump and final conditions for our pde will be equations for the contract value in terms of r at a particular time, our Cauchy data will always be parallel to the t-axis in the (r,t) plane, in the direction $(0,1)$. We will therefore be able to solve the pde uniquely as long as the characteristics are never parallel to the t-axis. The characteristic path is given, according to Equation (2.12), by $dr/dt = c$, i.e. in the direction $(1,c)$. Consequently, there will be a unique solution as long as c is finite. Equation (2.4) guarantees that this will always be the case.

The characteristics for the problem are shown in Figure 2.6. They span the whole (r,t) plane for $t \leq T$ and we expect a well-defined solution in this region.

We can solve Equations (2.10) and (2.11) to find

$$t = s + T \ , \ r = cs + p \text{ and } V = \Lambda(p)e^{\frac{1}{2}cs^2 + ps}.$$

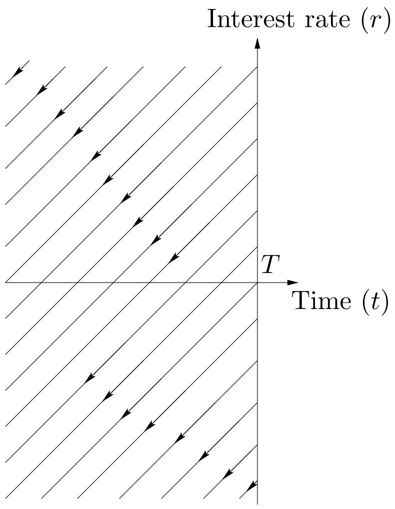


Figure 2.6: Characteristics for the linear problem

Since the Cauchy data is not parallel to the characteristics, we can invert the equations for (r, t) in terms of (s, p) to find (s, p) in terms of (r, t),

$$s = -(T - t) \text{ and } p = r + c(T - t).$$

We can then substitute for these into the equation for V to find

$$V = \Lambda(r + c(T - t))e^{-\frac{1}{2}c(T-t)^2 - r(T-t)}.$$

This solution holds for $s \leq 0$ and $-\infty < p < \infty$, i.e.

$$T - t \geq 0 \text{ and } -\infty < r + c(T - t) < \infty,$$

which covers the whole (r, t) space for $t \leq T$.

2.5 The characteristics for the nonlinear problem

The characteristics of Equation (2.6) are given by

$$\frac{dt}{1} = \frac{dr}{c(r, V_r)} = \frac{dV}{rV}. \tag{2.13}$$

The characteristic projections in the (r, t) plane are then

$$\frac{dr}{dt} = c(r, V_r). \tag{2.14}$$

We can solve this problem with final condition $V(r, T)$ as long as V_r is nonzero over the solution surface. If this is the case, the characteristics are well-defined by Equation (2.13) and span the solution space. We can then solve along these characteristics, using the final condition as Cauchy data.

If the Cauchy data is discontinuous, then the discontinuity will propagate along a characteristic. This is because any discontinuity in the solution of a hyperbolic partial differential equation must occur across a characteristic. To solve the problem, we simply 'patch together' the two classical solutions to the continuous problems either side of the characteristic [54].

However, as soon as there is a point at which $V_r = 0$, then we do not yet have a systematic method for constructing the characteristic path through the point. We cannot use Charpit's method [13] to improve the situation, as the approach does not simplify the dependence of c on V_r into a more tractable form. Instead, we must examine the various forms in which $V_r = 0$ can occur, and go back to the modelling of the problem to explain what happens to the characteristics.

We shall assume that the zero r-derivative occurs in our final data. If this not the case, then we can construct the characteristics and find a solution back until the time when $V_r(r,t) = 0$ first occurs and then consider this solution as our final data to proceed further back. To simplify matters, we shall initially only concern ourselves with the local problem around the point where the derivative is zero (i.e. away from the boundaries).

There are essentially two cases to consider:

- A maximum at r_T - where $V_r(r_T,T) = 0$, $V_r(r_T^-,T) > 0$ and $V_r(r_T^+,T) < 0$.
- A minimum at r_T - where $V_r(r_T,T) = 0$, $V_r(r_T^-,T) < 0$ and $V_r(r_T^+,T) > 0$,

where r_T^- and r_T^+ are an infinitessimal negative and positive distance away from r_T, respectively.

In the following work, we solve the pde backwards in time, from the final data. All discussion of the evolution of a solution, or the propagation of a turning point refers to the change as time to maturity increases (i.e. backwards in time).

2.5.1 $V_r = 0$ at a maximum

We consider the problem with $V(r,T) = f(r)$, where the final data has a maximum at (r_T,T). In this case, we have

$$f_r(r) > 0 \text{ for } r < r_T \text{ , } f_r(r) < 0 \text{ for } r > r_T \text{ and } f_r(r_T) = 0.$$

Therefore, for $r < r_T$, we have $c(r, V_r) = c^-$ and for $r > r_T$, we have $c(r, V_r) = c^+$. The characteristics are given by

$$\frac{dr}{dt} = c^-,$$

for $r < r_T$ at $t = T$ and by

$$\frac{dr}{dt} = c^+,$$

for $r > r_T$ at $t = T$.

There is a region, shown in Figure 2.7, in which points can be reached by two characteristics. Consequently, there will not be a unique solution in this region. In this and the following figures, we have set $c^+ = -c^-$ to be some positive constant, for ease of pictorial representation.

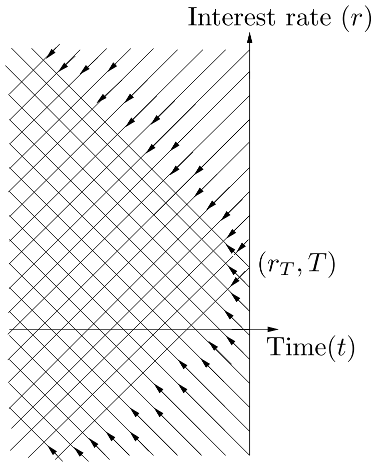


Figure 2.7: Multiple characteristics when there is a maximum at (r_T, T)

To find a unique solution, we must introduce a shock into the problem. The shock splits the solution space into two regions. In each region, there will only be a single set of characteristics and hence, a unique solution for V. To solve for the position of the shock, we have to go back to the modelling of the problem:

In the worst-case scenario, information (containing the solution) flows from interest rates where the contract value is low to those at which it is high. When there is a point at which the contract value has a maximum, then information flows into this point from both sides. This is shown in Figure 2.8, where we have ignored the effect of discounting and examine the evolution of the contract value in a worst-case scenario.

The information from each side leads to a different contract value and so there is a region which has a non-unique solution. Clearly, only one of these values is the worst-case value (the lower one). It is clear from the diagram that the changeover

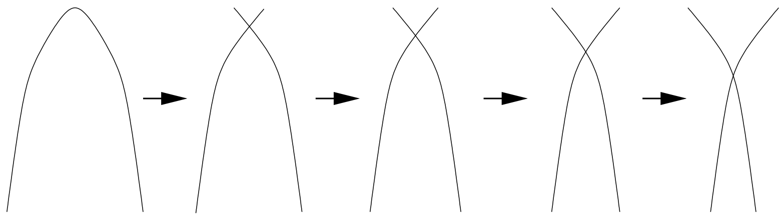


Figure 2.8: Evolution of a maximum without discounting

from one solution to the other, so that the contract value is always as low as possible, must occur such that the solution is continuous. The maximum will always be at the point where the two solutions meet.

Therefore, the condition that we must apply is that the solution for V is continuous. This is, in effect, an arbitrage argument, since the formation of a discontinuity would lead to arbitrage opportunities. With this information, we can use the following methodology to find the path of the shock. We split the Cauchy data into two sections, with the split at the maximum. We solve the two resulting problems individually. We then equate these two solutions and solve the resulting equation to find a relationship between r and t. This describes the path of the shock. On each side of the shock, we use the solution that came from the Cauchy data to that side. Across the shock, the solution is continuous, by definition. The path of the shock also describes the evolution of the maximum and so the characteristic picture is consistent with this evolution. In Figure 2.9, we show a typical set of characteristics for this problem.

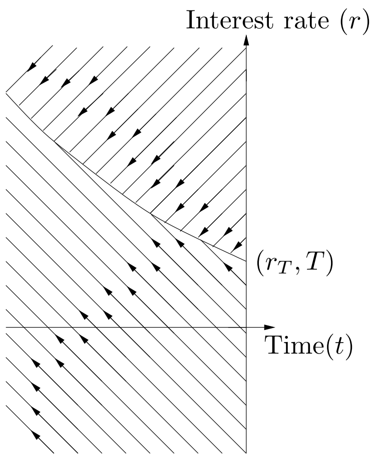


Figure 2.9: Characteristics when there is a maximum at (r_T, T) and we introduce a shock

2.5.2 The evolution of a maximum

Since the method of solution tracks the maximum, we can study the path that it takes. In the following work, we consider the case where c^+ and c^- are constants and examine the local problem around the maximum (i.e. away from the boundaries). The problem we solve is Equation (2.6) with final data

$$V(r,T) = \begin{cases} f_1(r) & \text{for } r \leq r_T \\ f_2(r) & \text{for } r > r_T, \end{cases}$$

where

$$\frac{df_1}{dr} > 0 \ , \ \frac{df_2}{dr} < 0 \text{ and } f_1(r_T) = f_2(r_T),$$

so that we have a continuous solution. This is shown in Figure 2.10.

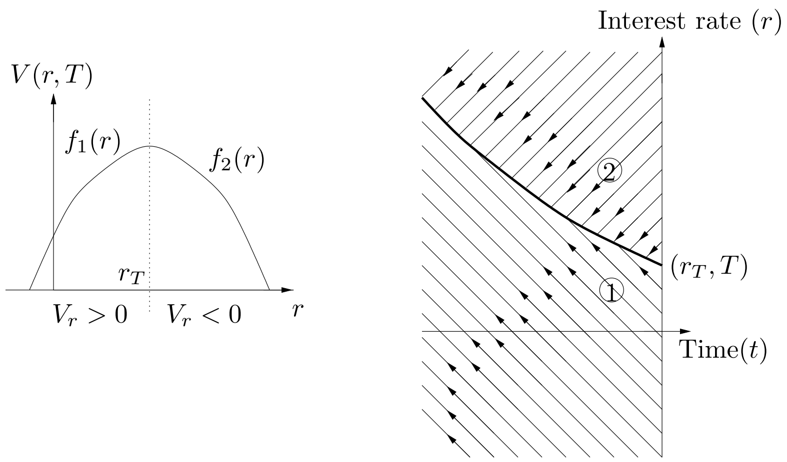


Figure 2.10: The local problem with a maximum at (r_T, T)

In region 1, the characteristics are defined by

$$dt = \frac{dr}{c^-} = \frac{dV_1}{rV_1} = ds,$$

and we have Cauchy data of

$$\Gamma_1(r,t,V_1) = (p, T, f_1(p)),$$

for $p < r_T$. We can solve this to find

$$V_1(r,t) = f_1(r + c^-(T-t))e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

for $r \leq r_T - c^-(T-t)$.

In region 2, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_2}{rV_2} = ds,$$

and we have Cauchy data of

$$\Gamma_2(r,t,V_2) = (p,T,f_2(p)),$$

for $p > r_T$. We can solve this to find

$$V_2(r,t) = f_2(r + c^+(T-t))e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

for $r \geq r_T - c^+(T-t)$.

To find the path of the maximum, we solve $V_1(r,t) = V_2(r,t)$. For small times before maturity, $T - t = \epsilon$, say, this is

$$f_1(r + c^-\epsilon)e^{-\frac{1}{2}c^-\epsilon^2 - r\epsilon} = f_2(r + c^+\epsilon)e^{-\frac{1}{2}c^+\epsilon^2 - r\epsilon},$$

i.e

$$f_1(r + c^-\epsilon) = f_2(r + c^+\epsilon)e^{-\frac{1}{2}(c^+ - c^-)\epsilon^2},$$

which, to $O(\epsilon)$, is

$$f_1(r + c^-\epsilon) = f_2(r + c^+\epsilon).$$

We can also find the first derivatives of the two solutions,

$$V_{1r}(r,t) = \left(\frac{df_1}{dr}(r + c^-(T-t)) - (T-t)f_1(r + c^-(T-t))\right)e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

and

$$V_{2r}(r,t) = \left(\frac{df_2}{dr}(r + c^+(T-t)) - (T-t)f_2(r + c^+(T-t))\right)e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

At a time ϵ before maturity, to $O(\epsilon)$,

$$V_{1r} = \left(\frac{df_1}{dr}(r + c^-\epsilon) - \epsilon f_1(r + c^-\epsilon)\right)e^{-r\epsilon},$$

and

$$V_{2r} = \left(\frac{df_2}{dr}(r + c^+\epsilon) - \epsilon f_2(r + c^+\epsilon)\right)e^{-r\epsilon}.$$

We consider two examples, a ‘linear’ maximum (with a discontinuous first derivative) and a ‘quadratic’ maximum (with a continuous first derivative).

2.5.2.1 A 'linear' maximum

In the linear case, we set

$$f_1(r) = a + b_1(r - r_T),$$

and

$$f_2(r) = a + b_2(r - r_T),$$

where $b_1 > 0$ and $b_2 < 0$. We then find that the equation of the shock and the evolution of the maximum are given by

$$r = r_T + \epsilon\left(\frac{c^+ b_2 - c^- b_1}{b_1 - b_2}\right),$$

We can express this movement locally, since the first derivative is discontinuous, as

$$r = r_T - \epsilon\frac{[cV_r]}{[V_r]},$$

where [.] represents the jump across the discontinuity.

The first derivatives of the two solutions are given by

$$V_{1r} = b_1 + \epsilon(-a - 2b_1 r + b_1 r_T),$$

and

$$V_{2r} = b_2 + \epsilon(-a - 2b_2 r + b_2 r_T).$$

We therefore find that the jump in derivative at the maximum is

$$V_{2r} - V_{1r} = (b_2 - b_1) - \epsilon(2r - r_T)(b_2 - b_1)),$$

which we can express as

$$= [V_r] - \epsilon([V] + r[V_r]).$$

2.5.2.2 A 'quadratic' maximum

In the quadratic case, we set

$$f_1(r) = a - b_1(r - r_T)^2,$$

and

$$f_2(r) = a - b_2(r - r_T)^2,$$

where $b_1 > 0$ and $b_2 > 0$. The equation of the shock and the evolution of the maximum are then given by

$$r = r_T + 2\epsilon\left(\frac{c^+ b_2 - c^- b_1}{b_1 - b_2}\right).$$

The first derivatives of the two solutions are

$$V_{1r} = -2b_1(r - r_T) + \epsilon\left(-a + b_1((r - r_T)^2 + r^2 - rr_T) - 2b_1c^-\right),$$

and

$$V_{2r} = -2b_2(r - r_T) + \epsilon\left(-a + b_2((r - r_T)^2 + r^2 - rr_T) - 2b_2c^+\right),$$

and we find that the jump in derivative at the maximum is given by

$$V_{2r} - V_{1r} = -2(b_2 - b_1)(r - r_T) + \epsilon\left((b_2 - b_1)(2r - r_T)(r - r_T) - 2b_2c^+ + 2b_1c^-\right).$$

2.5.3 $V_r = 0$ at a minimum

We now consider the problem with $V(r,T) = f(r)$, where the final data has a minimum at (r_T, T). In this case, we have

$$f_r(r) < 0 \text{ for } r < r_T\ ,\ f_r(r) > 0 \text{ for } r > r_T \text{ and } f_r(r_T) = 0.$$

Therefore, for $r < r_T$, we have $c(r, V_r) = c^+$ and for $r > r_T$, we have $c(r, V_r) = c^-$. The characteristics are given by

$$\frac{dr}{dt} = c^+,$$

for $r < r_T$ at $t = T$ and by

$$\frac{dr}{dt} = c^-,$$

for $r > r_T$ at $t = T$.

There is a region, shown in Figure 2.11, in which there are no characteristics. Consequently, solving along the characteristics does not find a solution in this region.

We must find a way of extending our solution into this middle section. Again, we refer back to the modelling of the situation. In the worst-case scenario, information flows away from the minimum, in both directions. This is shown in Figure 2.12, where we have ignored the effect of discounting and examine the evolution of the contract value in a worst-case scenario.

There is a gap in the middle of each of the diagrams. All the points in the gap can reach the minimum by maturity and so have a present value that is the value of this minimum. If the contract has positive value at the minimum, then the lowest of these will be the one associated with the highest interest rate, when we take discounting into account, and so the minimum propagates to higher interest rates. On the other hand, if the contract has negative value at the minimum, then it will propagate to lower interest rates.

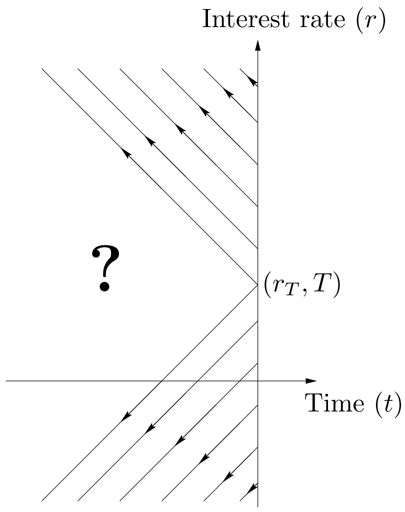


Figure 2.11: The incomplete set of characteristics originating from a minimum

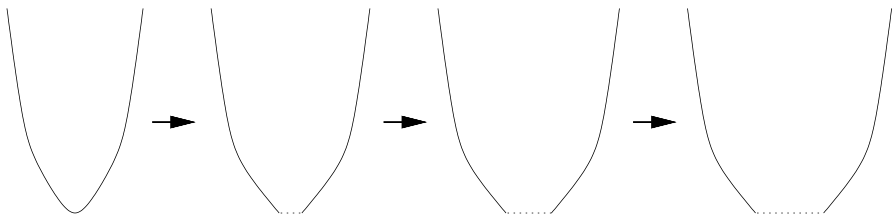


Figure 2.12: Evolution of a minimum without discounting

Consequently, we can consider the region in question as the area within which it is possible for the interest rate to evolve so that it reaches the minimum at maturity. Since this is a worst-case scenario, the worst-case interest rate paths will all start at this minimum. Moreover, the paths will follow the evolution of the minimum as far back from maturity as they can. They are constrained by their final condition (where in the solution space they must end up).

There will therefore be a set of characteristics which span the inner solution space. These characteristics will all propagate from the path taken by the minimum. We use the solution on this curve as Cauchy data for the characteristics. First of all, we must find the path for the evolution of the minimum. To do this we re-examine Equation (2.6).

We are interested in the evolution of the minimum from (r_T, T). This path has equation $r = R(t)$ say, where $R(T) = r_T$. To follow the path, we set

$$c(r, V_r) = \frac{dR(t)}{dt},$$

and find

$$V_t + \frac{dR}{dt} V_r - rV = 0,$$

with final data $V(r, T) = f(r)$.

We set $\tau = T - t$ and $\xi = r - R(T - \tau)$ to examine times close to maturity and the local problem around the minimum. The equation becomes

$$V_\tau + (\xi + R(T - \tau))V = 0,$$

which we can integrate to get

$$V = f(\xi + R(T))e^{\int_0^\tau (\xi + R(T-s))ds}.$$

We can then transform back to our original variables to find

$$V = f(r - R(t) + r_T)e^{-\int_{T-t}^T (r - R(t) + R(T-s))ds}.$$

On the path in question, $r = R(t)$, and

$$V = f(r_T)e^{-\int_{T-t}^T R(T-s)ds}.$$

For the worst case scenario, the evolution of the interest rate will minimise V. If $f(r_T) > 0$ we will want R to be as high as possible over the time period and therefore have

$$\frac{dR}{dt} = c^-,$$

(since we are constrained to end up at (r_T, T)) and if $f(r_T) < 0$ we will have

$$\frac{dR}{dt} = c^+.$$

In summary, to find the solution for V in the inner region, we find the solution along the path taken by the minimum and use this as Cauchy data for the inner region, as shown in Figure 2.13. The minimum itself propagates along the bounding characteristic of one of the outer two regions, and has equation

$$\frac{dr}{dt} = c^-,$$

when $V(r_T, T) > 0$, and

$$\frac{dr}{dt} = c^+,$$

when $V(r_T, T) < 0$.

If $V(r_T, T) = 0$ then the inner region has solution $V = 0$, because the initial minimum of zero is the lowest value attainable (it cannot be discounted to a lower level) and if we start anywhere within the region, we can reach the point (r_T, T) at maturity and have a final value of zero for the contract.

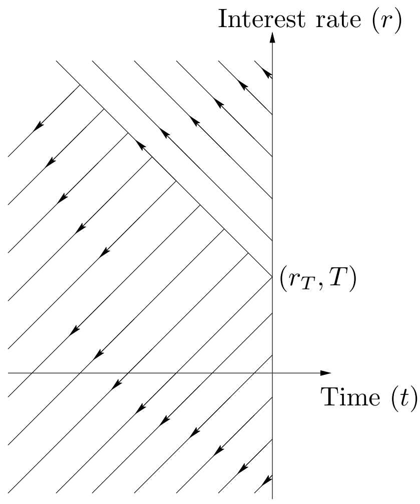


Figure 2.13: The complete set of characteristics with a minimum at (r_T, T)

2.5.4 The effect of a minimum on the solution

We can study the behaviour of the solution at the 'edges' of the region which the minimum can affect (along the bounding characteristics for the three regions). In the following work, we again consider the case where c^+ and c^- are constants and examine the local problem. We solve Equation (2.6) with final data

$$V(r,T) = \begin{cases} f_1(r) & \text{for } r \leq r_T \\ f_2(r) & \text{for } r > r_T, \end{cases}$$

where

$$\frac{df_1}{dr} < 0 \ , \ \frac{df_2}{dr} > 0 \text{ and } f_1(r_T) = f_2(r_T),$$

so that we have a continuous problem. This is shown in Figure 2.14. We will assume that the minimum occurs at a positive contract value (i.e. $f_1(r_T) > 0$).

In region 1, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_1}{rV_1} = ds,$$

and we have Cauchy data of

$$\Gamma_1(r,t,V_1) = (p,T,f_1(p)),$$

for $p < r_T$. We can solve this to find

$$V_1(r,t) = f_1(r + c^+(T-t))e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

for $r \leq r_T - c^+(T-t)$.

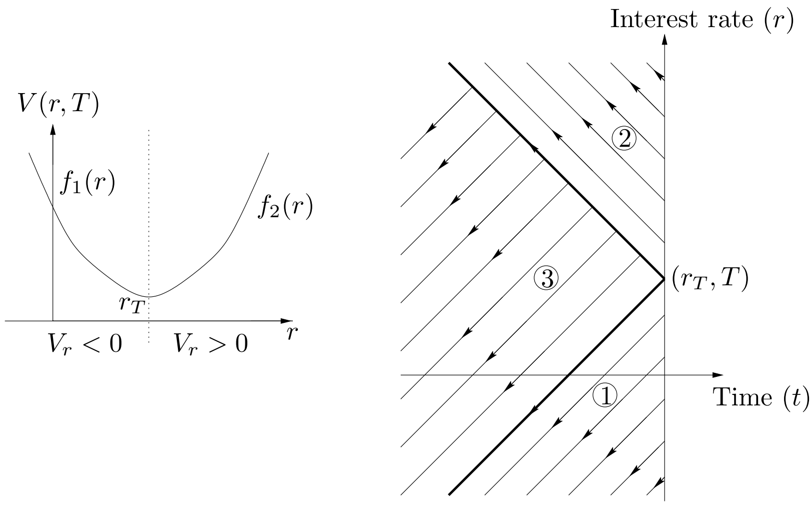


Figure 2.14: The local problem with a minimum at (r_T, T)

In region 2, the characteristics are defined by

$$dt = \frac{dr}{c^-} = \frac{dV_2}{rV_2} = ds,$$

and we have Cauchy data of

$$\Gamma_2(r, t, V_2) = (p, T, f_2(p)),$$

for $p > r_T$. We can solve this to find

$$V_2(r,t) = f_2(r + c^-(T-t))e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

for $r \geq r_T - c^-(T-t)$.

The minimum propagates along

$$r = r_T - c^-(T-t).$$

Substituting into the solution for V_2, we find that on this line, the solution is given by

$$f_2(r_T)e^{\frac{1}{2}c^-(T-t)^2 - r_T(T-t)}.$$

This gives us Cauchy data for region 3 of

$$\Gamma_3(r,t,V) = \left(r_T - c^-p, T - p, f_2(r_T)e^{\frac{1}{2}c^-p^2 - r_Tp}\right),$$

for $0 \leq p \leq T - t$. The characteristics in this region are given by

$$dt = \frac{dr}{c^+} = \frac{dV_3}{rV_3} = ds.$$

We can solve this to find

$$V_3(r,t) = f_2(r_T)e^{\frac{1}{2}c^-(T-t)^2 - r_T(T-t) + \frac{1}{2}\frac{(c^-(T-t)+(r-r_T))^2}{c^+-c^-}},$$

for

$$r_T - c^+(T-t) \leq r \leq r_T - c^-(T-t).$$

The first derivatives of these solutions are

$$V_{1r}(r,t) = \left(\frac{df_1}{dr}(r + c^+(T-t)) - (T-t)f_1(r + c^+(T-t)\right) e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

$$V_{2r}(r,t) = \left(\frac{df_2}{dr}(r + c^-(T-t)) - (T-t)f_2(r + c^-(T-t)\right) e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

and

$$V_{3r}(r,t) = f_2(r_T)\left(\frac{c^-(T-t) + (r - r_T)}{c^+ - c^-}\right) e^{\frac{1}{2}c^-(T-t)^2 - r_T(T-t) + \frac{1}{2}\frac{(c^-(T-t)+(r-r_T))^2}{c^+-c^-}}.$$

There are two lines along which we may expect to find some change in the first derivative of the solution. These are the characteristics that form the boundaries between the three solution regions. They are

$$r = r_T - c^+(T-t),$$

where the jump in derivative is $V_{3r} - V_{1r}$, between regions 1 and 3, and

$$r = r_T - c^-(T-t),$$

where the jump in derivative is $V_{2r} - V_{3r}$, between regions 2 and 3.

We find that

$$V_{3r} - V_{1r} = -\frac{df_1}{dr}(r_T)e^{-r_T(T-t) + \frac{1}{2}c^+(T-t)^2},$$

on $r = r_T - c^+(T-t)$, and

$$V_{2r} - V_{3r} = \left(\frac{df_2}{dr}(r_T) - (T-t)f_2(r_T)\right) e^{-r_T(T-t) + \frac{1}{2}c^-(T-t)^2},$$

on $r = r_T - c^-(T-t)$.

Again, we consider two examples, a 'linear' minimum (with a discontinuous first derivative) and a 'quadratic' minimum (with a continuous first derivative).

2.5.4.1 A 'linear' minimum

In the linear case, we set

$$f_1(r) = a + b_1(r - r_T),$$

and

$$f_2(r) = a + b_2(r - r_T),$$

where $a > 0$, $b_1 < 0$ and $b_2 > 0$. We find that

$$V_{3r} - V_{1r} = -b_1 e^{-r_T(T-t)+\frac{1}{2}c^+(T-t)^2}$$

along $r = r_T - c^+(T-t)$ and a small time before maturity, ϵ, this is equal to

$$-b_1 + b_1 r_T \epsilon + \ldots$$

We also find that along $r = r_T - c^-(T-t)$,

$$\begin{aligned} V_{2r} - V_{3r} &= (b_2 - a(T-t))\, e^{-r_T(T-t)+\frac{1}{2}c^-(T-t)^2} \\ &= b_2 - (a + r_T b_2)\epsilon + \ldots \end{aligned}$$

2.5.4.2 A 'quadratic' minimum

In the quadratic case, where

$$f_1(r) = a + b_1(r - r_T)^2,$$

and

$$f_2(r) = a + b_2(r - r_T)^2,$$

with $a > 0$, $b_1 > 0$ and $b_2 > 0$, we find that

$$V_{3r} - V_{1r} = 0,$$

(and hence there is a smooth join) along $r = r_T - c^+(T-t)$, and

$$\begin{aligned} V_{2r} - V_{3r} &= -a(T-t)e^{-r_T(T-t)+\frac{1}{2}c^-(T-t)^2} \\ &= -a\epsilon + \ldots \end{aligned}$$

along $r = r_T - c^-(T-t)$.

2.5.5 Multiple maxima and minima

We have shown how to construct the local characteristic surface around a maximum or minimum. Of course, it is possible that a problem could have more than a single point at which $V_r = 0$. In these circumstances, we piece together the characteristic picture using the knowledge we have about the characteristic behaviour around the points in question.

The exact nature of the characteristics will depend on the precise problem that we are to solve. As an example, we consider the effect of having a maximum and a minimum in close proximity. We assume that the minimum is at a positive contract value so that it propagates to higher interest rate values, and that the maximum is such that it propagates to lower interest rate values. There are two cases to consider, as shown in Figure 2.15.

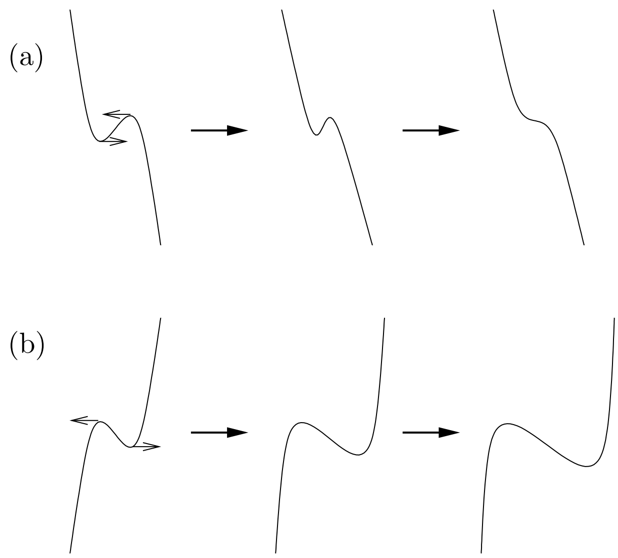


Figure 2.15: Evolution of a pair of a maximum and a minimum

In case (a), the maximum and minimum move towards each other, and 'cancel each other out' when they meet. In case (b), they move away from each other. The characteristics for both of these possibilities are shown in Figure 2.16. We remark that in case (a), the characteristic along which the minimum propagates meets the shock along which the maximum propagates. At this point, the two disappear and we are left with the simple characteristic picture for the situation when $V_r < 0$ everywhere.

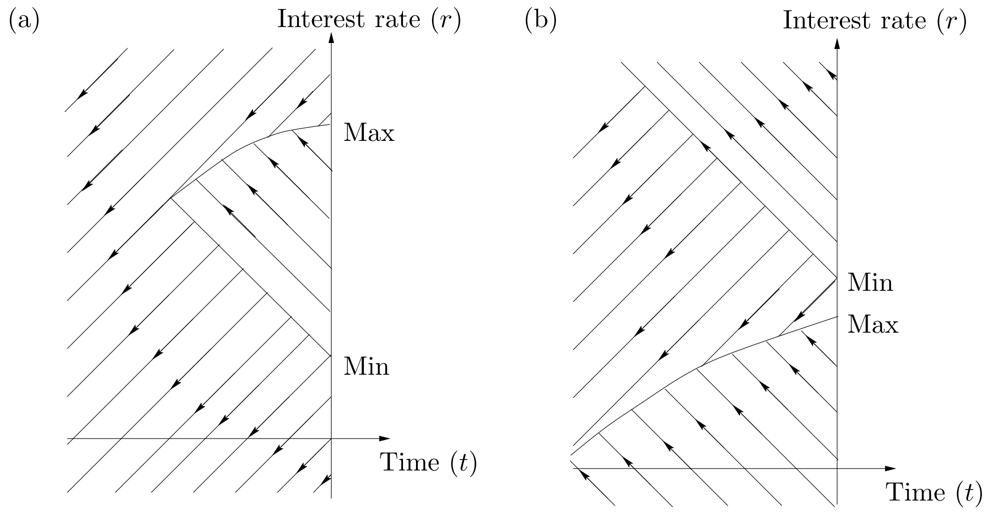


Figure 2.16: Characteristics for the pair of a maximum and a minimum

2.5.6 Other possible occurrences of $V_r = 0$

There are many other circumstances in which there can be a point or region where $V_r = 0$, without the existence of any maxima or minima.

For instance, consider the problem with $V(r, T) = f(r)$, where the final data has an inflection point at (r_T, T). In this case, we must have either

$$f_r(r) < 0 \text{ for } r \neq r_T \text{ and } f_r(r_T) = 0, \tag{2.15}$$

or

$$f_r(r) > 0 \text{ for } r \neq r_T \text{ and } f_r(r_T) = 0. \tag{2.16}$$

Alternatively, we could have a whole region with zero-derivative, e.g.

$$f_r(r) < 0 \text{ for } r < r_1 \text{ , } f_r(r) < 0 \text{ for } r_1 \leq r \leq r_2 \text{ and } f_r(r) > 0 \text{ for } r < r_2.$$

Naively, we may think that we can use a similar form of argument to that for the maxima and minima. If we apply this to the inflection point problem, we find the following:

In the first case (2.15), for $r \neq r_T$, we have $c(r, V_r) = c^+$. The characteristics are given by

$$\frac{dr}{dt} = c^+,$$

for $r \neq r_T$ at $t = T$.

The characteristic originating from the inflection point must then be of the same form as those just above and below it. If it were not, then the characteristics would cross, creating a non-unique solution. Hence, the characteristic originating from (r_T, T) has equation $\frac{dr}{dt} = c^+$. Financially, it is clear that in the worst-case, the interest rate will always decrease and so information travels to lower interest rates as we go backwards in time. The information from the inflection point logically has to travel in the same direction.

The characteristics for the problem are shown in Figure 2.17.

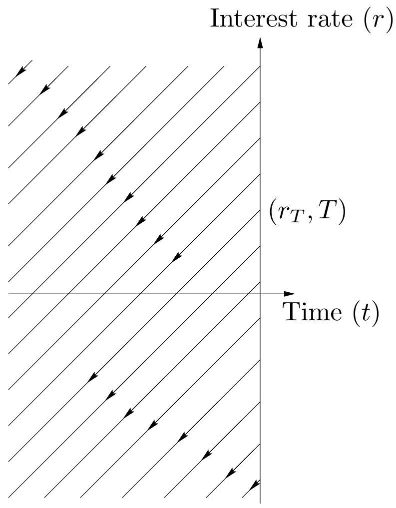


Figure 2.17: Characteristics with an inflection point at (r_T, T)

Similarly, in the second case (2.16), where $V_r(r, T) < 0$ except at the inflection point, the characteristic originating from the inflection point must be of the same form as those just above and below it. Hence, the characteristic originating from (r_T, T) has equation $\frac{dr}{dt} = c^-$.

However, this method only predicts the correct characteristic picture for an instant in time. The method works for maxima and minima because it tracks their propagation and because maxima or minima persist until they either reach a boundary or collide. Inflection points, in contrast, only exist for a moment in time. As soon as the contract value is discounted slightly, they disappear.

To illustrate this, we consider Figure 2.18. This diagram shows the evolution of various contract values with inflection points, where the inflection point occurs at either a positive or negative contract value.

In all four cases, the inflection point instantaneously disappears. in cases (a) and (d) we are left with a problem where V_r is one-signed, which we can easily solve. In the other two cases, the inflection point turns into a pair of a maximum and a minimum which then propagate away from each other. The characteristics for the four cases are shown in Figure 2.19.

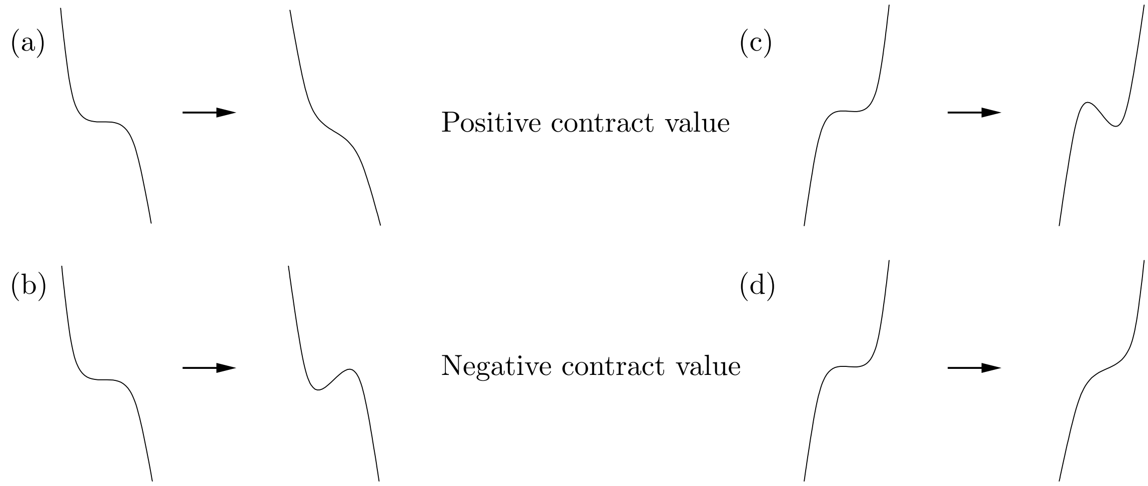


Figure 2.18: Evolution of various inflection points

We could compare cases (a) and (d) to (b) and (c) by thinking of them as cases where the inflection point also turns into a pair of a maximum and a minimum. The only difference is that in (a) and (d), they propagate towards each other and so immediately 'cancel out'.

In summary, if there is an inflection point, or region with zero derivative, in our problem, then we must piece together a suitable characteristic picture. This picture will depend on the exact specification of the problem to be solved.

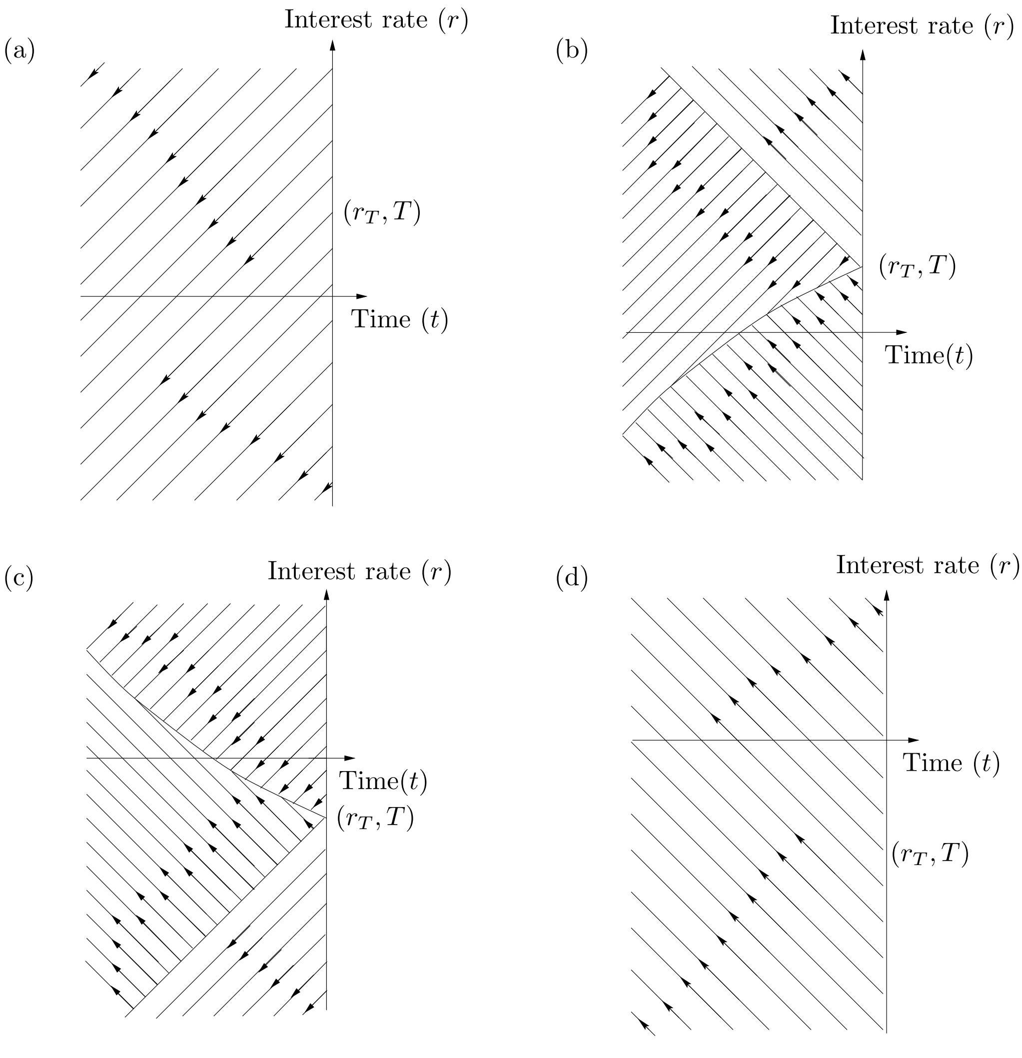


Figure 2.19: Characteristics with various inflection points

2.6 Behaviour at the interest rate boundaries

We do not have to impose interest rate boundary conditions on our pde, per se, since the first-order hyperbolic problem is well-posed without them. However, we must still determine the characteristic picture near to a boundary. There are two situations to consider: when the characteristics are flowing into the boundary, and when they are flowing away from it. Figure 2.20 includes examples of both of these situations. (This is in fact a possible characteristic picture for a zero-coupon bond).

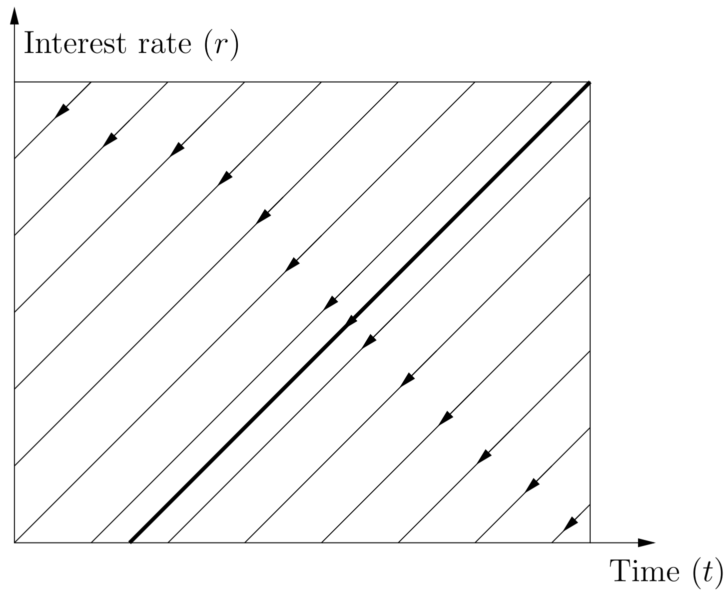


Figure 2.20: Characteristics with boundaries

At the lower end, the characteristics flow from our final data into the lower boundary. If we solve the pde with Cauchy data from the final condition, then we can find a unique solution on the lower boundary.

However, at the upper end, the characteristics flow out from the boundary. We must therefore specify Cauchy data on the upper boundary and use this to find a solution in the inner region by propagating characteristics from the boundary. The bold characteristic in the figure is the bounding characteristic that separates the solution found from the Cauchy data at maturity and that from the Cauchy data on the upper boundary. In this case, the latter data will be the discounted value of the contract when the interest rate remains at the upper boundary until maturity.

It is possible for minima or maxima to propagate into or away from a boundary, in which case, we just incorporate the relevant characteristic picture from the previous section. In the next chapter, we determine the analytical solution for a number of problems, which include examples of this behaviour.

Chapter 3

Analytical solutions of the bounded problem

This chapter contains examples of the techniques of the previous chapter applied in practice. We solve our hyperbolic partial differential equation on a bounded interval, with various forms of positive final data. The examples we choose are of a derivative solely of one sign (both positive and negative) and final data containing an internal maximum or minimum.

We assume that c^+ and c^- are positive and negative constants, respectively, and use the method of characteristics to solve our partial differential equation.

3.1 The general methodology

We examine the final data for our problem and construct the relevant characteristic picture, according to the 'rules' of the previous chapter. We then solve the pde along these characteristics. Finally, we check that our solution does not contain any point at which $V_r = 0$ (excluding those already accounted for). Otherwise we must identify the turning point in question and adjust our characteristic picture accordingly.

In practice, it is possible to inspect the final data and have some idea of where additional turning points might arise. We do this by considering how the final data will discount. Clearly, if the data has an internal turning point, then this will evolve as discussed. However, there are other possibilities which we now consider.

All of our examples have positive final data. When we discount a positive contract value, it decreases faster at higher interest rates. Consequently, V_r also decreases. If the final data for our problem is such that $V_r(r,T) < 0$ everywhere, then we can be sure that we will have $V_r(r,t) < 0$ at all times.

This means that if we have a maximum at the lower boundary then it will remain at the lower boundary (since the effect of discounting is least at the lower interest rate boundary on a positive contract value). Similarly, if there is a minimum at the upper boundary, then it will remain there (since the effect of discounting is always greatest at the highest interest rate).

On the other hand, if we have a minimum at the lower boundary, then it will propagate to higher interest rates, as discussed in Section 2.5.3. The situation is not quite so clear for a maximum at the upper boundary. In this case, information flows into the maximum from lower interest rates. Without discounting, the maximum would therefore remain at the upper boundary. However the effect of discounting is greatest at this boundary. Consequently, the contract value may decrease to a lower value at the upper boundary than that of nearby lower interest rates, in which case the maximum propagates into the solution region. We must therefore check to see if there is a point where $V_r = 0$ on the upper boundary. If there is, then we have to consider the possibility that the maximum may propagate into the region.

3.1.1 A note on the figures

In the characteristic pictures of this chapter, we use bold to highlight the lines on which we specify Cauchy data and dashed bold for the boundaries between solution regions.

In the solution figures, since the contract values are all positive and discount to lower values, the lower the curve, the higher the time to maturity.

3.2 $V_r(r, T) < 0$ everywhere

We first consider the solution of our equation when our final data has a negative derivative. In this case, $V(r, T) = f(r)$, where $\frac{df}{dr} < 0$. We do not expect there to be any turning points in the solution and can draw the characteristic picture, as shown in Figure 3.1.

There are two solution regions. In both, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV}{rV} = ds.$$

In region 1, we have Cauchy data, from the final data, of

$$\Gamma_1(r, t, V_1) = (p, T, f(p)),$$

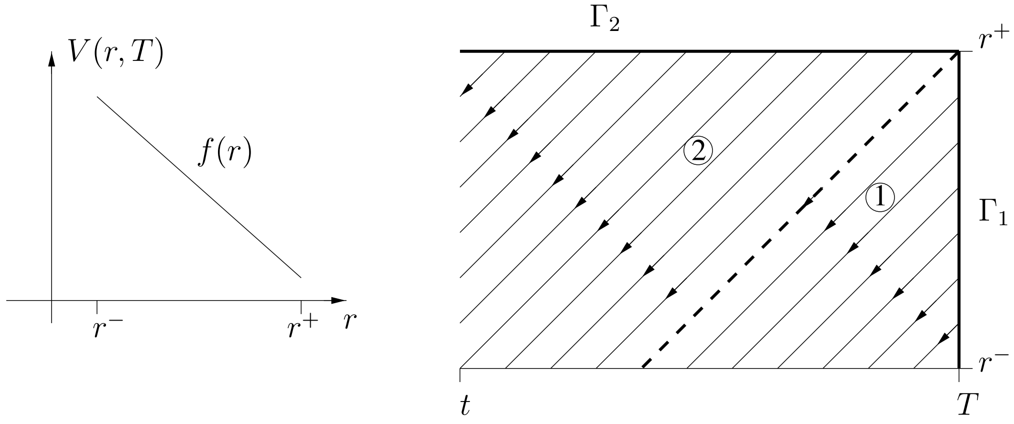


Figure 3.1: Characteristics when $V_r(r,T) < 0$

for $r^- \leq p \leq r^+$. We can solve this to find

$$V_1(r,t) = f(r + c^+(T-t))e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

for $r^- - c^+(T-t) \leq r \leq r^+ - c^+(T-t)$.

In region 2, the Cauchy data comes from the boundary at r^+. The contract value on this boundary is the value of the contract at r^+ at time T, $f(r^+)$, discounted at the rate r^+ over the time to maturity. This is

$$V(r^+,t) = f(r^+)e^{-r^+(T-t)}.$$

The Cauchy data on the boundary is therefore

$$\Gamma_2(r,t,V_2) = \left(r^+, p, f(r^+)e^{-r^+(T-p)}\right),$$

for $0 \leq p \leq T$. We can solve this to find

$$V_2(r,t) = f(r^+)e^{-r^+(T-t) + \frac{1}{2c^+}(r^+ - r)^2},$$

for $r^+ - c^+(T-t) \leq r \leq r^+ + c^+t$.

The solution to the problem is then

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \leq r \leq \max(r^-, r^+ - c^+(T-t)) \\ V_2(r,t) & \text{for } \max(r^-, r^+ - c^+(T-t)) \leq r \leq r^+, \end{cases}$$

since we are on the bounded interval, $r^- \leq r \leq r^+$.

In Figure 3.2 we show the solution with various times to maturity, where

$$f(r) = 0.3 - r,$$

and

$$r^- = 0.03\ ,\ r^+ = 0.2\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

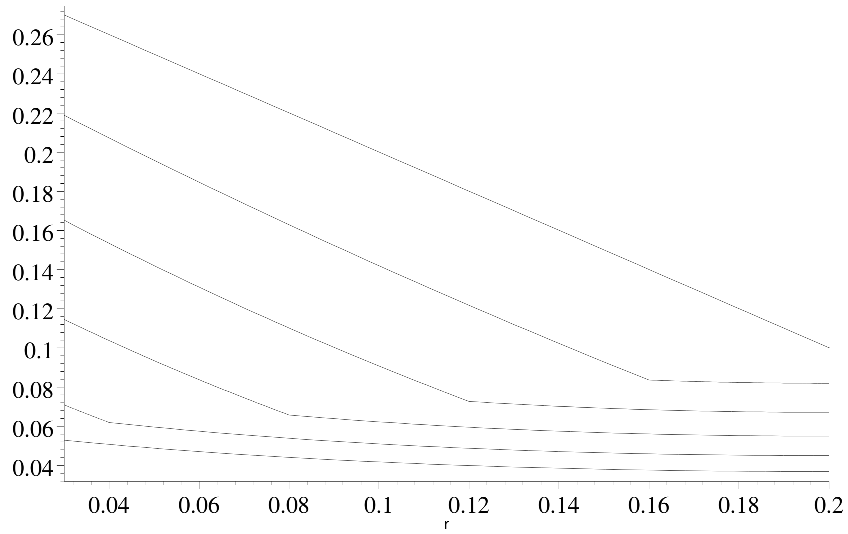


Figure 3.2: Contract value with 0, 1, 2, 3, 4, 5 years to maturity

3.3 $V_r(r, T) > 0$ everywhere

We now consider the solution to our problem when we have final data with a positive derivative, i.e. $V(r, T) = f(r)$, where $\frac{df}{dr} > 0$.

We expect the minimum at the lower boundary to propagate towards the upper boundary. The maximum at the upper boundary may remain there until the minimum reaches the upper boundary (Case I) or may propagate into the solution region at a time before the minimum reaches the upper boundary (Case II). In this case, the minimum and maximum will collide in the interior region.

We solve the problem assuming that we are in Case I and check that we do not have $V_r = 0$ on the upper boundary. If we do find a point where $V_r = 0$, then we must resolve the problem assuming that we are in Case II.

3.3.1 Case I

The characteristic picture for Case I is shown in Figure 3.3.

In region 1, the characteristics are defined by

$$dt = \frac{dr}{c^-} = \frac{dV_1}{rV_1} = ds,$$

and we have Cauchy data, from the final data, of

$$\Gamma_1(r, t, V_1) = (p, T, f(p)),$$

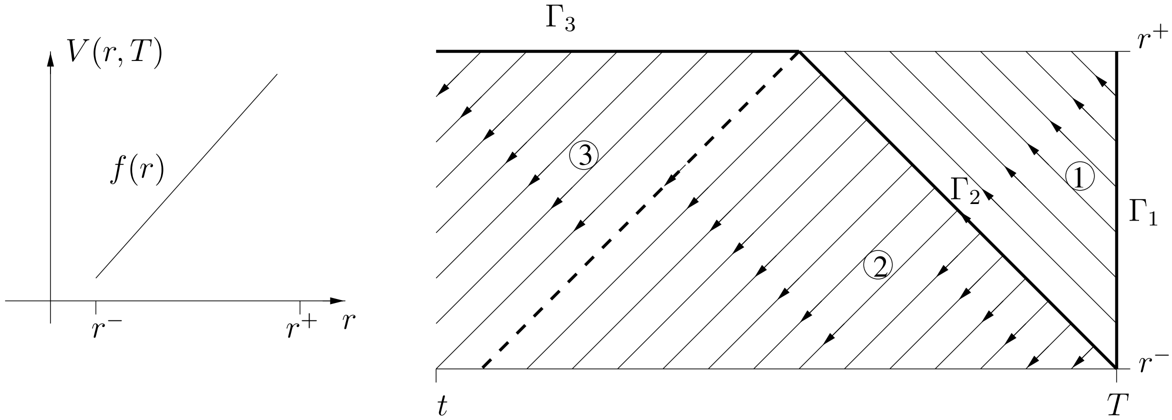


Figure 3.3: Possible characteristics when $V_r(r,T) > 0$: Case I

for $r^- \leq p \leq r^+$. We can solve this to find

$$V_1(r,t) = f(r + c^-(T-t))e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

for $r^- - c^-(T-t) \leq r \leq r^+ - c^-(T-t)$.

In region 2, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_2}{rV_2} = ds,$$

and we have Cauchy data specified along $r = r^- - c^-(T-t)$ from $t = T$ to $t = T + \frac{r^+ - r^-}{c^-}$. Along this line, the solution from region 1 is

$$V = f(r^-)e^{\frac{1}{2}c^-(T-t)^2 - r^-(T-t)}.$$

We therefore have Cauchy data of

$$\Gamma_2(r,t,V_2) = \left(r^- - c^-(T-p), p, f(r^-)e^{\frac{1}{2}c^-(T-p)^2 - r^-(T-p)}\right),$$

for $T + \frac{r^+ - r^-}{c^-} \leq p \leq T$. We can solve this to find

$$V_2(r,t) = f(r^-)e^{\frac{1}{2}c^-(T-t)^2 - r^-(T-t) + \frac{1}{2}(r - r^- + c^-(T-t))^2/(c^+ - c^-)},$$

for $r^- - c^+(T-t) \leq r \leq r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r^-)$.

In region 3, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_3}{rV_3} = ds,$$

and we have Cauchy data specified along $r = r^+$. The bounding characteristic at the end of region 1 reaches r^+ at time $T + \frac{r^+ - r^-}{c^-}$. The solution at this point is

$$V\left(r^+, T + \frac{r^+ - r^-}{c^-}\right) = f(r^-)e^{-\frac{1}{2c^-}(r^+ - r^-)^2 + r^+\left(\frac{r^+ - r^-}{c^-}\right)}.$$

The Cauchy data for region 3 is then this value discounted at a rate of r^+,

$$\Gamma_3(r, t, V_3) = \left(r^+, p, V\left(r^+, T + \frac{r^+ - r^-}{c^-}\right) e^{-r^+\left(T + \frac{r^+ - r^-}{c^-} - p\right)}\right),$$

for $0 \leq p \leq T + \frac{r^+ - r^-}{c^-}$. We can solve this to find

$$V_3(r, t) = f(r^-) e^{-r^+(T-t) - \frac{1}{2c^-}(r^+ - r^-)^2 + \frac{1}{2c^+}(r^+ - r)^2},$$

for $r^+ - c^+(T - t) - \frac{c^+}{c^-}(r^+ - r^-) \leq r \leq r^+ + c^+ t$.

The solution is therefore

$$V(r, t) = \begin{cases} V_1(r, t) & \text{for } r^- - c^-(T - t) \leq r \leq r^+ \\ V_2(r, t) & \text{for } r^- \leq r \leq r^- - c^-(T - t), \end{cases}$$

if $T + \frac{r^+ - r^-}{c^-} < t \leq T$, and

$$V(r, t) = \begin{cases} V_2(r, t) & \text{for } r^- \leq r \leq \max(r^-, r^+ - c^+(T - t) - \frac{c^+}{c^-}(r^+ - r^-)) \\ V_3(r, t) & \text{for } \max(r^-, r^+ - c^+(T - t) - \frac{c^+}{c^-}(r^+ - r^-)) \leq r \leq r^+, \end{cases}$$

if $t \leq T + \frac{r^+ - r^-}{c^-}$.

The solution with various times to maturity is shown in Figure 3.4, where

$$f(r) = 0.1 + r,$$

and

$$r^- = 0.03 \;,\; r^+ = 0.2 \;,\; c^- = -0.04 \text{ and } c^+ = 0.04.$$

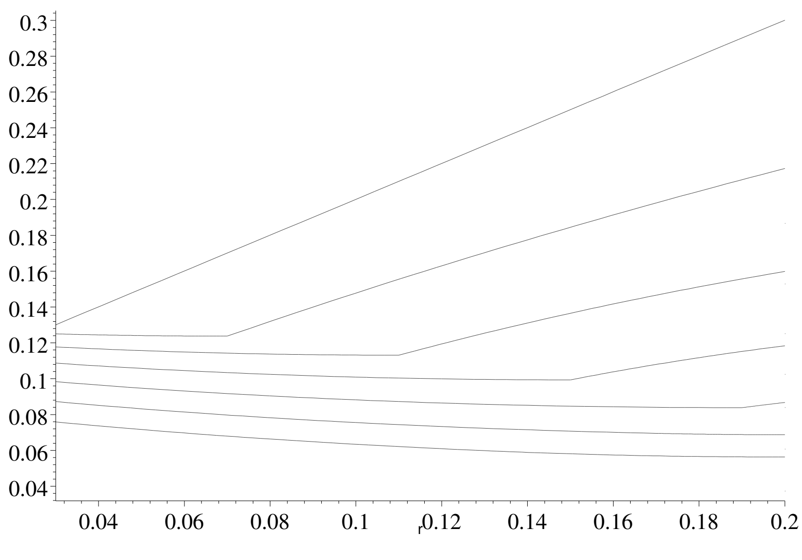


Figure 3.4: Contract value with 0, 1, 2, 3, 4, 5, 6 years to maturity

3.3.2 Case II

The characteristics when the maximum propagates into the region from the upper boundary are shown in Figure 3.5, where we assume that we first find $V_r = 0$ at (r^+, t^*) say.

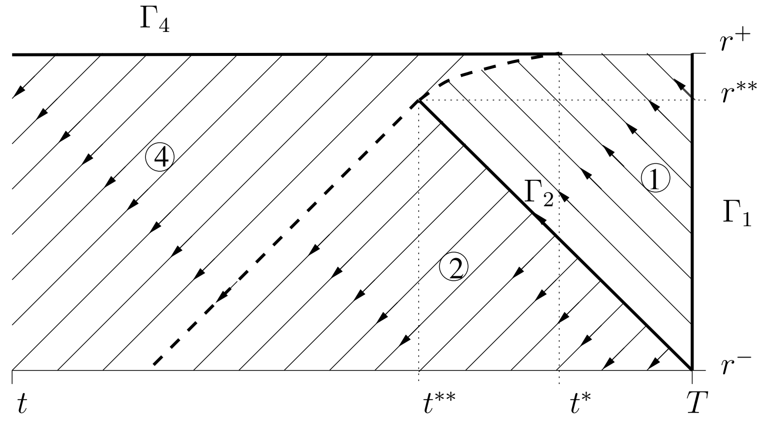


Figure 3.5: Possible characteristics when $V_r(r, T) > 0$: Case II

The solutions found for regions 1 and 2 are the same as those for Case I. We are only left to solve for region 4. In this region, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_4}{rV_4} = ds,$$

and we have Cauchy data specified along $r = r^+$, which is the value of the maximum, $V_1(r^+, t^*)$, discounted at the rate r^+, i.e.

$$\Gamma_4(r, t, V_4) = \left(r^+, p, V_1(r^+, t^*)e^{-r^+(t^*-p)}\right),$$

for $0 \leq p \leq t^*$. We can solve this to find

$$V_4(r, t) = V_1(r^+, t^*)e^{-r^+(t^*-t)+\frac{(r^+-r)^2}{2c^+}},$$

for $r^+ - c^+(t^* - t) \leq r \leq r^+ + c^+t$.

We must then solve $V_1(r, t) = V_4(r, t)$ for $t \leq t^*$ to find the position of the shock, $R(t)$ say. Above the shock, $V = V_4$ and below the shock, $V = V_1$. The shock (and therefore the maximum) meets the minimum (which propagates along the bounding characteristic separating regions 1 and 2, $r^- - c^-(T - t)$) at (r^{**}, t^{**}), say. From this point, the boundary

$$r = r^{**} - c^+(t^{**} - t),$$

separates the solutions of regions 2 (below the line) and 4 (above the line).

The solution is therefore

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \le r \le r^+ \\ V_2(r,t) & \text{for } r^- \le r \le r^- - c^-(T-t), \end{cases}$$

if $t^* < t \le T$,

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \le r \le R(t) \\ V_2(r,t) & \text{for } r^- \le r \le r^- - c^-(T-t) \\ V_4(r,t) & \text{for } R(t) \le r \le r^+, \end{cases}$$

if $t^{**} < t \le t^*$, and

$$V(r,t) = \begin{cases} V_2(r,t) & \text{for } r^- \le r \le \max(r^-, r^{**} - c^+(t^{**}-t)) \\ V_4(r,t) & \text{for } \max(r^-, r^{**} - c^+(t^{**}-t)) \le r \le r^+, \end{cases}$$

if $t \le t^{**}$.

The solution with various times to maturity is shown in Figure 3.6, where

$$f(r) = 0.1 + r,$$

and

$$r^- = 0.03 \text{ , } r^+ = 0.5 \text{ , } c^- = -0.04 \text{ and } c^+ = 0.04.$$

In this particular example, the maximum starts to propagate into the solution region 1.91 years from maturity, and collides with the minimum 6.06 years from maturity, at an interest rate of 0.272.

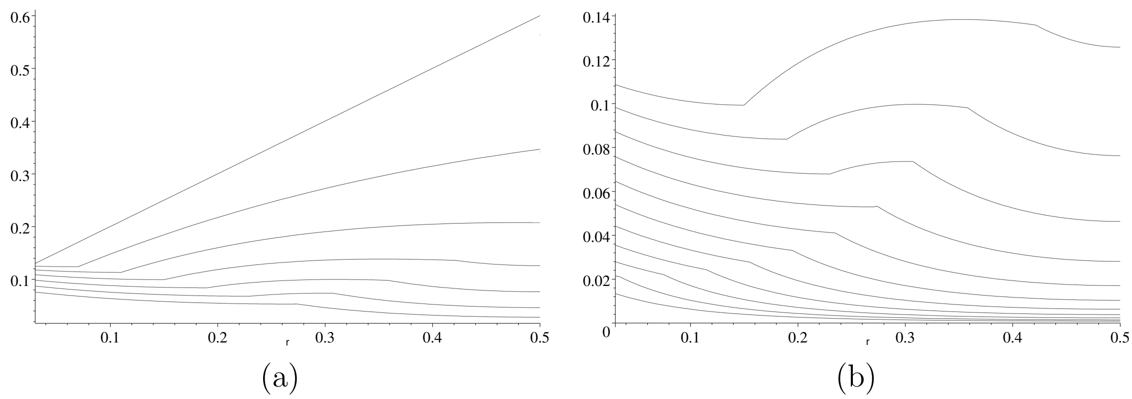


Figure 3.6: (a) Contract value with 0, 1, 2, 3, 4, 5, 6 years to maturity, (b) Contract value with 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 years to maturity

We remark that there is only one difference between the two examples that we have chosen to illustrate the solutions. The upper interest rate boundary is at 0.2 in the first case and 0.5 in the second. This has two effects. First of all, the minimum has further to travel to reach the upper boundary, which means that there is a larger interval of time between maturity and the minimum reaching the upper boundary. Second, the effect of discounting is greater at the upper boundary in the latter example, since it is at a higher interest rate. This means that $V_r = 0$ will occur on the upper boundary at a time closer to maturity than in the former example. These two effects combined cause the change from Case I to Case II.

3.4 $V_r(r,T)$ has an interior maximum

In this example, we have a maximum in the final data at an interior point of the region, r_T. We therefore have final data $V(r,T)$, where,

$$V(r,T) = \begin{cases} f_1(r) & \text{for } r \leq r_T \\ f_2(r) & \text{for } r > r_T, \end{cases}$$

and

$$\frac{df_1}{dr} > 0 \ , \ \frac{df_2}{dr} < 0 \text{ and } f_1(r_T) = f_2(r_T),$$

We expect the minimum at the lower boundary to propagate towards the upper boundary. This minimum may collide with the interior maximum before the maximum reaches the upper boundary (Case I). Otherwise, the maximum reaches the upper boundary (Case II).

We will find the same set of solutions for both cases. However, the regions in which these solutions are valid will be case-dependent. In practice, we first find this set of solutions and then solve for the position of the shock. We determine whether or not the minimum reaches this shock to decide which case we are in and can then calculate the boundaries for each solution region.

3.4.1 Case I

The characteristics when the minimum reaches the maximum before the upper boundary are shown in Figure 3.7.

In region 1, the characteristics are defined by

$$dt = \frac{dr}{c^-} = \frac{dV_1}{rV_1} = ds,$$

and we have Cauchy data, from the final data, of

$$\Gamma_1(r,t,V_1) = (p,T,f_1(p)),$$

for $r^- \leq p \leq r_T$. We can solve this to find

$$V_1(r,t) = f_1(r + c^-(T-t))e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

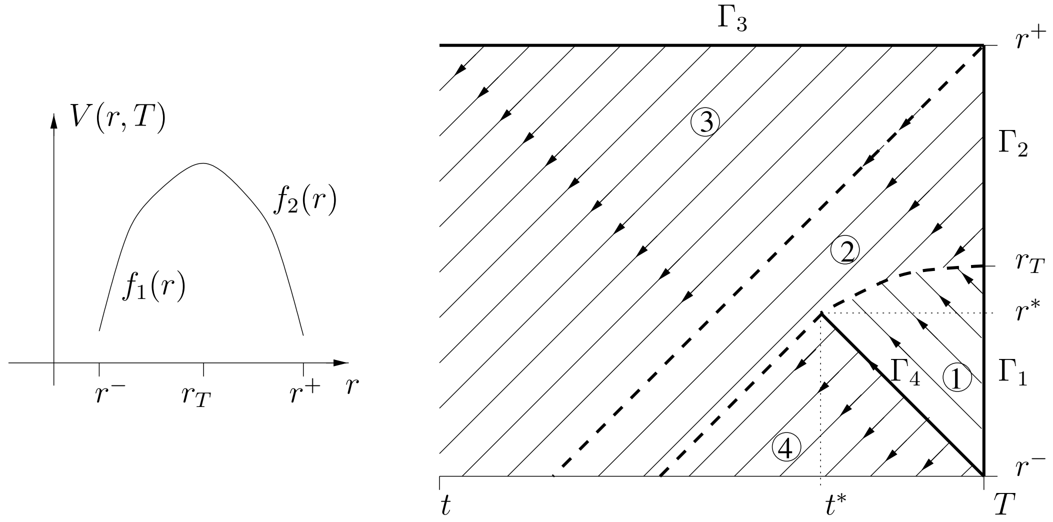


Figure 3.7: Possible characteristics when there is an interior maximum: Case I

for $r^- - c^-(T-t) \le r \le r_T - c^-(T-t)$.

In regions 2, 3 and 4, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV}{rV} = ds.$$

In region 2, we have Cauchy data, from the final data, of

$$\Gamma_2(r,t,V_2) = (p, T, f_2(p)),$$

for $r_T \le p \le r^+$. We can solve this to find

$$V_2(r,t) = f_2(r + c^+(T-t))e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

for $r_T - c^+(T-t) \le r \le r^+ - c^+(T-t)$.

In region 3, we have Cauchy data on the upper boundary of

$$\Gamma_3(r,t,V_3) = \left(r^+, p, f_2(r^+)e^{-r^+(T-p)}\right),$$

for $0 \le p \le T$. We can solve this to find

$$V_3(r,t) = f_2(r^+)e^{-r^+(T-t) + \frac{1}{2c^+}(r^+-r)^2},$$

for $r^+ - c^+(T-t) \le r \le r^+ + c^+t$.

In region 4, we have Cauchy data along the bounding characteristic, $r = r^- - c^-(T-t)$. Along this line, the solution from region 1 is

$$V = f_1(r^-)e^{\frac{1}{2}c^-(T-t)^2 - r^-(T-t)}.$$

We therefore have Cauchy data of

$$\Gamma_4(r,t,V_4) = \left(r^- - c^-(T-p), p, f_1(r^-)e^{\frac{1}{2}c^-(T-p)^2 - r^-(T-p)}\right),$$

for $T + \frac{r^+ - r^-}{c^-} \leq p \leq T$. We can solve this to find

$$V_4(r,t) = f_1(r^-)e^{\frac{1}{2}c^-(T-t)^2 - r^-(T-t) + \frac{1}{2}(r - r^- + c^-(T-t))^2/(c^+ - c^-)},$$

for $r^- - c^+(T-t) \leq r \leq r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r^-)$.

We then solve $V_1(r,t) = V_2(r,t)$ to find the position of the shock, $R(t)$ say. Below the shock, $V = V_1$ and above the shock, $V = V_2$. The shock meets the minimum (which propagates along the bounding characteristic separating regions 1 and 4) at (r^*, t^*), say. From this point, the boundary

$$r = r^* - c^+(t^* - t),$$

separates the solutions of regions 2 (above the line) and 4 (below the line).

The solution is therefore

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \leq r \leq R(t) \\ V_2(r,t) & \text{for } R(t) \leq r \leq r^+ - c^+(T-t) \\ V_3(r,t) & \text{for } r^+ - c^+(T-t) \leq r \leq r^+ \\ V_4(r,t) & \text{for } r^- \leq r \leq r^- - c^-(T-t), \end{cases}$$

if $t^* \leq t \leq T$, and

$$V(r,t) = \begin{cases} V_2(r,t) & \text{for } \max(r^-, r^* - c^+(t^*-t)) \leq r \leq \max(r^-, r^+ - c^+(T-t)) \\ V_3(r,t) & \text{for } \max(r^-, r^+ - c^+(T-t)) \leq r \leq r^+ \\ V_4(r,t) & \text{for } r^- \leq r \leq \max(r^-, r^* - c^+(t^*-t)), \end{cases}$$

if $t < t^*$.

The solution with a 'linear' maximum and various times to maturity is shown in Figure 3.8, where

$$f_1(r) = 0.5 + (r - r_T)\ ,\ f_2(r) = 0.5 - (r - r_T),$$

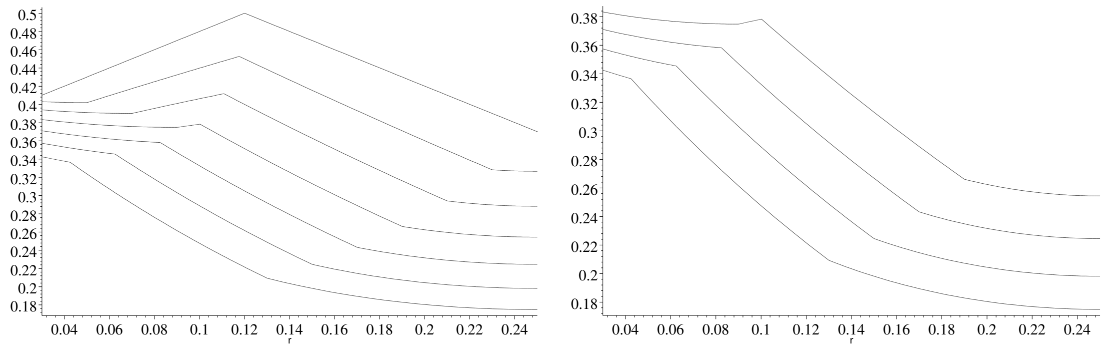
and

$$r^- = 0.03\ ,\ r^+ = 0.25\ ,\ r_T = 0.12\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

In this example, the maximum collides with the minimum 1.66 years from maturity, at an interest rate of 0.096.

The solution with a 'quadratic' maximum and various times to maturity is shown in Figure 3.9, where

$$f_1(r) = f_2(r) = 0.5 - 10(r - r_T)^2,$$



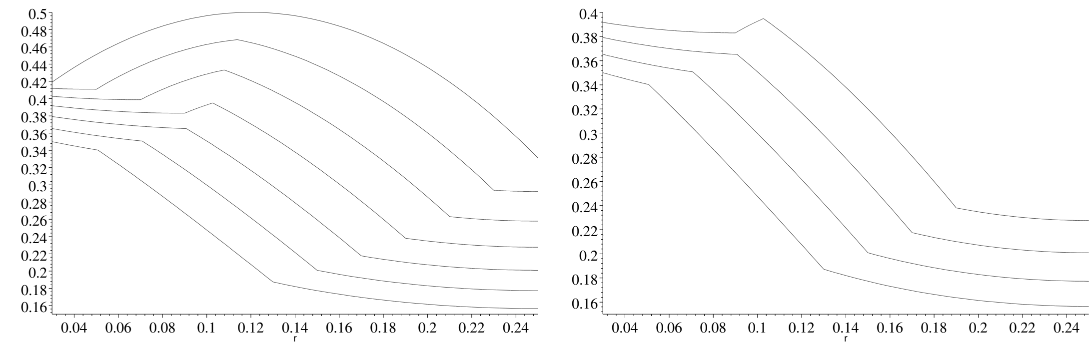
(a) (b)

Figure 3.8: (a) Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity, (b) Contract value with 1.5, 2, 2.5, 3 years to maturity

and

$$r^- = 0.03\ ,\ r^+ = 0.25\ ,\ r_T = 0.12\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

In this example, the maximum collides with the minimum 1.76 years from maturity, at an interest rate of 0.100.



(a) (b)

Figure 3.9: (a) Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity, (b) Contract value with 1.5, 2, 2.5, 3 years to maturity

Alternatively, the shock could reach the minimum after the bounding characteristic from the upper boundary. In this case, our characteristic picture would be that shown in Figure 3.10.

The solution that we have calculated is still valid, but the regions in which each solution holds have changed. If the equation of the shock is $R(t)$, and the shock collides with the minimum at (r^*, t^*), then our solution becomes

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \le r \le R(t) \\ V_2(r,t) & \text{for } R(t) \le r \le \max(R(t), r^+ - c^+(T-t)) \\ V_3(r,t) & \text{for } \max(R(t), r^+ - c^+(T-t)) \le r \le r^+ \\ V_4(r,t) & \text{for } r^- \le r \le r^- - c^-(T-t), \end{cases}$$

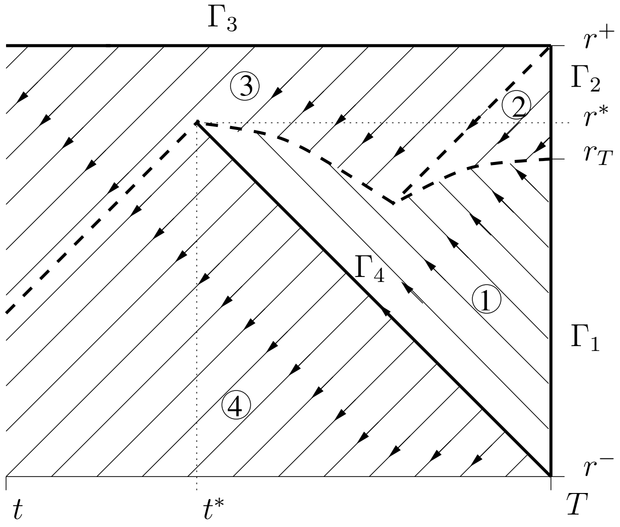


Figure 3.10: Possible characteristics when there is an interior maximum: Case I

if $t^* \leq t \leq T$, and

$$V(r,t) = \begin{cases} V_3(r,t) & \text{for } \max(r^-, r^* - c^+(t^* - t)) \leq r \leq r^+ \\ V_4(r,t) & \text{for } r^- \leq r \leq \max(r^-, r^* - c^+(t^* - t)), \end{cases}$$

if $t < t^*$.

3.4.2 Case II

The characteristic picture for the second case is shown in 3.11.

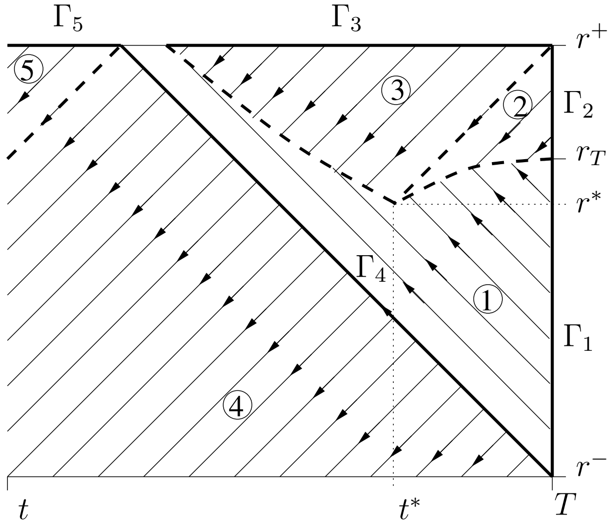


Figure 3.11: Possible characteristics when there is an interior maximum: Case II

The solutions that we have found for regions 1, 2, 3 and 4 are still valid. However, the regions in which the solutions are valid have changed. We must also find the solution for region 5. In this region, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_5}{rV_5} = ds.$$

We have Cauchy data along the upper boundary of

$$\Gamma_5(r,t,V_5) = \left(r^+, p, V\left(r^+, T + \frac{r^+ - r^-}{c^-}\right) e^{-r^+\left(T + \frac{r^+-r^-}{c^-} - p\right)}\right),$$

where $0 \le p \le T + \frac{r^+-r^-}{c^-}$ and

$$V\left(r^+, T + \frac{r^+ - r^-}{c^-}\right) = f_1(r^-) e^{-\frac{1}{2c^-}(r^+-r^-)^2 + r^+\left(\frac{r^+-r^-}{c^-}\right)}.$$

We can solve this to find

$$V_5(r,t) = f_1(r^-) e^{-r^+(T-t) - \frac{1}{2c^-}(r^+-r^-)^2 + \frac{1}{2c^+}(r^+-r)^2},$$

for $r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r^-) \le r \le r^+ + c^+t$.

Again, we solve $V_1(r,t) = V_2(r,t)$, and then $V_1(r,t) = V_3(r,t)$ to find the position of the shock, $R(t)$ say. Below the shock, $V = V_1$ and above the shock, $V = V_2$ or V_3, depending which side of the bounding characteristic, $r = r^+ - c^+(T-t)$, we are. (The shock meets this bounding characteristic at time t^* say).

The solution is therefore

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \le r \le R(t) \\ V_2(r,t) & \text{for } R(t) \le r \le r^+ - c^+(T-t) \\ V_3(r,t) & \text{for } r^+ - c^+(T-t) \le r \le r^+ \\ V_4(r,t) & \text{for } r^- \le r \le r^- - c^-(T-t), \end{cases}$$

if $t^* < t \le T$, and

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- - c^-(T-t) \le r \le \min(r^+, R(t)) \\ V_3(r,t) & \text{for } \min(r^+, R(t)) \le r \le r^+ \\ V_4(r,t) & \text{for } r^- \le r \le r^- - c^-(T-t), \end{cases}$$

if $T + \frac{r^+-r^-}{c^-} < t \le t^*$, and

$$V(r,t) = \begin{cases} V_4(r,t) & \text{for } r^- \le r \le \max(r^-, r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r^-)) \\ V_5(r,t) & \text{for } \max(r^-, r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r^-)) \le r \le r^+, \end{cases}$$

if $t < T + \frac{r^+-r^-}{c^-}$.

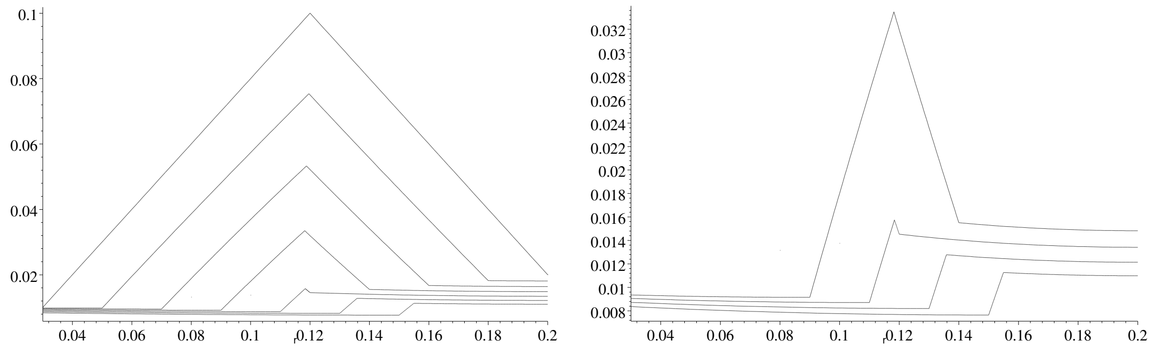
The solution with a ‘linear’ maximum and various times to maturity is shown in Figure 3.12, where

$$f_1(r) = 0.1 + (r - r_T)\ ,\ f_2(r) = 0.1 - (r - r_T),$$

and

$$r^- = 0.03 \; , \; r^+ = 0.2 \; , \; r_T = 0.12 \; , \; c^- = -0.04 \text{ and } c^+ = 0.04.$$

In this example, the maximum meets the minimum from the upper boundary 2.04 years from maturity, at an interest rate of 0.118.



(a) (b)

Figure 3.12: (a) Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity, (b) Contract value with 1.5, 2, 2.5, 3 years to maturity

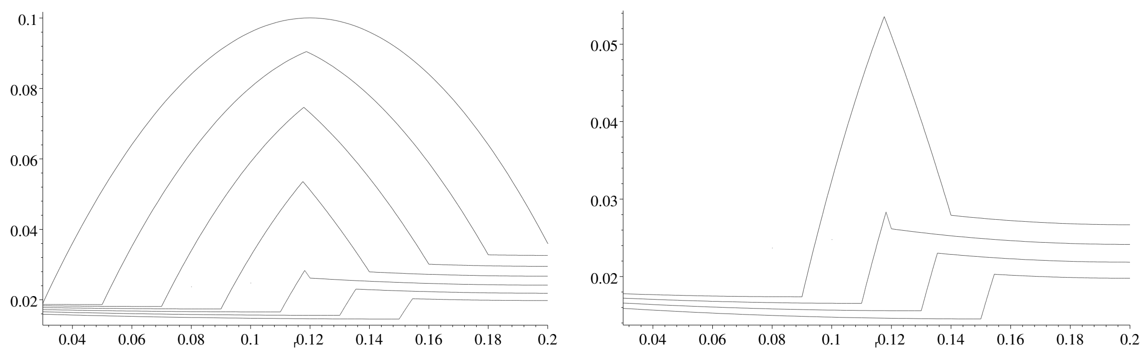
The solution with a 'quadratic' maximum and various times to maturity is shown in Figure 3.13, where

$$f_1(r) = f_2(r) = 0.1 - 10(r - r_T)^2,$$

and

$$r^- = 0.03 \; , \; r^+ = 0.2 \; , \; r_T = 0.12 \; , \; c^- = -0.04 \text{ and } c^+ = 0.04.$$

In this example, the maximum meets the minimum from the upper boundary 2.04 years from maturity, at an interest rate of 0.118.



(a) (b)

Figure 3.13: (a) Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity, (b) Contract value with 1.5, 2, 2.5, 3 years to maturity

3.5 $V_r(r,T)$ has an interior minimum

In our final example, we have a minimum in the final data at an interior point of the region, r_T. This takes the form,

$$V(r,T) = \begin{cases} f_1(r) & \text{for } r \leq r_T \\ f_2(r) & \text{for } r > r_T, \end{cases}$$

where

$$\frac{df_1}{dr} < 0 \ , \ \frac{df_2}{dr} > 0 \text{ and } f_1(r_T) = f_2(r_T),$$

We expect the minimum to propagate to the upper boundary (Case I). It is possible that the maximum at the upper boundary may propagate into the region before the minimum reaches the upper boundary, in which case they collide in the inner region (Case II).

We solve the problem assuming that we are in Case I and then check that we do not have $V_r = 0$ on the upper boundary. If we find a point at which $V_r(r^+,t) = 0$, then we resolve the problem assuming that we are in Case II.

3.5.1 Case I

The characteristic picture when the minimum reaches the upper boundary is shown in Figure 3.14.

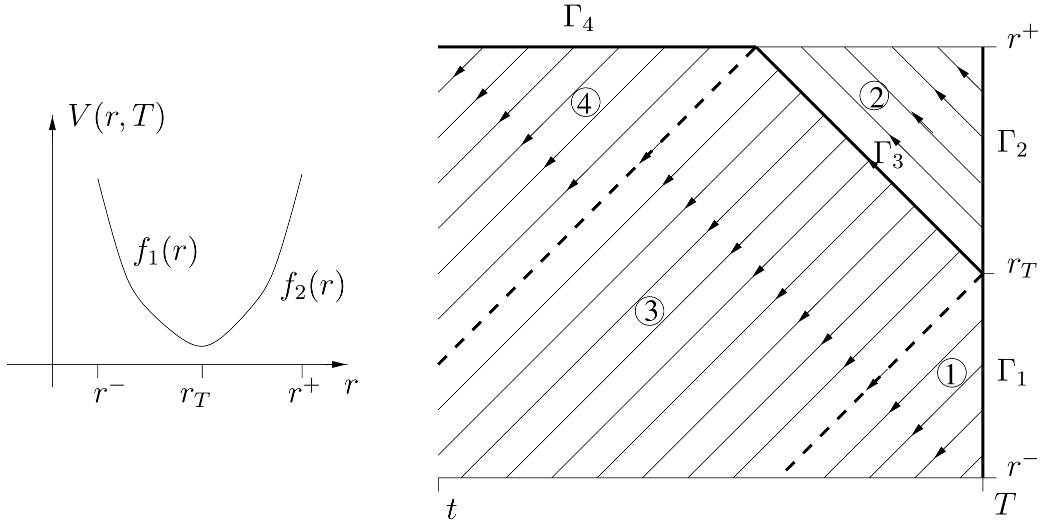


Figure 3.14: Possible characteristics when there is an interior minimum: Case I

In region 1, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_1}{rV_1} = ds,$$

and we have Cauchy data, from the final data, of

$$\Gamma_1(r, t, V_1) = (p, T, f_1(p)),$$

for $r^- \leq p \leq r_T$. We can solve this to find

$$V_1(r, t) = f_1(r + c^+(T - t))e^{-\frac{1}{2}c^+(T-t)^2 - r(T-t)},$$

for $r^- - c^+(T - t) \leq r \leq r_T - c^+(T - t)$.

In region 2, the characteristics are defined by

$$dt = \frac{dr}{c^-} = \frac{dV_2}{rV_2} = ds,$$

and we have Cauchy data, from the final data, of

$$\Gamma_2(r, t, V_2) = (p, T, f_2(p)),$$

for $r_T \leq p \leq r^+$. We can solve this to find

$$V_2(r, t) = f_2(r + c^-(T - t))e^{-\frac{1}{2}c^-(T-t)^2 - r(T-t)},$$

for $r_T - c^-(T - t) \leq r \leq r^+ - c^-(T - t)$.

In regions 3 and 4, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV}{rV} = ds.$$

In region 3, we have Cauchy data along the bounding characteristic of region 1, $r = r_T - c^-(T - t)$, of $V = V_2(r_T - c^-(T - t), t)$. This gives us

$$\Gamma_3(r, t, V_3) = \left(r_T - c^-(T - p), p, f_2(r_T)e^{\frac{1}{2}c^-(T-p)^2 - r_T(T-p)}\right),$$

for $T + \frac{r^+ - r_T}{c^-} \leq p \leq T$. We can solve this to find

$$V_3(r, t) = f_2(r_T)e^{\frac{1}{2}c^-(T-t)^2 - r_T(T-t) + \frac{1}{2}(r - r_T + c^-(T-t))^2/(c^+ - c^-)},$$

for $r_T - c^+(T - t) \leq r \leq r_T - c^+(T - t) - (c^+ - c^-)\frac{r^+ - r_T}{c^-}$.

In region 4, we have Cauchy data along the upper boundary of the discounted value of the contract when the minimum reaches the upper boundary,

$$\Gamma_4(r, t, V_4) = \left(r^+, p, V_2\left(r^+, T + \frac{r^+ - r_T}{c^-}\right)e^{-r^+\left(T + \frac{r^+ - r_T}{c^-} - p\right)}\right),$$

for $0 \le p \le T + \frac{r^+ - r_T}{c^-}$. We can solve this to find

$$V_4(r,t) = f_2(r_T) e^{-r^+(T-t) - \frac{1}{2c^-}(r^+ - r_T)^2 + \frac{1}{2c^+}\left(r^+ - r\right)^2},$$

for $r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r_T) \le r \le r^+ + c^+ t$.

The solution is therefore

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \le r \le \max(r^-, r_T - c^+(T-t)) \\ V_2(r,t) & \text{for } \min(r^+, r_T - c^-(T-t)) \le r \le r^+ \\ V_3(r,t) & \text{for } \max(r^-, r_T - c^+(T-t)) \le r \le \min(r^+, r_T - c^-(T-t)), \end{cases}$$

if $T + \frac{r^+ - r_T}{c^-} \le t \le T$, and

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \le r \le \max(r^-, r_T - c^+(T-t)) \\ V_3(r,t) & \text{for } \max(r^-, r_T - c^+(T-t)) \le r \\ & \qquad \le \max(r^-, r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r_T)) \\ V_4(r,t) & \text{for } \max(r^-, r^+ - c^+(T-t) - \frac{c^+}{c^-}(r^+ - r_T)) \le r \le r^+, \end{cases}$$

if $t < T + \frac{r^+ - r_T}{c^-}$.

The solution with a 'linear' minimum and various times to maturity is shown in Figure 3.15, where

$$f_1(r) = 0.1 - (r - r_T)\ ,\ f_2(r) = 0.1 + (r - r_T),$$

and

$$r^- = 0.03\ ,\ r^+ = 0.2\ ,\ r_T = 0.12\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

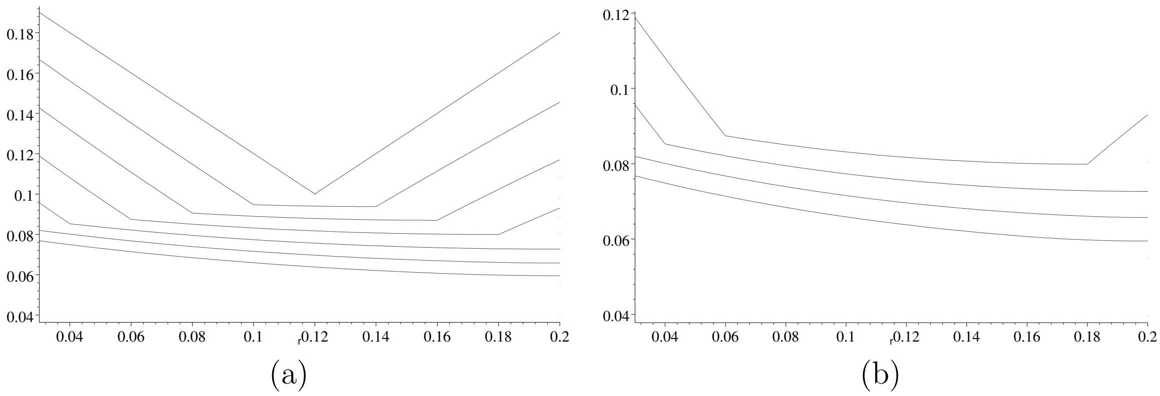


Figure 3.15: (a) Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity, (b) Contract value with 1.5, 2, 2.5, 3 years to maturity

The solution with a 'quadratic' minimum and various times to maturity is shown in Figure 3.16, where

$$f_1(r) = f_2(r) = 0.1 + 10(r - r_T)^2,$$

and

$$r^- = 0.03\ ,\ r^+ = 0.2\ ,\ r_T = 0.12\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

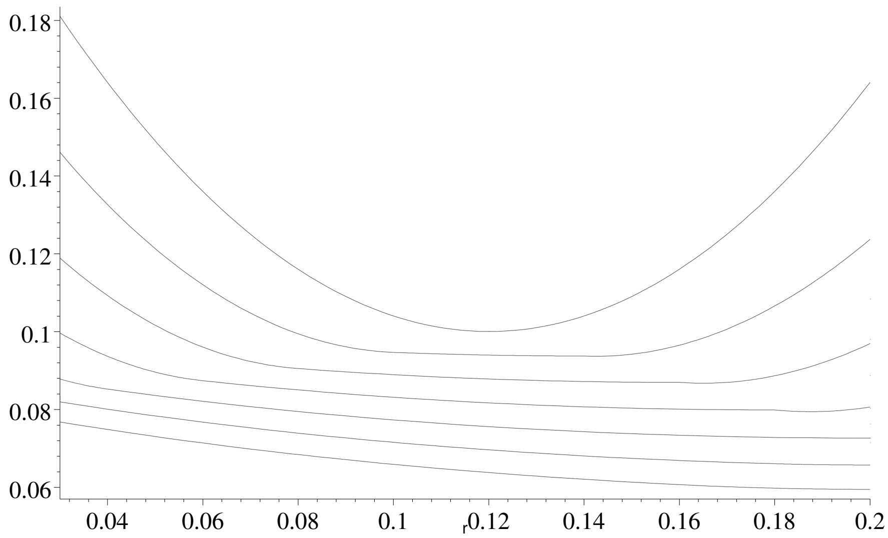


Figure 3.16: Contract value with 0, 0.5, 1, 1.5, 2, 2.5, 3 years to maturity

3.5.2 Case II

The characteristics when the maximum propagates into the region from the upper boundary are shown in Figure 3.17, where we assume that we first find $V_r = 0$ at (r^+, t^*) say.

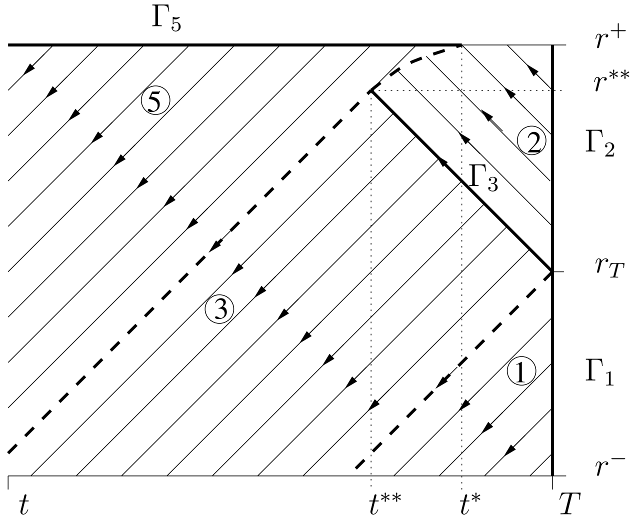


Figure 3.17: Possible characteristics when there is an interior minimum: Case II

The solutions that we found in Case I for regions 1, 2 and 3 are still valid. We must now find the solution for region 5 along with the new regions in which these solutions hold.

In region 5, the characteristics are defined by

$$dt = \frac{dr}{c^+} = \frac{dV_5}{rV_5} = ds,$$

and the Cauchy data for this region is

$$\Gamma_5(r,t,V_5) = \left(r^+, p, V_2(r^+,t^*)e^{-r^+(t^*-p)}\right),$$

for $0 \leq p \leq t^*$. We can solve this to find

$$V_5(r,t) = V_2(r^+,t^*)e^{-r^+(t^*-t)+\frac{1}{2c^+}(r^+-r)^2},$$

for $r^+ - c^+(t^*-t) \leq r \leq r^+ + c^+t$.

We then solve $V_2(r,t) = V_5(r,t)$ to find the path of the shock, $R(t)$. This meets the minimum (which travels along $r = r_T - c^-(T-t)$) at (r^{**}, t^{**}) say. The bounding characteristic between regions 3 and 5 is then $r = r^{**} - c^+(t^{**}-t)$.

The solution is therefore

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \leq r \leq \max(r^-, r_T - c^+(T-t)) \\ V_2(r,t) & \text{for } \min(r^+, r_T - c^-(T-t)) \leq r \leq r^+ \\ V_3(r,t) & \text{for } \max(r^-, r_T - c^+(T-t)) \leq r \leq \min(r^+, r_T - c^-(T-t)), \end{cases}$$

if $t^* \leq t \leq T$,

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \leq r \leq \max(r^-, r_T - c^+(T-t)) \\ V_2(r,t) & \text{for } \min(r^+, r_T - c^-(T-t)) \leq r \leq R(t) \\ V_3(r,t) & \text{for } \max(r^-, r_T - c^+(T-t)) \leq r \leq \min(r^+, r_T - c^-(T-t)) \\ V_5(r,t) & \text{for } R(t) \leq r \leq r^+, \end{cases}$$

if $t^{**} < t < t^*$, and

$$V(r,t) = \begin{cases} V_1(r,t) & \text{for } r^- \leq r \leq \max(r^-, r_T - c^+(T-t)) \\ V_3(r,t) & \text{for } \max(r^-, r_T - c^+(T-t)) \leq r \leq \max(r^-, r^{**} - c^+(t^{**}-t)) \\ V_5(r,t) & \text{for } \max(r^-, r^{**} - c^+(t^{**}-t)) \leq r \leq r^+, \end{cases}$$

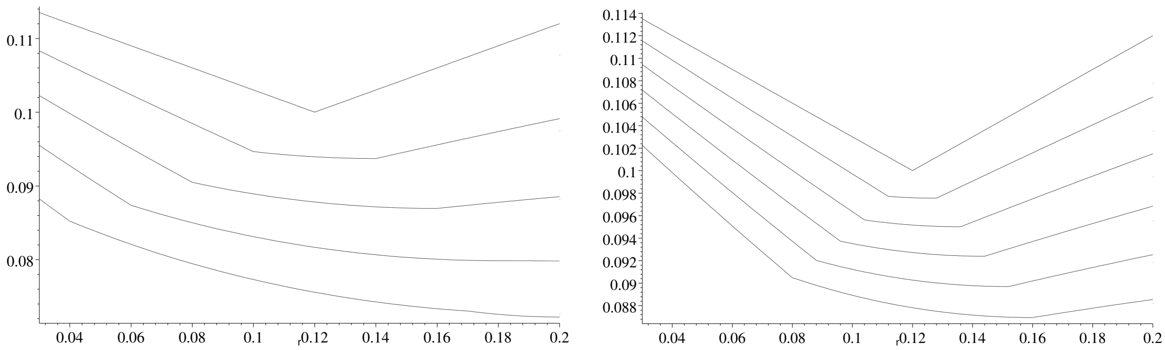
if $t < t^{**}$.

The solution with a 'linear' minimum and various times to maturity is shown in Figure 3.18, where

$$f_1(r) = 0.1 - 0.15(r - r_T) \ , \ f_2(r) = 0.1 + 0.15(r - r_T),$$

and

$$r^- = 0.03 \ , \ r^+ = 0.2 \ , \ r_T = 0.12 \ , \ c^- = -0.04 \text{ and } c^+ = 0.04.$$



(a) (b)

Figure 3.18: (a) Contract value with 0, 0.5, 1, 1.5, 2 years to maturity, (b) Contract value with 0, 0.2, 0.4, 0.6, 0.8, 1 year to maturity

The maximum starts to propagate into the region 1.45 years from maturity and the minimum meets the maximum 1.63 years from maturity at an interest rate of 0.185.

The solution with a 'quadratic' minimum and various times to maturity is shown in Figures 3.19, where

$$f_1(r) = f_2(r) = 0.1 + (r - r_T)^2,$$

and

$$r^- = 0.03\ ,\ r^+ = 0.2\ ,\ r_T = 0.12\ ,\ c^- = -0.04 \text{ and } c^+ = 0.04.$$

The maximum starts to propagate into the region 0.879 years from maturity and the minimum meets the maximum 0.944 years from maturity at an interest rate of 0.158.

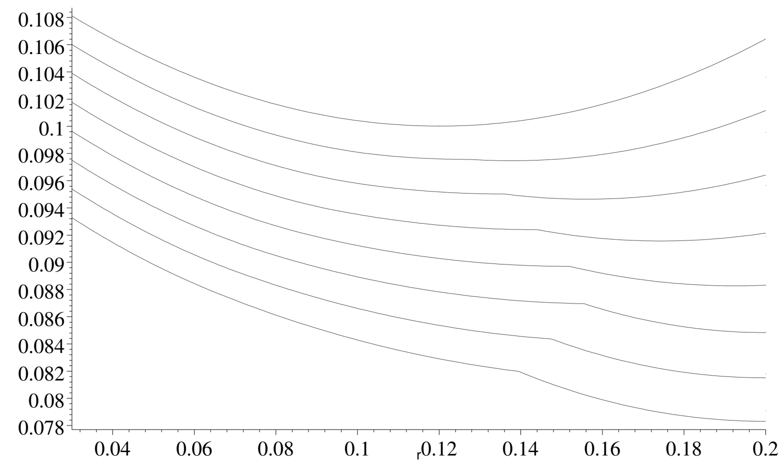


Figure 3.19: Contract value with 0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4 years to maturity

Chapter 4

Pricing and hedging simple products

In this chapter, we apply our model to the pricing and hedging of simple fixed-income products. We begin with a discussion of the consequences of the existence of a nonlinear pricing equation. We then solve the valuation problem of the best-case scenario. With this information, and using the zero-coupon bond as a case-example, we explain how to price and hedge a product. We apply these procedures to value other simple products - swaps, caps and floors. Finally we discuss possible applications of the methodology and provide a 'real-world' example.

4.1 Consequences of our nonlinear model

We have derived a first-order, nonlinear, hyperbolic partial differential equation for the value of a contract in a worst-case scenario and demonstrated the analytical solution of this equation. In practice, however, the complexity of the characteristic picture requires us to solve the pde numerically. Details of the numerical solution of the problem can be found in Appendix A. Subsequent results in this thesis will be of a numerical nature.

In addition to the complexity of solution, there are a number of other significant properties associated with the nonlinearity of our pricing equation.

4.1.1 Spreads for prices

Since our uncertain interest rate model solely places bounds on the short-term interest rate, it is not surprising that the best we can do is to find bounds for the value of a contract. We have derived a partial differential equation for the value of a contract in a worst-case scenario (a lower bound). As we shall show in the next section, it is also possible to derive an equation for the upper bound. We therefore find a spread for the possible price of a contract. Consequently, long and short positions in a contract have different values (since the worst-case value for the long position will be equivalent to the best-case for the short position and vice-versa).

Finding a spread for prices is not necessarily a disadvantage of the model. After all, the market itself has such a property (the bid-offer spread). In some sense, spreads are therefore a more realistic result than a single price. However, it becomes a disadvantage when the spreads are so large that the results become meaningless. We require a method to reduce large spreads to more sensible levels - this is the process of static hedging.

4.1.2 Static hedging

The concept of static hedging relies on another property associated with the presence of a nonlinear pricing equation, that the value of a portfolio of contracts is not necessarily equal to the sum of their individual values. Hence the value of a contract depends on what else it is priced with. The financial reasoning for this is as follows:

Imagine that we have two contracts, A and B. If we value A in a worst-case scenario, then we can find the lowest possible amount that A is worth, and the corresponding worst-case interest rate path. Similarly, we can value B in a worst-case scenario and find the lowest possible amount that this contract is worth, along with its worst-case path. The crucial point is that these two paths do not have to be the same. If we follow the worst-case path for A, then B may be worth more than its lowest value. Similarly, if we follow the path for B, then A may be worth more than its worst-case value. Hence, if we value the combined portfolio of A and B in a worst-case scenario, we may find that the interest rate path follows some sort of 'compromise' between the two separate worst-case paths. Our overall portfolio may then have a worst-case scenario value that is higher than the two separate worst-case values added together. In fact, this will always be the case except when the worst-case scenario interest rate paths for A and B coincide.

Suppose that we want to buy or sell an OTC contract. We can price the contract under our model, to find a worst- and best-case value. To guarantee that we never lose money on the deal, we would then buy at the worst-case and sell at the best-case price. However, the spread between these prices may be too large to make the values of any practical use in the marketplace. We must use the information available in the market (the market prices) to reduce the spread.

Although our OTC contract does not exist in the market, there may be similar contracts available. We can static hedge our contract with these market-traded products to alter the marginal worst- and best-case values of our contract. (The marginal value of a contract in a portfolio is the overall value of the portfolio minus the cost of the other instruments in the portfolio).

The idea of static hedging for spread reduction was originally due to Avellaneda and Paras [6] in the uncertain volatility model for equity derivatives. It will be possible to find an optimal static hedge for which the marginal worst-case value of the contract is as high as possible. Similarly, there will be a (possibly different) optimal static hedge for which the marginal best-case value of the contract will be as low as possible.

If we price a contract that is actually a market-traded instrument and then set up a static hedge using all the available instruments in the market, we find that the optimal static hedge in both worst- and best-case scenarios will be to hedge the contract one-for-one with itself. This will close the position and leave no residual portfolio, a perfect replication hedge strategy. Both the worst- and best-case values after hedging will then be equal to the market price (and there is zero spread). In this way, the contract price replicates the observed market price, without the need for any fitting or calibration of model parameters.

This approach can successfully reduce the spread in contract prices to realistic levels. Before illustrating the theory of static hedging, we first develop the idea of the best-case scenario.

4.2 Contract value in a best-case scenario

So far, we have concentrated on finding the value of a contract in a worst-case scenario. This is the scenario in which, under our given constraints on the interest rate, the contract value is as low as possible. We can be sure that the contract is worth at least this much. If we were interested in buying a contract, for instance, then paying this price would guarantee that we could not make a loss on the transaction. But say we wanted to sell the contract. In this case, we would be interested in an upper bound for the contract value.

Since all possible interest rate movements are bounded under our model, we can find such a bound. This upper bound is the value of the contract in a best-case scenario (the scenario in which the evolution of the interest rate is such that no other possible evolution would give the contract a higher value).

There are a number of possible approaches to the solution of this best-case problem. For instance, we can repeat the analysis of Section 2.3.1, but maximise where previously we had minimised. The result is that the contract value in a best-case scenario, V, satisfies

$$\max_{dr}(V_r dr + V_t dt) = rV dt,$$

and since dr is bounded by

$$c^- dt \leq dr \leq c^+ dt,$$

we obtain

$$V_t + \bar{c}(r, V_r)V_r - rV = 0, \tag{4.1}$$

where

$$\bar{c}(r, X) = \begin{cases} c^+ & \text{if } X > 0 \\ c^- & \text{if } X < 0 \end{cases} = c(r, -X).$$

Alternatively, we can view the best-case scenario valuation as equivalent to a worst-case scenario valuation in which we hold the contract short instead of long. The financial reasoning behind this is that the best-case scenario for the holder of a contract will always be the worst-case scenario for the writer, and vice-versa. There is also a mathematical argument behind this result. If we substitute $-W$ for V in Equation (2.6), then we obtain Equation (4.1) for W. Consequently, we find

$$V_{\text{best case}} = -(-V)_{\text{worst case}}. \tag{4.2}$$

To work out the value of a contract in a best-case scenario, we can therefore either solve Equation (4.1) for long the contract, or solve Equation (2.6) for short the contract and take the negative of the result.

4.3 The zero-coupon bond

We will use the zero-coupon bond as a case example to demonstrate the pricing and hedging of a contract. To price a zero-coupon bond under our model, we solve our partial differential equation, either (2.6) or (4.1), with final condition

$$V(r, T) = P,$$

where T is the maturity and P the principal of the bond.

To price a coupon bond, we would just add each coupon in as a jump condition. For example, to include a coupon of size cP at time T_c, we would add the condition

$$V(r, T_c^-) = cP + V(r, T_c^+).$$

Figure 4.1 shows a worst-case scenario valuation for a zero-coupon bond with principal 1, with varying time to maturity and initial short-term interest rate. Figure 4.2 shows a best-case scenario valuation for the same bond. Figures 4.3 and 4.4 show the yield for these bond prices respectively.

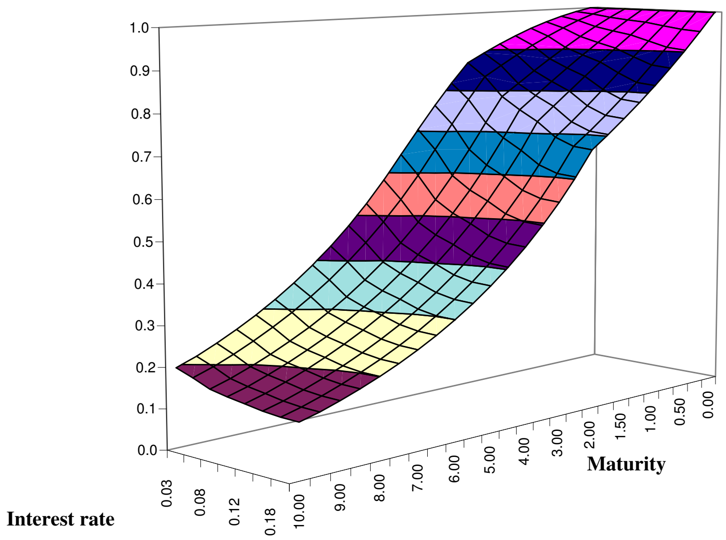


Figure 4.1: Zero-coupon bond value in a worst-case scenario

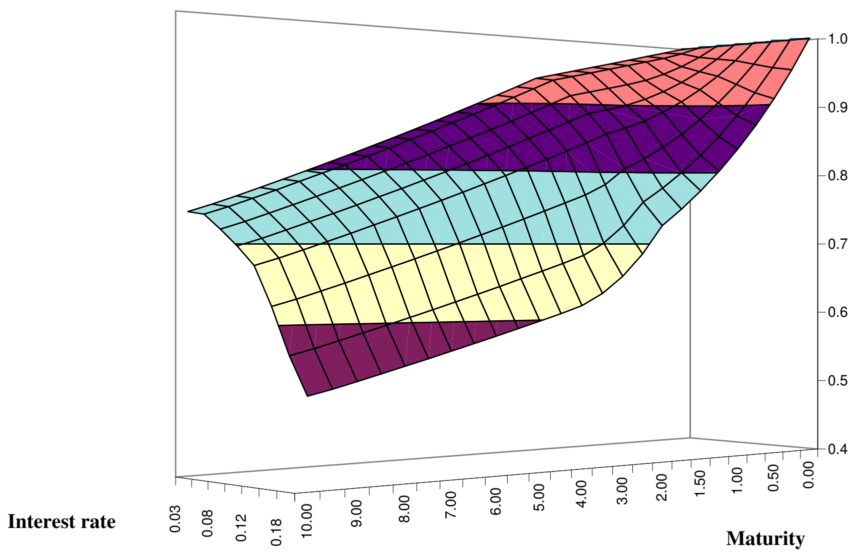


Figure 4.2: Zero-coupon bond value in a best-case scenario

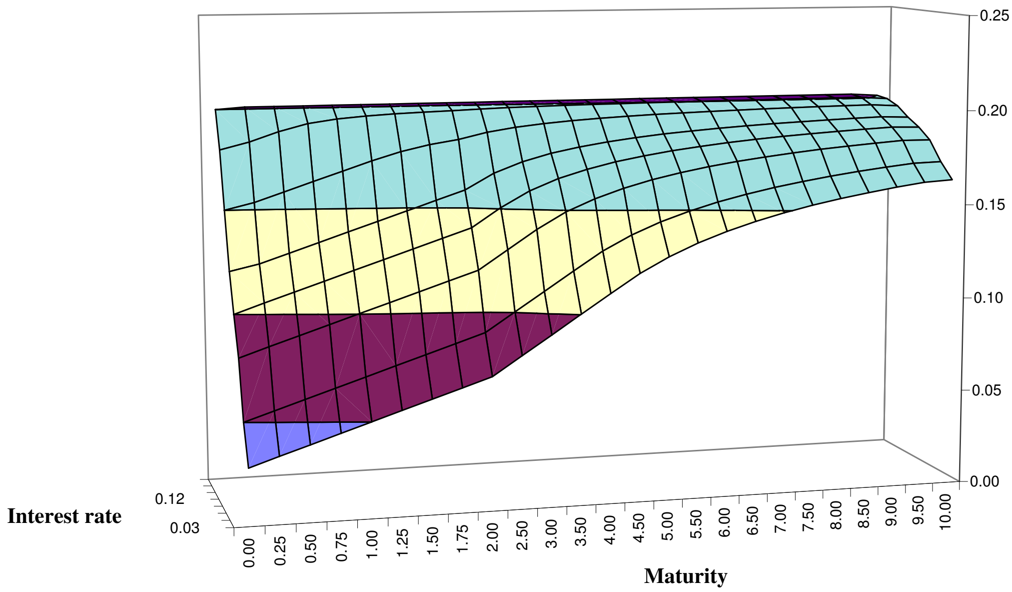


Figure 4.3: Yield in a worst-case scenario

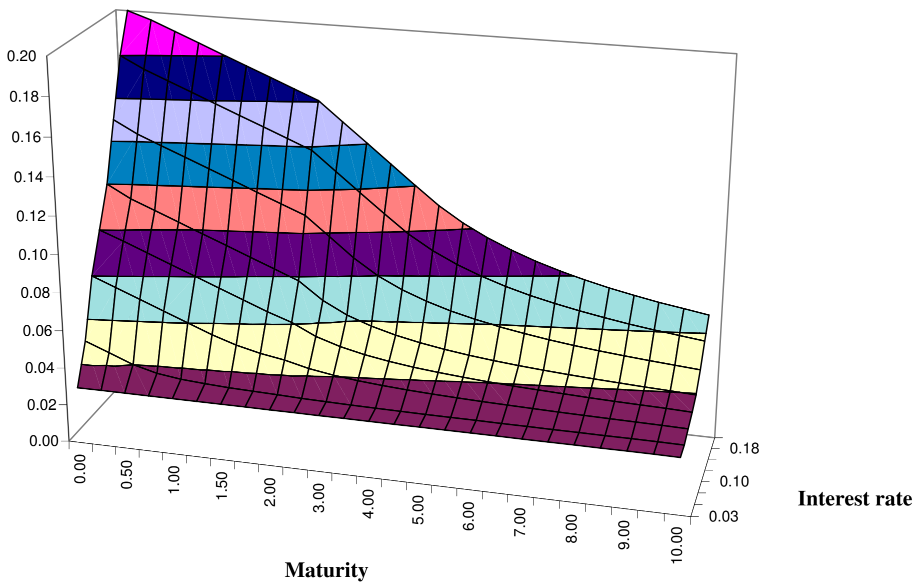


Figure 4.4: Yield in a best-case scenario

(The interest rate bounds for these figures are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa).

We note that there is a significant spread between these worst- and best-case yields. This is illustrated in Figure 4.5 which shows the worst- and best-case yields for the zero-coupon bond with principal 1, maturity in 5 years and varying initial interest rate, as well as the spread between these values. (The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa in this figure).

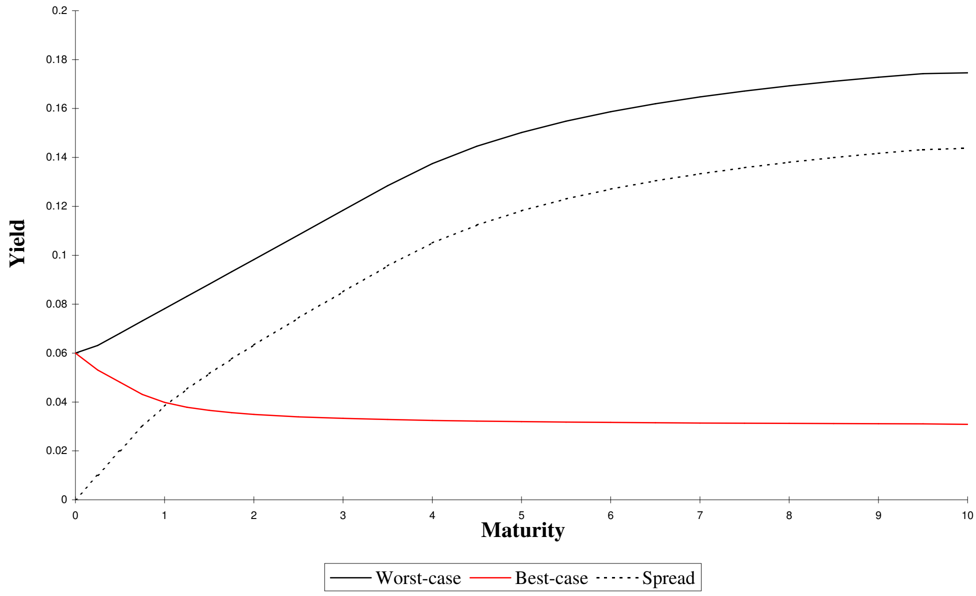


Figure 4.5: Spread in zero-coupon bond yields

4.4 Hedging a contract

We now demonstrate how to use the fixed income products available in the market as hedging instruments to reduce the exposure of our contract to changes in the interest rate. We will perform an optimisation on our hedged contract, using the same philosophy as practised by Avellaneda, Levy and Paras to hedge volatility risk with derivatives [4]. We concentrate on the worst-case scenario valuation for a contract V and solve Equation (2.6). To perform a best case scenario valuation, we would just hedge a contract of $-V$ (from Equation (4.2)) and solve Equation (2.6) or hedge a contract of V and solve Equation (4.1).

We omit the bid-offer spreads to simplify the arguments involved, and note that their inclusion would not affect the discussion.

4.4.1 Hedging with one instrument

Consider a contract consisting of a set of cash flows. We wish to value this contract in a worst-case scenario. Suppose that there exists a market-traded instrument, with known market price (a zero-coupon bond, for instance). We hedge with this instrument and price the resulting portfolio in a worst-case scenario. The value of the overall portfolio is

$$value(\text{contract} + \text{hedging instrument}).$$

where '*value*' means the solution of the nonlinear partial differential equation, Equation (2.6), with relevant final and jump conditions.

The cost of setting up this static hedge is equal to the current market value of the hedging instrument. The marginal value of our hedged contract is therefore the value of the overall portfolio minus the cost of the static hedge,

$$value(\text{hedged contract}) = value(\text{contract} + \text{hedging instrument}) - \text{cost of hedge}.$$

We show this type of hedging in Figure 4.6, where we hedge a set of known cashflows with λ of a zero-coupon bond.

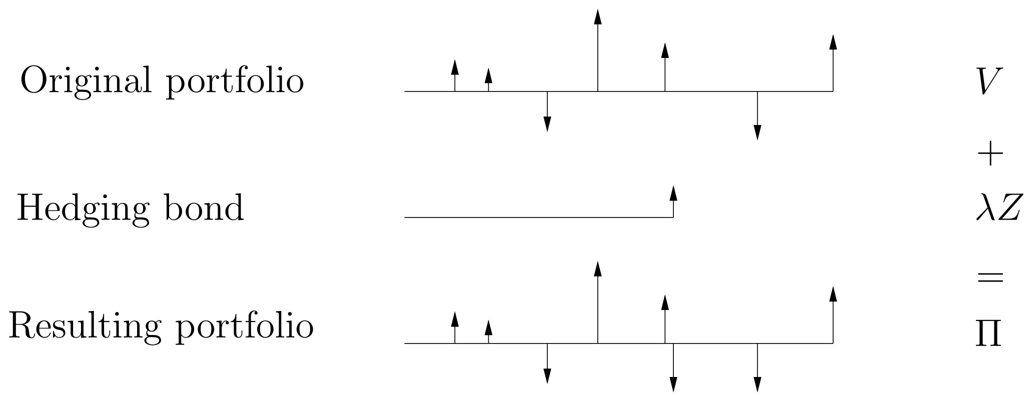


Figure 4.6: Hedging a contract with a zero-coupon bond

We assume that the market value of a hedging instrument is contained within the best- and worst-case scenario prices for the instrument, obtained from our model. This is in fact an assumption that there are no arbitrage opportunities. If this were not true, then we could make a risk-free profit by buying (selling) the instrument at a price we are certain is below (above) its minimum (maximum) possible value (assuming that r moves within our specified constraints). If the market value was actually equal to one of the two bounding prices, then this would still be an arbitrage opportunity, as there would be the possibility of making a risk-free profit, with the guarantee of no loss - a weak arbitrage opportunity. (If the market price were outside of the bounding prices, then there would be a guarantee of a risk-free profit - a strong arbitrage opportunity) [52].

If we hedge a contract, with λ of a zero-coupon bond with positive principal, then the value of the contract in a worst-case scenario reaches its maximum at a finite value of λ.

For instance, when we hedge our portfolio of cashflows with the zero-coupon bond:

The portfolio V has cash flows C_i at times T_i, for $1 \leq i \leq n$. The value of this portfolio in a worst case scenario is

$$V = \sum_{i=1}^{n} C_i e^{-\int_t^{T_i} r(\tau) d\tau},$$

where the interest rate realised for the worst case scenario is $r(t)$.

Suppose that there also exists a zero-coupon bond in the market, with known market price Z. This bond has a single principal payment of 1 at time S. We hedge V with this zero-coupon bond. We then price the resulting portfolio, Π, in a worst case scenario.

We hedge with λ of the bond. The value of the overall portfolio is

$$\Pi = \sum_{i=1}^{n} C_i e^{-\int_t^{T_i} \bar{r}(\tau) d\tau} + \lambda e^{-\int_t^{S} \bar{r}(\tau) d\tau},$$

where the interest rate realised for this worst case scenario is $\bar{r}(t)$.

The cost of setting up this static hedge is the equal to the current market value of the hedging instrument,

$$\lambda Z.$$

The marginal value of our contract, V, is therefore the value of the overall portfolio, Π, minus the cost of the static hedge,

$$V = \sum_{i=1}^{n} C_i e^{-\int_t^{T_i} \bar{r}(\tau) d\tau} + \lambda \left(e^{-\int_t^{S} \bar{r}(\tau) d\tau} - Z \right).$$

Note that $\bar{r}(t)$ will depend implicitly on the value of λ. The particular evolution of the interest rate in a worst case scenario for the portfolio will depend on the exact makeup of the portfolio, and consequently on the amount, λ, of the hedging bond contained in the portfolio.

As λ tends to ∞, the presence of the hedging bond dominates the overall portfolio. The evolution of the interest rate in a worst case scenario for the overall portfolio becomes the evolution of the interest rate in a worst case scenario for the zero-coupon bond. We have assumed that the market value for the zero-coupon bond, Z, is greater than the worst case scenario value. We then have that

$$e^{-\int_t^S \bar{r}(\tau)d\tau} - Z < 0.$$

The value of our portfolio after we have hedged is

$$V = \sum_{i=1}^{n} C_i e^{-\int_t^{T_i} \bar{r}(\tau)d\tau} + \lambda \left(e^{-\int_t^S \bar{r}(\tau)d\tau} - Z \right).$$

The worst case scenario value for V therefore tends to $-\infty$ as λ tends to ∞.

Similarly, as λ tends to $-\infty$, the evolution of the interest rate in a worst case scenario for the overall portfolio becomes the evolution of the interest rate in a best case scenario for the zero-coupon bond. We have assumed that the market value for the zero-coupon bond, Z, is less than the best case scenario value. We then have that

$$e^{-\int_t^S \bar{r}(\tau)d\tau} - Z > 0.$$

The value of our portfolio after we have hedged is

$$V = \sum_{i=1}^{n} C_i e^{-\int_t^{T_i} \bar{r}(\tau)d\tau} + \lambda \left(e^{-\int_t^S \bar{r}(\tau)d\tau} - Z \right).$$

The worst case scenario value for V therefore tends to $-\infty$ as λ tends to $-\infty$.

There will consequently be a finite optimal value of λ which maximises the value of V in a worst case scenario. Similarly, there will be a finite value of λ which minimises the value of V in a best-case scenario.

Example: We hedge a 5 year zero-coupon bond, with principal 1, with a 1 year zero-coupon bond, with principal 1 and market price 0.905. The spot short-term interest rate is 10%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa.

Without hedging, the worst-case value of the 5 year bond is 0.417 and the best-case value is 0.810. If we hedge on the worst-case scenario, the optimal static hedge is -1.949 of the 1 year bond. In this case, the worst-case value is 0.444 and the best-case value is 0.777.

If we hedge on the best-case scenario, then the optimal static hedge is -2.470 of the 1 year bond. In this case, the worst-case value is 0.443 and the best-case value is 0.775.

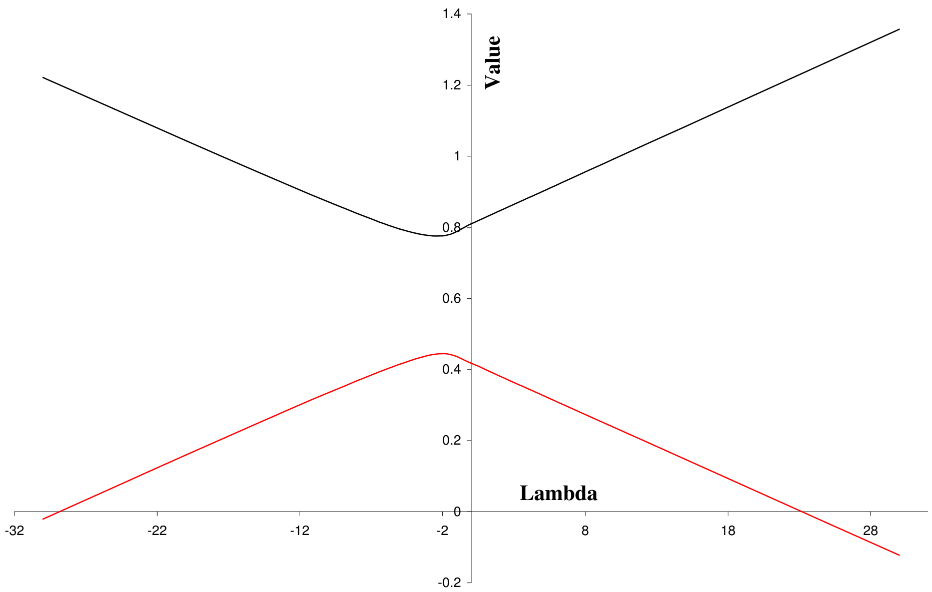


Figure 4.7: Value of a 5 year zero-coupon bond when we hedge with λ of a 1 year zero-coupon bond

Figure 4.7 shows the effect of different hedge quantities on the value of the 5 year bond in both a worst- and best-case scenario. (The lower curve represents the worst-case value and the upper curve, the best-case value).

We remark that the optimal static hedges for the two scenarios are similar but not identical. They are of the same sign and order of magnitude since the form of the interest rate risk is similar in both circumstances. However, the nonlinearity of the problem means that the actual hedge quantities still vary for the two different scenarios.

4.4.2 Hedging with multiple instruments

Suppose that there exist m instruments in the market, and each has a known market price P_j, for $1 \leq j \leq m$. We hedge our original contract with these instruments. We then price the resulting portfolio, Π, in a worst-case scenario.

We hedge with λ_j of the jth instrument. The value of the overall portfolio is

$$\Pi = value(\text{original contract} + \text{hedging instruments}).$$

The cost of setting up this static hedge is equal to the current market value of the hedging instruments,

$$\sum_{j=1}^{m} \lambda_j P_j.$$

The marginal value of our hedged contract, V, is therefore the value of the overall portfolio, Π, minus the cost of the static hedge,

$$V = value(\text{original contract} + \text{hedging instruments}) - \sum_{j=1}^{m} \lambda_j P_j.$$

We show this hedge in Figure 4.8.

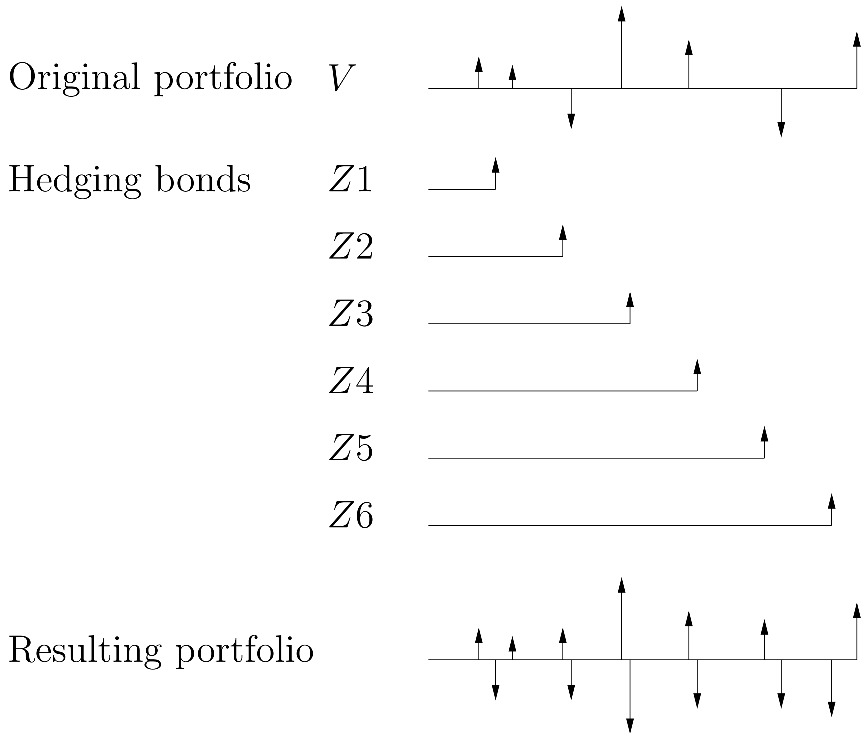


Figure 4.8: Hedging a contract with market-traded instruments

To avoid arbitrage opportunities, we must again assume that the market value of each hedging instrument is contained within the best- and worst-case scenario values for that instrument, obtained from our model.

There will be an optimal static hedge, for which we obtain the maximum possible worst-case scenario value for V. To find this, we maximise the value of the portfolio with respect to the hedge quantities, λ_j:

$$V = \max_{\lambda_j} \left(value(\text{original contract} + \text{hedging instruments}) - \sum_{j=1}^{m} \lambda_j P_j \right).$$

Similarly, there will be an optimal static hedge, for which we obtain the minimum possible best-case scenario value for V. To find this, we minimise the value of the portfolio with respect to the hedge quantities, λ_j:

$$V = \min_{\lambda_j} \left(value(\text{original contract} + \text{hedging instruments}) - \sum_{j=1}^{m} \lambda_j P_j \right),$$

where 'value' now denotes the solution of Equation (4.1).

To include a bid-offer spread in the market price for a hedging instrument, we simply make P_j dependent on the sign of λ_j. If $\lambda_j > 0$ then we are long the bond and the market price is the offer price. If $\lambda_j < 0$ then we are short the bond and the market price is the bid price, i.e.

$$P_j(\lambda_j) = \begin{cases} P_j^+ & \text{if } \lambda_j > 0 \\ P_j^- & \text{if } \lambda_j < 0, \end{cases}$$

where P_j^+ is the offer price and P_j^- is the bid price.

By hedging with market-traded instruments, we can significantly decrease the spread between the worst- and best-case valuations for our contract. For instance,

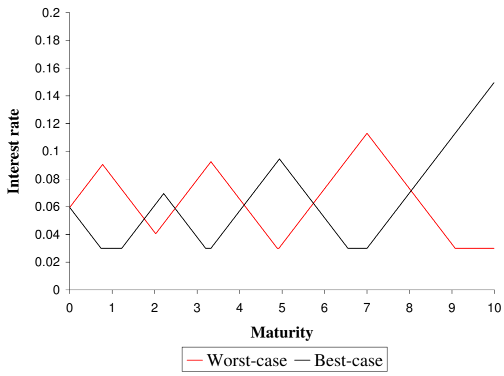
Example: We hedge a 4 year zero-coupon bond, with principal 1, with zero-coupon bonds, with principal 1. These hedging bonds are shown in Table 4.1. The spot short-term interest rate is 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa.

Hedging bond	Maturity (yrs)	Market price
Z_1	0.5	0.970
Z_2	1	0.933
Z_3	2	0.868
Z_4	3	0.805
Z_5	5	0.687
Z_6	7	0.579
Z_7	10	0.449

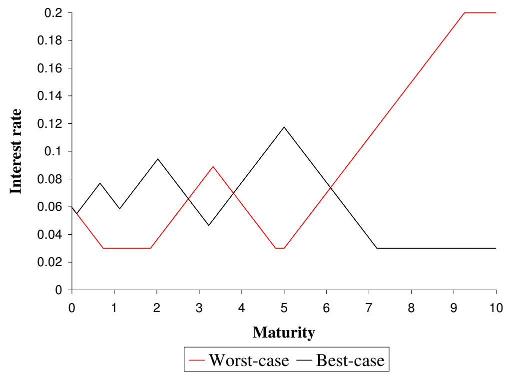
Table 4.1: The zero-coupon bonds with which we hedge

The results of the valuation for the 4 year bond, with and without hedging, are shown in Table 4.2. The optimal static hedges for the worst- and best-case valuations are shown in Table 4.3. The short-term interest rate paths for the optimally-hedged worst- and best-case scenarios are shown in Figure 4.9.

We note that the hedge quantities are different (although similar) for worst and best case scenarios, and that not all hedging instruments are used in the optimal hedging strategy. Only the hedging bonds most similar in form to our contract figure



(a)



(b)

Figure 4.9: Interest rate paths for the 4 yr bond optimally hedged in (a) a worst-case scenario and (b) a best-case scenario

	Worst case	Best case
No hedge	0.575	0.877
Optimal hedge on worst-case	**0.730**	0.761
Optimal hedge on best-case	0.728	**0.758**

Table 4.2: Value of a 4 year zero-coupon bond

Hedging bond	Worst case hedge quantity	Best case hedge quantity
Z_1	0.000	0.002
Z_2	-0.004	-0.004
Z_3	0.169	0.117
Z_4	-0.699	-0.653
Z_5	-0.468	-0.481
Z_6	0.020	0.000
Z_7	0.000	0.000

Table 4.3: The optimal static hedges for a 4 year zero-coupon bond

noticeably in the static hedge. Hedging has significantly reduced the spread in the value of the 4 year bond, from a spread of 0.302 to 0.028. The latter figure is the difference between the two bold values in Table 4.2, i.e. the difference between the two optimally-hedged values. It is important to note that the hedges for these two values are different. Which hedge we create is dependent on whether we want to guarantee that the worst- or best-case price is optimal (i.e. on whether we are buying or selling the contract). When we have set up a particular hedge, the relevant row in the table describes the consequent possible spread for the contract price.

4.5 The Yield Envelope

The tools we have developed allow us to calculate the Yield Envelope. This is an extension of the yield curve. At a maturity for which there are no traded instruments, we obtain a yield spread.

We first calculate the lowest and highest possible values of a zero-coupon bond with principal 1, and maturity at time T, that is, the worst- and best-case scenario valuations. We can then calculate the lowest and highest possible yields at this time, using the formula

$$Y = -\frac{\log Z}{T}.$$

We can reduce the yield spread using hedging. If we hedge our zero-coupon bond with market-traded zero-coupon bonds of known value, and various maturities, then we can reduce the spread between the worst- and best-case scenario bond prices at time T, and therefore reduce the yield spread at time T.

Example: We hedge our zero-coupon bond (with maturity at time T and a principal of 1) with the zero-coupon bonds in Table 4.1. The spot short-term interest rate is currently 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa. The results are shown in Table 4.4, where bold denotes that a bond of that maturity is available in the market, and Figure 4.10.

For a maturity at which there is a market-traded zero-coupon bond, we find that both our worst- and best-case scenario valuations equal this market value. This is because we can completely hedge our zero-coupon bond with the market-traded bond and this will be the optimal static hedge in both cases (assuming that the market value of the bond is within its worst- and best-case scenario valuations). For this reason, we find that the Yield Envelope closes up to the observable yield at a maturity for which there exists a market-traded instrument. Therefore, the Yield Envelope is consistent with observable yield data.

Maturity (yrs)	Worst case bond value	Best case bond value	Yield in worst case	Yield in best case
0.00	**1.000**	**1.000**	**0.060**	**0.060**
0.25	0.985	0.986	0.059	0.057
0.50	**0.970**	**0.970**	**0.061**	**0.061**
0.75	0.952	0.953	0.066	0.065
1.00	**0.933**	**0.933**	**0.069**	**0.069**
1.25	0.913	0.916	0.073	0.070
1.50	0.895	0.902	0.074	0.069
1.75	0.881	0.886	0.073	0.069
2.00	**0.868**	**0.868**	**0.071**	**0.071**
2.50	0.832	0.840	0.074	0.070
3.00	**0.805**	**0.805**	**0.072**	**0.072**
3.50	0.765	0.784	0.077	0.069
4.00	0.730	0.758	0.079	0.069
4.50	0.705	0.725	0.078	0.071
5.00	**0.687**	**0.687**	**0.075**	**0.075**
5.50	0.648	0.677	0.079	0.071
6.00	0.618	0.643	0.080	0.073
6.50	0.595	0.613	0.080	0.075
7.00	**0.579**	**0.579**	**0.078**	**0.078**
7.50	0.542	0.567	0.082	0.076
8.00	0.512	0.552	0.084	0.074
8.50	0.488	0.531	0.084	0.075
9.00	0.469	0.506	0.084	0.076
9.50	0.457	0.477	0.083	0.078
10.00	**0.449**	**0.449**	**0.080**	**0.080**

Table 4.4: Hedging a zero-coupon bond with maturity T

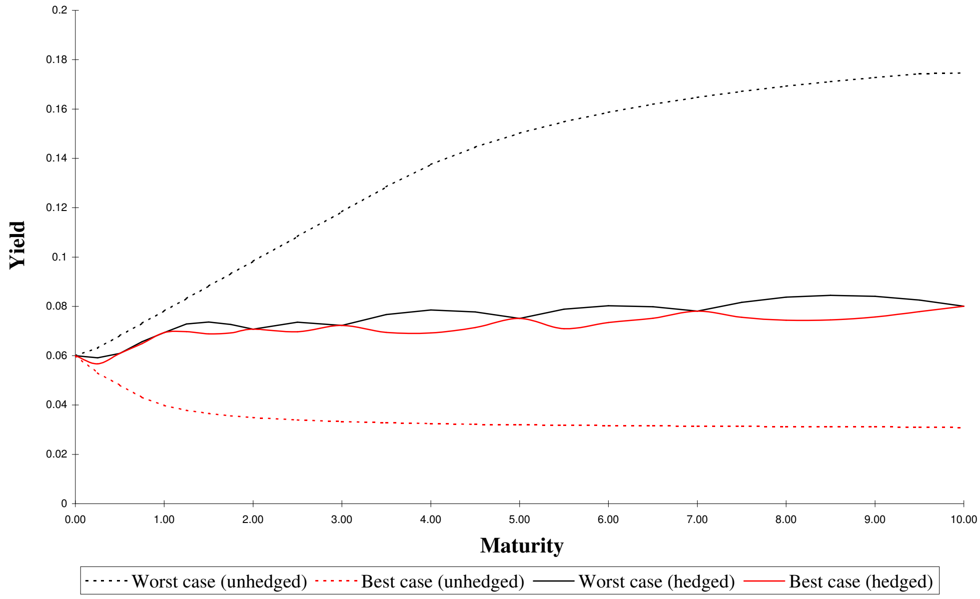


Figure 4.10: Yield Envelope with hedging

Figures 4.11 and 4.12 show the yield at varying maturities and with varying initial spot short-term interest rate, in a worst-case scenario and in a best case scenario, respectively. In both cases, we have hedged with the zero-coupon bonds (with unit principal) in Table 4.5. (The interest rate bounds for these figures are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa).

Hedging bond	Maturity (yrs)	Market price
Z_1	0.5	0.950
Z_2	1	0.899
Z_3	2	0.803
Z_4	3	0.712
Z_5	5	0.566
Z_6	7	0.448
Z_7	10	0.304

Table 4.5: The hedging bonds for Figures 4.11 and 4.12

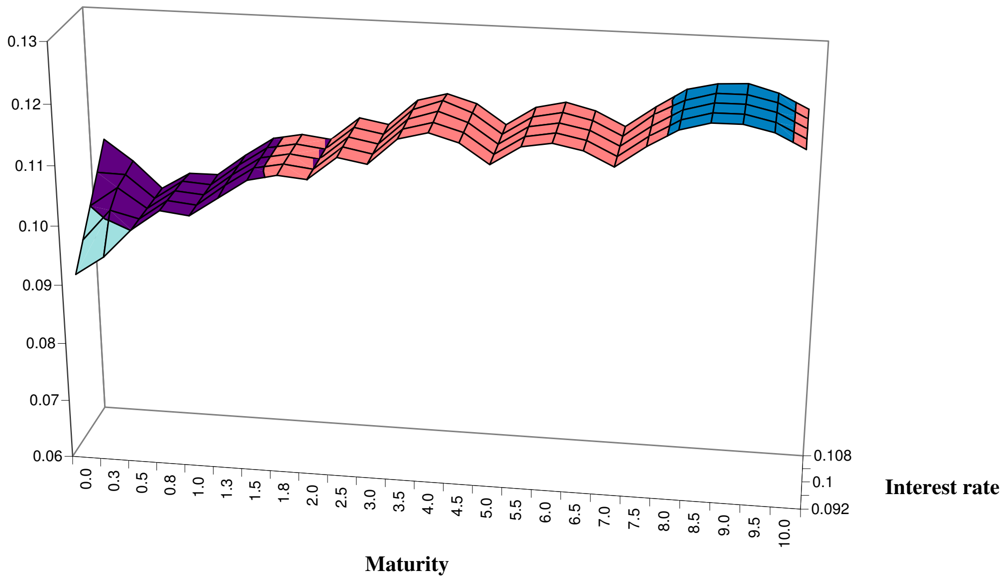


Figure 4.11: Yield in a worst-case scenario

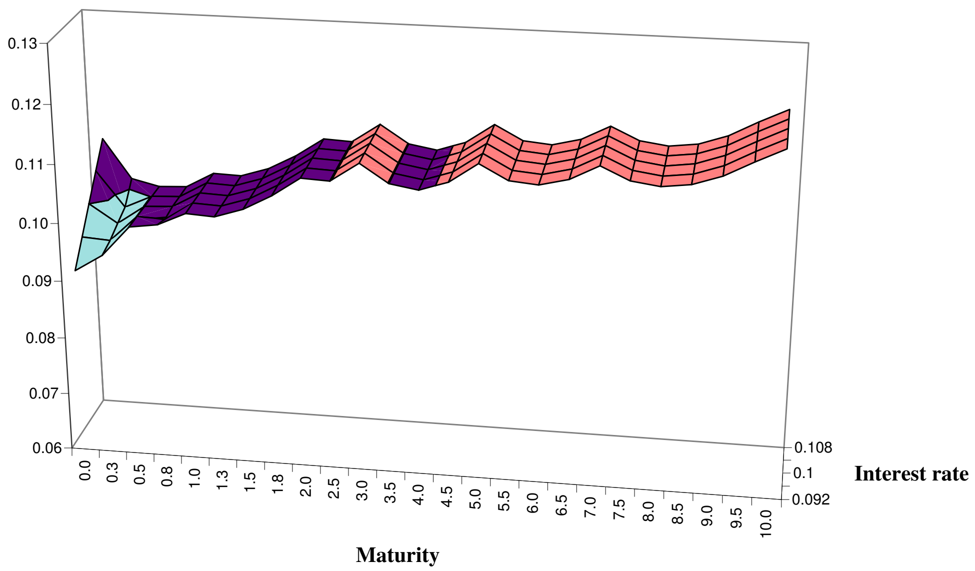


Figure 4.12: Yield in a best-case scenario

4.6 Swaps

With the methodology that we have built up, we can price and hedge a variety of simple fixed-income products. To value any instrument whose cashflows solely depend on the short-term interest rate, we simply solve our partial differential equation and include these cashflows as jump conditions. We now show how to price swaps using this approach, and in the next section, apply the method to caps and floors.

We will consider a swap with a τ-period reference rate (measured τ before the payment date), an annualised fixed rate r_f, maturity at time T, a principal of P and swaplet dates every τ. There are two approaches to the pricing of this contract. Market practice is to decompose the swap into a portfolio of zero-coupon bonds, as shown in Section 1.1.2. These can then be priced using our model for the short-term interest rate.

We solve Equation (2.6), for the worst-case scenario value, or Equation (4.1), for the best-case scenario value, with a final condition of,

$$V(r,T) = -P(1 + r_f/\tau),$$

jump conditions of,

$$V(r,t_s^-) = -P(r_f/\tau) + V(r,t_s^+),$$

at each swaplet date (before maturity), t_s, and a jump condition of,

$$V(r,t_r^-) = P + V(r,t_r^+),$$

at a time τ before the first swaplet date, t_r say.

An alternative method, more often seen in academia, is to approximate the τ-period rate by substituting our annualised short-term interest rate (which is instantaneously compounding) in its place. We can then price the cashflows directly using our model since they now only depend on our short-term interest rate. The method is only valid when the period τ is short enough to make this a reasonable assumption.

To solve the problem using this latter approach, we solve Equation (2.6) or Equation (4.1) with a final condition of,

$$V(r,T) = P(r - r_f)/\tau,$$

and the jump condition,

$$V(r,t_s^-) = P(r - r_f)/\tau + V(r,t_s^+),$$

at each swaplet date, t_s, before maturity.

(Note that we divide by τ because the fixed rate, r_f, and (in the second approach) the short-term interest rate, r, are annualised rates, and we require rates that hold for a period of τ only).

Example: We price and hedge a 2 year swap, on the three month interest rate, with a principal of 1. Swaplets occur every three months and the last payment is 3 years from today. The spot short-term interest rate is 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa.

We use the yield curve determined from the hedging bonds of table 4.1 and the decomposition approach to find the 'fair value' for r_f. We linearly interpolate to construct the yield curve, as shown in Figure 4.13. We can then price the relevant zero-coupon bonds and find that the fixed interest rate should be 7.44% per annum.

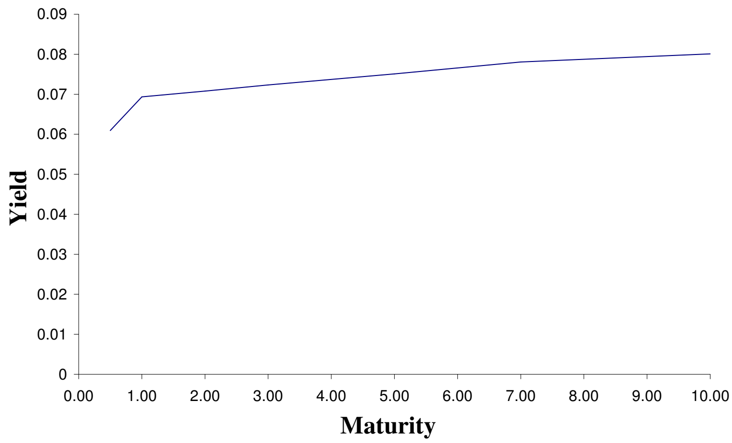


Figure 4.13: The yield curve for the bonds from Table 4.1

We first value the swap using the decomposition into zero-coupon bonds. The results of this valuation, with and without hedging, are shown in Table 4.6. The optimal static hedges for the worst- and best-case valuations are shown in Table 4.7 (where we hedge with the bonds in Table 4.1).

	Worst case	Best case
No hedge	-0.0822	0.1056
Optimal hedge on worst-case	-0.0002	0.0003
Optimal hedge on best-case	-0.0003	0.0002

Table 4.6: Value of the decomposed swap

Hedging bond	Worst case hedge quantity	Best case hedge quantity
Z_1	0.000	0.000
Z_2	-0.977	-0.983
Z_3	0.092	0.097
Z_4	1.037	1.035
Z_5	0.000	0.000
Z_6	0.000	0.000
Z_7	0.000	0.000

Table 4.7: The optimal static hedges for the decomposed swap

We also value the swap using the short-term interest rate to approximate the three month rate. The results of the valuation, with and without hedging, are shown in Table 4.8. The optimal static hedges for the worst- and best-case valuations are shown in Table 4.9.

	Worst case	Best case
No hedge	-0.0824	0.1095
Optimal hedge on worst-case	-0.0060	0.0056
Optimal hedge on best-case	-0.0169	0.0036

Table 4.8: Value of the approximated swap

Hedging bond	Worst case hedge quantity	Best case hedge quantity
Z_1	-0.050	1.625
Z_2	-0.746	-1.863
Z_3	-0.312	-0.036
Z_4	1.237	1.163
Z_5	0.000	0.000
Z_6	0.000	0.000
Z_7	0.000	0.000

Table 4.9: The optimal static hedges for the approximated swap

The value of the swap before hedging is similar for both the decomposition and approximation methods. The small difference between the two prices could be due to the choice of fixed rate via the former rather than the latter approach. We could therefore remark that the latter method is a reasonable approximation to make.

However, the effect of hedging is far more effective for the decomposition method, where the spread is reduced from 0.1878 to 0.0004, than for the approximation method, where the spread is reduced from 0.1919 to 0.0096. (Even so, hedging has still had a significant effect in the latter case). A plausible explanation for the greater spread reduction could be that, in the first case, we are hedging cashflows with cashflows of the same form (both are zero-coupon bonds) whereas in the second case, the cashflows to be hedged are of a different form and consequently, the hedge will not be as effective.

We conclude that in choosing which instruments to use in the setting up of a static hedge, we should identify those products most similar in form to our contract as the most suitable hedging instruments. (For example, when we come to price the convertible bond, in Section 5.4, zero-coupon bonds are a natural choice of hedging instrument, but when we price the index amortising rate swap, in Section 5.3, we choose to hedge with swaps).

4.7 Caps and floors

We consider an interest rate agreement with a τ-period reference rate, an annualised strike rate r_s, with maturity at time T, a principal of P and cashflow dates every τ. As with the swap valuation, there are two approaches to the pricing of these contracts. Market practice is to decompose the contract into a portfolio of bond options, as shown in Section 1.1.5. However, to value a bond option, we will need to extend our pricing methodology. We discuss the valuation of such options in the next chapter.

Alternatively, we can again approximate the τ-period rate by substituting our short-term interest rate in its place. We then price the cashflows directly using our model. The method is only valid when the period τ is short enough to make this a reasonable assumption.

To solve the problem using the latter approach, we solve Equation (2.6), for the worst-case scenario value, or Equation (4.1), for the best-case scenario value, with jump and final conditions dependent on the particular specification of the interest rate agreement.

For a cap, we apply a final condition of

$$V(r,T) = P\max(r - r_s, 0)/\tau,$$

and the jump condition

$$V(r,t_c^-) = P\max(r - r_s, 0)/\tau + V(r,t_c^+),$$

at each caplet date, t_c, before maturity. We must again divide by τ because the short-term interest rate, r, and the strike rate, r_s, are annualised rates, and we require rates that hold for a period of τ only.

For a floor, we apply a final condition of

$$V(r,T) = P \max(r_s - r, 0)/\tau,$$

and the jump condition

$$V(r,t_f^-) = P \max(r_s - r, 0)/\tau + V(r, t_f^+),$$

at each floorlet date (before maturity), t_f.

Example: We price and hedge 2 year contracts, on the three month interest rate, with a principal of 1. Cashflows occur every three months and the last payment is 2 years from today. The spot short-term interest rate is 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa.

We hedge the contracts with the zero-coupon bonds in Table 4.1. The results for the valuation are shown in Tables 4.10 and 4.11.

Cap (5% strike)	Worst case	Best case
No hedge	0.000	0.096
Optimal hedge on worst-case	0.035	0.072
Optimal hedge on best-case	0.003	0.046

Cap (6% strike)	Worst case	Best case
No hedge	0.000	0.078
Optimal hedge on worst-case	0.018	0.072
Optimal hedge on best-case	0.000	0.036

Cap (7% strike)	Worst case	Best case
No hedge	0.000	0.060
Optimal hedge on worst-case	0.003	0.072
Optimal hedge on best-case	-0.058	0.022

Table 4.10: Value of a cap with varying strike

Floor (5% strike)	Worst case	Best case
No hedge	0.000	0.031
Optimal hedge on worst-case	0.000	0.031
Optimal hedge on best-case	-0.035	0.005

Floor (6% strike)	Worst case	Best case
No hedge	0.000	0.050
Optimal hedge on worst-case	0.000	0.050
Optimal hedge on best-case	-0.050	0.011

Floor (7% strike)	Worst case	Best case
No hedge	0.000	0.069
Optimal hedge on worst-case	0.002	0.063
Optimal hedge on best-case	-0.054	0.020

Table 4.11: Value of a floor with varying strike

We remark that hedging appears to be more effective at reducing the best-case price than raising the worst-case price. A cause of this could be that, in the worst-case, the interest rate will move to ensure that the caplets or floorlets have zero value. To have an effect hedging must either move the worst-case interest rate path so that these cashflows have value, or take advantage of the worst-case path to make a profit on the hedging bonds. On the other hand, to reduce the best-case price, the hedging only has to counteract the positive value of the cashflows.

4.8 Applications of the model

There are a number of different ways in which the theory and techniques that we have developed can be applied in the marketplace.

4.8.1 Identifying arbitrage opportunities

Assuming that the movements of the interest rate conform to our constraints, the realised value of any contract must lie within our bounds. We have therefore obtained a spread for the possible price of a contract. If we find a contract whose value lies outside of these bounds, then we have identified an arbitrage opportunity. We should obviously buy the contract if its current market value is below this range, and sell the contract if its value is above the range.

4.8.2 Establishing prices for the market maker

If we are making a market in a contract, then setting our bid price at the low end and our offer price at the high end of the spread range guarantees that we cannot lose money on any deal. This technique is particularly appropriate in the OTC contracts business where spreads are usually much higher because of the often exotic nature and illiquidity of the product.

4.8.3 Static hedging to reduce interest rate risk

In both of the above applications, we can create a static hedge, using market-traded instruments, to reduce the spread in our prices and make them more competitive. It is important to note that these new prices are only valid as long as we create the static hedge in reality (on trading the contract).

We can static hedge with any product in the market. It does not have to be similar to the contract we are pricing. However, the most effective static hedges are likely to be those which are made up of contracts similar in form to that being priced. Hence, the more exotic a contract, the fewer similar contracts that will be available and the larger the spread, as we would expect.

This form of hedging is 'static' in that, once we have set up a hedge, we should leave it until maturity of our contract. However, if the market changes significantly, then a re-hedging of the contract may yield a different static hedge, with a higher worst-case (or lower best-case) price. Subject to transaction costs, updating our static hedge guarantees a better worst-case (or best-case) price and 'locks-in' an additional profit of the difference between this price and the old one.

4.8.4 Risk management - a measure of absolute loss

Conventional risk management models produce results of the form - a maximum loss of X with probability Y or model the possibility of extremal events [29], [45]. Since our spreads for prices are absolute, they give us a good measure of the total risk in a portfolio, and we can predict the absolute maximum loss possible [14].

Furthermore, by static hedging, we can generate an optimal static hedge to reduce this interest rate risk. This would not be possible with a linear model calibrated to the yield curve. This is because the price of a contract would be invariant to the addition of hedging instruments if they had originally been used to construct the yield curve. Hence, a further use of the model is to construct static hedges for portfolios, regardless of which interest rate model we choose to finally price them.

4.8.5 A final remark on the application of the model

In Chapter 6 we describe various extensions to our model. These extensions allow for interest rate evolutions that are indistinguishable from those observed in practice. With these extensions in place, we can be confident that actual interest rate movements will definitely lie within our bounds. The applications that we have described above then become feasible in the marketplace.

We end this chapter by considering a real world application.

4.9 A real portfolio

We use our model to price and hedge a fixed-rate lease portfolio, consisting of various set cashflows, owned by Dresdner Kleinwort Benson (DKB) [49]. There are 15,300 cashflows, over a period of 1,830 days, as shown in Figure 4.14. The current yield curve, as of 8th January 1998, is shown in Figure 4.15. The benchmark contracts that make up this yield curve are shown in Table 4.12.

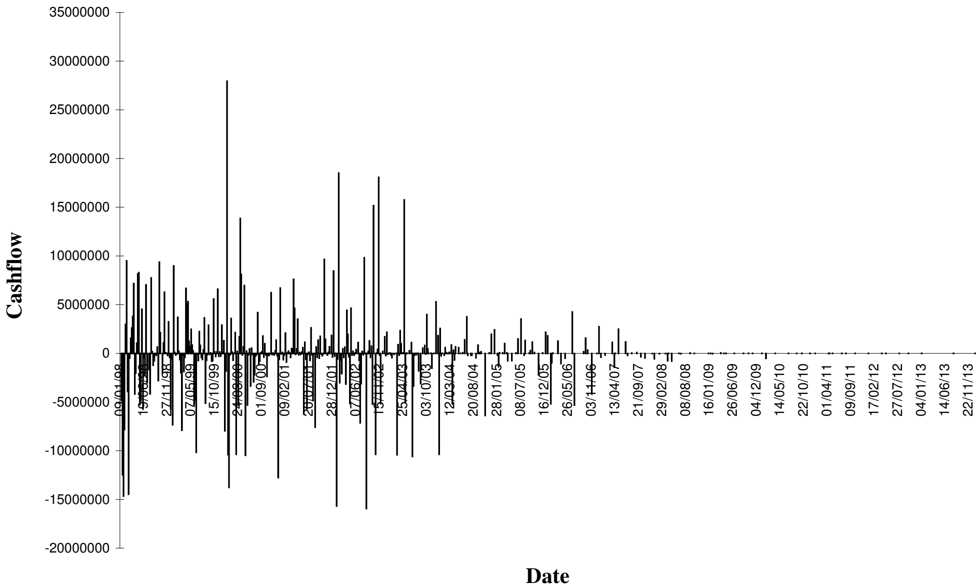


Figure 4.14: Cashflows of the leasing portfolio

First of all, we price the portfolio off the yield curve. The present value of the portfolio is found to be $-£3,539,362$. We can then examine the sensitivity of this price to changes in the yield curve. Table 4.13 shows the sensitivity of the price to instantaneous parallel shifts in the yield curve. This is a traditional measure of the extent to which the portfolio is hedged against future interest rate movements. From

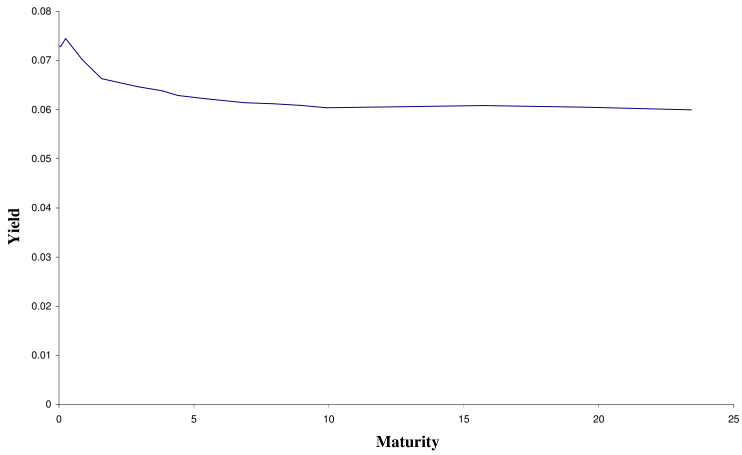


Figure 4.15: The current yield curve

Hedging bond	Coupon rate	Maturity	Market price (£)
1M T-Bill	0	04 FEB 1998	99.46
3M T-Bill	0	08 APR 1998	98.15
1Y GILT	0.12	20 NOV 1998	104.125
2Y GILT	0.06	10 AUG 1999	99.125
3Y GILT	0.08	07 DEC 2000	104.03125
4Y GILT	0.07	06 NOV 2001	102.09375
5Y GILT	0.07	07 JUN 2002	102.75
6Y GILT	0.08	10 JUN 2003	108.09375
7Y GILT	0.0675	26 NOV 2004	103.40625
8Y GILT	0.085	07 DEC 2005	114.75
9Y GILT	0.075	07 DEC 2006	109.625
10Y GILT	0.0725	07 DEC 2007	108.96875
15Y GILT	0.08	27 SEP 2013	119.21875
20Y GILT	0.0875	25 AUG 2017	130.75
25Y GILT	0.08	07 JUN 2021	125.03125

Table 4.12: The benchmark bonds (principal £100)

the point of view of the owners of the portfolio (DKB), the concern was to hedge the portfolio to try and prevent significantly increased loss (hence we are interested in the worst-case scenario value). We note that a 2% downward shift reduces the value of the portfolio to $-£4,432,153$, a loss of $£892,791$.

Using the uncertain interest rate model, with bounds of 3% and 20% on the interest rate, and -4% pa and 4% pa on its growth rate, we find that the worst-case scenario value for the portfolio is $-£6,135,878$. (The best-case value is $-£1,898,173$). Our worst-case scenario value is much lower than the price predicted off the yield curve.

Yield shift	Present value (£)
2%	-2,620,568
1%	-3,083,010
0%	-3,539,362
-1%	-3,989,194
-2%	-4,432,153

Table 4.13: Present value of the portfolio with parallel shifts in the yield curve

However, we can static hedge the portfolio, to increase this worst-case scenario value. We use the benchmark bonds (those used to calculate the yield curve) to construct an optimal static hedge for the worst-case scenario value of the portfolio. The hedge quantities for this static hedge are shown in Table 4.14 and the hedging cashflows are shown in Figure 4.16. The marginal worst-case value of the portfolio, with this hedge in place, is $-£4,009,582$. This value is only $-£470,220$ less than the original price predicted from the yield curve and is comparable to the present value with a 1% downwards yield shift. (The best-case value with this static hedge in place is $-£3,112,337$).

Hedging bond	Hedge quantity
1M T-Bill	763
3M T-Bill	557
1Y GILT	745
2Y GILT	-7113
3Y GILT	-123014
4Y GILT	7064
5Y GILT	-30844
6Y GILT	1689
7Y GILT	3135
8Y GILT	-6945
9Y GILT	-285
10Y GILT	11674
15Y GILT	-123
20Y GILT	1457
25Y GILT	-1299

Table 4.14: The optimal hedge for the worst-case scenario valuation

The optimal static hedge has significantly increased the worst-case price. If we set up this static hedge, then we are guaranteed to lose no more than the equivalent of a 1% yield shift down. The static hedge therefore reduces the downside risk in a

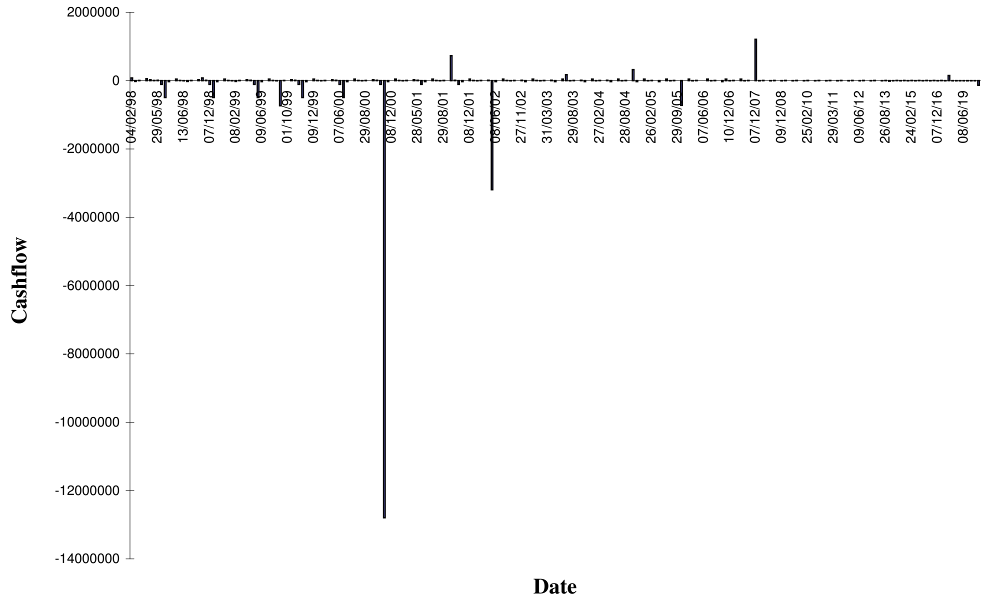


Figure 4.16: The hedging cashflows

robust fashion (a guaranteed maximum loss) but still leaves some upside (the new best-case value is approximately equivalent to a 1% upwards yield shift). In addition, if we re-price our hedged portfolio off the yield curve, then we find the same result as before. This is because we have hedged with the benchmark bonds that were used to calibrate the yield curve in the first place. Since pricing off the curve is linear, addition of these bonds will have no effect on the price of the portfolio.

Examining worst-case scenarios and yield shifts are both reasonable approaches to interest rate risk management. But there is a fundamental difference between these two methodologies. The advantage of the worst-case scenario approach is that the actual realised value of the portfolio can never be lower than this hedged worst-case value (assuming that the interest rate stays within the constraints), when we set up the optimal static hedge. On the other hand, if the actual shift in the yield curve is larger than -1%, then the realised value when we price off of the yield curve will be far lower than we first thought.

Also, our method gives us a systematic approach for creating a static hedge of liquid, market-traded products. This static hedge decreases the interest rate risk in our position and hence improves the worst-case price.

Chapter 5

Pricing and hedging complex products

In this chapter, we apply our uncertain interest rate model to the pricing and hedging of more complex products. We are unable to price these instruments directly using our current methodology. This is because their specification is such that we require a further factor to complete our model (asset price, for instance) or must take into account some form of optionality.

5.1 Bond options

We begin by considering the bond option. This is the first contract that we have encountered which includes optionality in its specification. We first demonstrate how to price a European option on a zero-coupon bond. This approach will still be appropriate when we hedge the option with the underlying bond. However, if we wish to hedge with a different instrument, or price the American option, we will need to modify our approach to ensure that we have a consistent and optimal worst- or best-case interest rate path. This will be a more general method of solution and, consequently, computationally more intensive.

5.1.1 Pricing a European option on a zero-coupon bond

We consider a European option with a zero-coupon bond as the underlying. The bond has a principal payment of P at time T_Z. The option expires at time $T_O < T_Z$ and has payoff $\Lambda(Z)$,

e.g. for a long call option,

$$\Lambda(Z) = \max(Z - E, 0),$$

where E is the exercise price of the option.

We will consider the option value in a worst-case scenario and price the option in two stages. We first ascertain the spread for the zero-coupon bond price at expiry of the option.

Consider a fixed value of r, r^* say. For this value of r, we find the worst-case scenario price, $Z^-(r^*, T_O)$, and the best-case scenario price, $Z^+(r^*, T_O)$, for the zero-coupon bond, at time T_O. (The solution to this problem is discussed in Section 4.3). We then know that at T_O, the actual bond price, $Z(r^*, T_O)$, lies between these two values, i.e.

$$Z^-(r^*, T_O) \leq Z(r^*, T_O) \leq Z^+(r^*, T_O).$$

We can thus find the spread in price for the zero-coupon bond for each value of r between r^- and r^+, at time T_O.

Figure 5.1 shows the form that these results take.

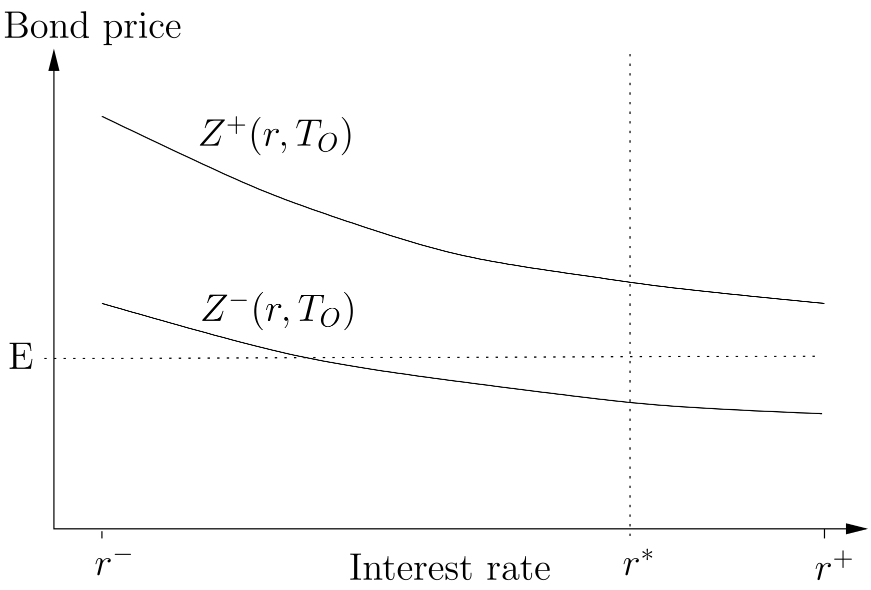


Figure 5.1: Worst- and best-case prices for the underlying zero-coupon bond

We then consider the value of the option at expiry, for this fixed value of r, r^*. In general, this is $\Lambda(Z(r^*, T_O))$. Figure 5.2 shows the extremal possible values for a long call option.

To find the option value in a worst-case scenario, we determine the minimum possible value of $\Lambda(Z(r^*, T_O))$, when $Z(r^*, T_O)$ can vary between $Z^-(r^*, T_O)$ and $Z^+(r^*, T_O)$. This minimum value is $\Lambda^-(r^*)$, say, where

$$\Lambda^-(r^*) = \min_{Z(r^*, T_O)} \left(\Lambda(Z(r^*, T_O))\right),$$

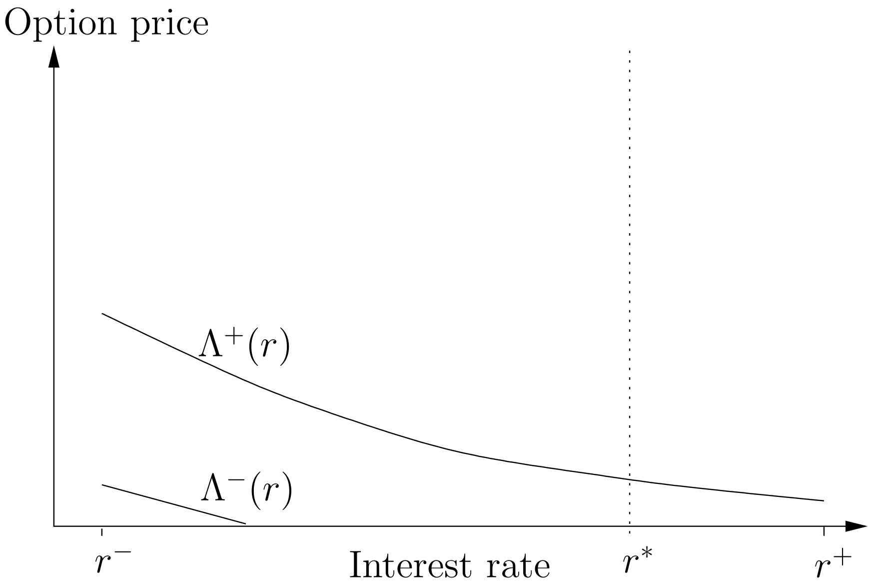


Figure 5.2: Value of the call option

e.g. for a long call option,

$$\Lambda^-(r^*) = \min_{Z(r^*,T_O)} \left(\max(Z(r^*,T_O) - E, 0)\right).$$

The minimum value will occur, for this call option, when the bond price $Z(r^*, T_O)$ is as low as possible. We therefore take $Z(r^*, T_O) = Z^-(r^*, T_O)$ and find that,

$$\Lambda^-(r^*) = \max(Z^-(r^*, T_O) - E, 0).$$

This holds for each possible value of r^* between r^- and r^+. We have therefore determined the worst-case scenario value of the option, $\Lambda^-(r)$, at its expiry, for all possible r.

We solve Equation (2.6) with $\Lambda^-(r)$ as our final data, to determine the worst-case value of the option, $V^-(r,t)$ say, at earlier times $t \leq T_O$. The method of solution is shown schematically in Figure 5.3.

We can also value the option in a best-case scenario. Here, we find the maximum possible value of $\Lambda(Z(r, T_O))$, $\Lambda^+(r)$ say, where

$$\Lambda^+(r^*) = \max_{Z(r^*,T_O)} \left(\Lambda(Z(r^*, T_O))\right),$$

and solve Equation (4.1) with this as our final data to determine the best-case value of the option, $V^+(r,t)$, at earlier times.

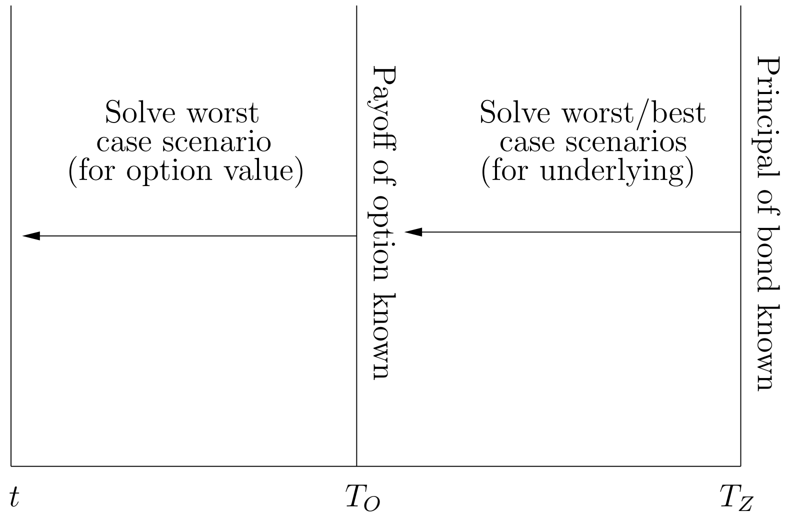


Figure 5.3: Pricing a bond option in a worst-case scenario

5.1.2 Hedging the European option with the underlying zero-coupon bond

We can hedge our option with the underlying bond to reduce the spread between the worst- and best-case prices. We will buy λ of the bond, which has a current market price of Z_M and price the resulting portfolio in a worst-case scenario.

The first stage of the pricing process remains the same. For each possible fixed value of r, r^*, we find the worst and best-case prices for the zero-coupon bond at time T_O, $Z^-(r^*, T_O)$ and $Z^+(r^*, T_O)$ respectively. We then value our combined portfolio of the option and the hedging bond, Π, at expiry of the option. The portfolio is worth,

$$\Pi(Z(r^*, T_O), T_O) = \Lambda(Z(r^*, T_O)) + \lambda Z(r^*, T_O).$$

We want to value this portfolio in a worst-case scenario. We therefore determine the minimum possible value of the portfolio, when $Z(r^*, T_O)$ can vary between $Z^-(r^*, T_O)$ and $Z^+(r^*, T_O)$. This value is $\Pi^-(r^*, T_O)$ say, where

$$\Pi^-(r^*, T_O) = \min_{Z(r^*, T_O)} \left(\Lambda(Z(r^*, T_O)) + \lambda Z(r, T_O)\right).$$

We perform this calculation for each value of r^* between r^- and r^+. We then have the minimum possible value of the portfolio, $\Pi^-(r, T_O)$, for all r at time T_O. This is the worst-case value of the portfolio at expiry of the option.

We solve Equation (2.6) with $\Pi^-(r, T_O)$ as final data to find the value of the portfolio in a worst-case scenario at earlier times, $\Pi^-(r, t)$. This is our minimum possible value for the portfolio when we hedge with λ of the bond.

To find the marginal worst-case value of the option, $V^-(r,t)$, we must subtract the cost of the static hedge, λZ_M, to obtain,

$$V^-(r,t) = \Pi^-(r,t) - \lambda Z_M.$$

We can maximise the worst-case value of the option, with respect to λ, to find the optimal static hedge of the underlying bond, λ, and the best worst-case value,

$$V^-(r,t) = \max_{\lambda} \left(\Pi^-(r,t) - \lambda Z_M \right).$$

This is the minimum possible value for the optimally hedged bond option.

We can also value the hedged option in a best-case scenario. Here, we find the maximum possible value of the portfolio at expiry of the option, $\Pi^+(r, T_O)$ say, where,

$$\Pi^+(r^*, T_O) = \max_{Z(r^*,T_O)} \left(\Lambda(Z(r^*, T_O)) + \lambda Z(r, T_O) \right).$$

We can then work out the current value of the portfolio in a best-case scenario, by solving Equation (4.1) with $\Pi^+(r, T_O)$ as final data. To determine the optimal static hedge and the minimal best-case value for the bond option, we minimise with respect to λ to find,

$$V^+(r,t) = \min_{\lambda} \left(\Pi^+(r,t) - \lambda Z_M \right).$$

This is the maximum possible value for the optimally hedged bond option.

Example: We price vanilla European call and put options with expiry in 1 year and exercise price E, on a zero-coupon bond with principal 1 and maturity in 5 years. The current market price of the bond is 0.687. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

The results for the option valuation, without hedging and with the optimal static hedges for both worst- and best-case valuations, are shown in Tables 5.1 and 5.2.

We see that hedging significantly reduces the spread in price and that the optimal hedges for both worst- and best-case valuations are similar. When the option is significantly 'in the money', the hedge is almost exactly one of the underlying (short for the call, or long for the put). To reduce the spread further, however, we will need to hedge the option with contracts other than the underling bond.

$E = 0.4$	Worst case	Best case	Hedge quantity
No hedge	0.102	0.468	-
Optimal hedge on worst-case	0.303	0.318	-0.993
Optimal hedge on best-case	0.303	0.318	-0.996

$E = 0.5$	Worst case	Best case	Hedge quantity
No hedge	0.009	0.372	-
Optimal hedge on worst-case	0.207	0.226	-0.993
Optimal hedge on best-case	0.207	0.226	-0.996

$E = 0.6$	Worst case	Best case	Hedge quantity
No hedge	0.000	0.276	-
Optimal hedge on worst-case	0.111	0.215	-0.993
Optimal hedge on best-case	0.077	0.161	-0.743

Table 5.1: Value of a European call option hedged with the underlying

$E = 0.8$	Worst case	Best case	Hedge quantity
No hedge	0.000	0.268	-
Optimal hedge on worst-case	0.052	0.164	0.996
Optimal hedge on best-case	0.021	0.117	0.710

$E = 0.9$	Worst case	Best case	Hedge quantity
No hedge	0.012	0.361	-
Optimal hedge on worst-case	0.144	0.177	0.996
Optimal hedge on best-case	0.144	0.177	0.993

$E = 1.0$	Worst case	Best case	Hedge quantity
No hedge	0.108	0.453	-
Optimal hedge on worst-case	0.237	0.273	0.996
Optimal hedge on best-case	0.237	0.273	0.993

Table 5.2: Value of a European put option hedged with the underlying

5.1.3 Hedging the European option with other instruments

Unfortunately, this approach to option pricing is no longer appropriate when we try to hedge the option with a contract that is not the underlying. This is because there will now be two quantities that we need to determine at expiry of the option. The first of these is the spread for prices for the underlying zero-coupon bond. Without this spread, we cannot determine the worst-case payoff for the option. The second quantity of interest is the value of the hedging instruments in a worst-case scenario. The worst-case value of the overall portfolio at expiry of the option will be the sum of the value of these instruments and the worst-case value of the option payoff.

We could perform two separate valuations to find these two quantities. To achieve this we would solve Equation (2.6) for the underlying bond price and then resolve the same equation, with different final data and jump conditions, to find the value of the hedging instruments, at expiry of the option. However, there is no guarantee that these two solutions would have the same interest rate path. This is because of the nonlinearity of the pricing equation. If the interest rate paths were different, then there would be an inconsistency in our pricing methodology.

There are a number of ways in which this could manifest itself. Our spread for the underlying bond price may be be too large - the presence of the hedging bonds should narrow the spread in zero-coupon bond price at expiry of the option (as shown in Section 4.4). Alternatively, our worst-case value for the portfolio of hedging instruments may be too low. In either occurrence, our eventual option price will not be optimal. It will be lower than the actual worst-case value (hence it will still be a valid lower bound, just not the best one).

We will develop a more general approach to the pricing of contracts with optionality by considering the cases in which we exercise and do not exercise the option separately, i.e. we consider all of our options individually and then choose the appropriate course of action. The drawback to this approach will be that for each instance of 'either/or' optionality, we double the number of cases to be considered.

Let Π_0 be the overall portfolio of cashflows that we would have if we chose to exercise the option at expiry. This consists of the cashflows due to the hedging instruments plus the cashflows that we would receive if we were to exercise the option. In the case of a call option, the latter cashflows would be those of the underlying bond (for a put option, they would be the cashflows for the short bond). We also let Π_1 be the portfolio of cashflows that we would have if we did not exercise the option (i.e. just those from the hedging instruments). We remark that this approach is only appropriate for options whose payoff we can express as a series of cashflows (e.g.

vanilla calls and puts). We discuss an area for further research that may enable us to price more exotic options in Section 7.2.2.

We solve Equation (2.6) with the appropriate final and jump conditions (dependent on the nature of the hedging instruments and option payoff) to find the value of the portfolio in a worst-case scenario at expiry, when we do exercise the option. This is $\Pi_0^-(r, T_0)$. We also solve Equation (2.6) with the appropriate final and jump conditions to find the value of the portfolio in a worst-case scenario at expiry, when we do not exercise the option. This is $\Pi_1^-(r, T_0)$.

Since we are long the option, we have control over whether or not to exercise, and so we set the value of the portfolio at expiry to be the more valuable of the two courses of action, where we take the exercise price into account. For a call option, this is the maximum of the value of the portfolio when we do exercise minus the exercise price and the value when we do not exercise, i.e.

$$\Pi^-(r, T_O) = \max\left(\Pi_0^-(r, T_0) - E, \Pi_1^-(r, T_0)\right).$$

(For a put option, we add the exercise price, since the holder of the option receives the exercise price at expiry).

We then solve Equation (2.6) with $\Pi^-(r, T_O)$ as final data and apply appropriate jump conditions (for the hedging instruments) to find the current worst-case scenario value of the portfolio. This method is shown schematically in Figure 5.4. To find the marginal worst-case value of the option, we then subtract the cost of the static hedge. Finally, we can maximise the marginal option value with respect to the hedge quantities to find the optimal worst-case scenario value for the option.

We can also find the value of the option in a best-case scenario. We solve Equation (4.1) with appropriate final and jump conditions to find the best-case values of Π_0 and Π_1 at expiry of the option, $\Pi_0^+(r, T_0)$ and $\Pi_1^+(r, T_0)$ respectively. We then set the value of the portfolio at expiry of the option to be the most valuable course of action. For example, for a call option,

$$\Pi^+(r, T_O) = \max\left(\Pi_0^+(r, T_0) - E, \Pi_1^+(r, T_0)\right).$$

We solve Equation (4.1) with $\Pi^+(r, T_O)$ as final data and appropriate jump conditions to find the current best-case scenario value of the portfolio. We then subtract the cost of the static hedge to find the marginal best-case value of the option. Again, we can optimise the result and minimise with respect to the hedge quantities to find the optimal best-case value.

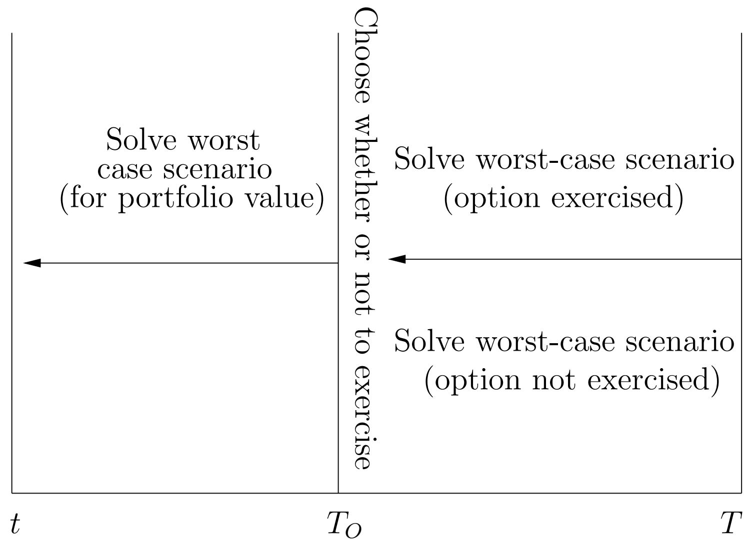


Figure 5.4: A more general approach to option pricing

Example: We price vanilla European call and put options with expiry in 1 year, on a zero-coupon bond with principal 1 and maturity in 5 years. The current market price of the bond is 0.687. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

We hedge with the hedging bonds of Table 4.1.

The results when we price a call option, with exercise price 0.5, are shown in Tables 5.3 and 5.4 and those when we price a put option, with exercise price 0.9, are shown in Tables 5.5 and 5.6. Figure 5.5 show the value of the put option in a worst-case scenario under the various hedging strategies.

From the figure, we can see that although the extra hedging instruments have not had a particularly noticeable effect in raising the worst-case price at the spot short rate (over and above that when we hedged with the underlying), they have 'flattened out' the curve and this must correspond to a significant decrease in the interest rate risk in the portfolio.

If we examine the static hedges, we see that they still include approximately one of the underlying bond (short for the call, long for the put) although the specific quantities of this bond have altered slightly. The hedges also include sizeable amounts of the one year bond. This bond matures at the same time as the option expires and is consequently an effective hedging tool for the option payoff. With the extra hedging instruments it has been possible to reduce the option price spread to a level which is of the same magnitude as the bid-offer spread seen in practice.

Call, $E = 0.5$	Worst case	Best case
No hedge	0.009	0.372
Optimal hedge on worst-case	0.220	0.221
Optimal hedge on best-case	0.220	0.221

Table 5.3: Value of the optimally-hedged European call option

Hedging bond	Maturity (yrs)	Worst case hedge quantity	Best case hedge quantity
Z_1	0.5	0.012	0.082
Z_2	1	0.488	0.456
Z_3	2	0.001	0.002
Z_4	3	-0.002	-0.008
Z_5	5	-1.004	-0.994
Z_6	7	0.006	0.000
Z_7	10	0.000	0.000

Table 5.4: The optimal static hedges for the European call option

Put, $E = 0.9$	Worst case	Best case
No hedge	0.012	0.361
Optimal hedge on worst-case	0.152	0.154
Optimal hedge on best-case	0.152	0.153

Table 5.5: Value of the optimally-hedged European put option

Hedging bond	Maturity (yrs)	Worst case hedge quantity	Best case hedge quantity
Z_1	0.5	-0.136	-0.034
Z_2	1	-0.826	-0.874
Z_3	2	-0.003	-0.003
Z_4	3	0.008	0.008
Z_5	5	0.996	0.995
Z_6	7	-0.001	-0.001
Z_7	10	0.000	0.000

Table 5.6: The optimal static hedges for the European put option

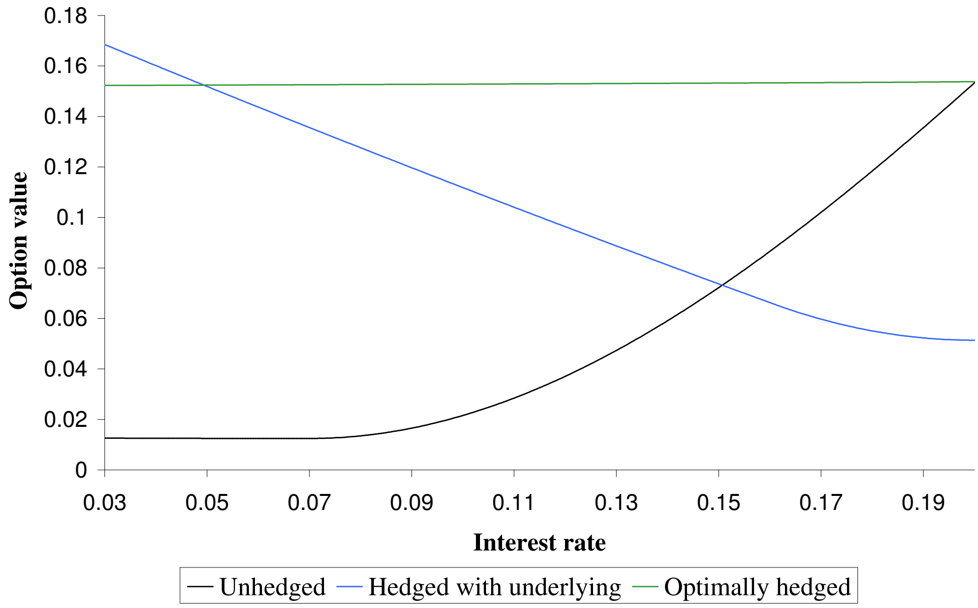


Figure 5.5: European put option value in a worst-case scenario

Finally, we remark that if we were to construct the yield curve from the hedging instruments and then price the options off this, the call would be worth

$$0.687 - 0.5 \times 0.933 = 0.2205,$$

and the put would be worth

$$0.9 \times 0.933 - 0.687 = 0.1527.$$

Both of these values are contained within their respective spreads for the prices.

Alternatively, we could use the Black approximation to the bond option value, an approach popular with practitioners [10]. We value the bond option using the Black–Scholes equity option pricing methodology, where we have assumed that the bond price behaves in a lognormal fashion. The price of a European call option, expiring at time T_1, on a bond maturing at time T_2, is then given by

$$Z(t;T_1)(FN(d_1) - EN(d_2)),$$

where the forward price of the bond at expiry of the option, F, is

$$F = \frac{Z(t;T_2)}{Z(t;T_1)},$$

$$d_1 = \frac{\log(F/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

and,

$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

The corresponding put option has value

$$Z(t; T_1)(EN(-d_2) - FN(-d_1)).$$

In our example, with a volatility of 8%, the call option is then worth 0.2205 and the put option, 0.1528. Again, both of these prices are contained within the spreads predicted.

5.1.4 Pricing and hedging American options

For the European option, we compared the value of two portfolios at expiry of the option and just picked the course of action that had the higher value. However, for the American option, we may also exercise at earlier times. We must therefore be aware that at any time before expiry, it may be optimal to exercise rather than hold the option. This presents itself as a constraint on the value of the portfolio in which we continue to hold the option.

We again consider two portfolios - one containing the cashflows we would have if we were to exercise the option, the other containing those we would have if we continued to hold the option. We consider an option with the same specification as before, with the one exception that the holder now has the right to exercise the option at any time before T_O. We let $\Pi_0(r, t)$ be the overall portfolio of cashflows that we would have at time t if we were to exercise the option at time t and $\Pi_1(r, t)$ be the portfolio of cashflows that we would have if we continued to hold the option at time t (this does not include any cashflow due to the option payoff at expiry). We remark that when $t = T_O$, these are the same portfolios as for the European option in Section 5.1.3.

We solve Equation (2.6) with the appropriate final data and jump conditions to find the worst-case value of the portfolio when we do exercise at time t, $\Pi_0^-(r, t)$. This tells us what our portfolio payoff would be if we decided to exercise at time t.

We then solve Equation (2.6) with the appropriate final data and jump conditions to find the worst-case value of the portfolio when we continue to hold the option at time t, $\Pi_1^-(r, t)$. In absence of arbitrage, we would exercise the option if the value of the consequent portfolio, with the exercise price taken into account, were greater than the current value of the portfolio when we continue to hold the option. For the call option, this gives us the additional constraint

$$\Pi_1^-(r,t) \geq \Pi_0^-(r,t) - E,$$

during the period in which we are allowed to exercise the option (in this case, for $t \leq T_O$). We show this method schematically in Figure 5.6. For a put option, the constraint is

$$\Pi_1^-(r,t) \geq \Pi_0^-(r,t) + E.$$

The marginal worst-case value of the option at time t is then the value of the portfolio in which we still hold the option, $\Pi_1^-(r,t)$, minus the cost of the static hedge. We can maximise with respect to the hedge quantities to find the optimal marginal value.

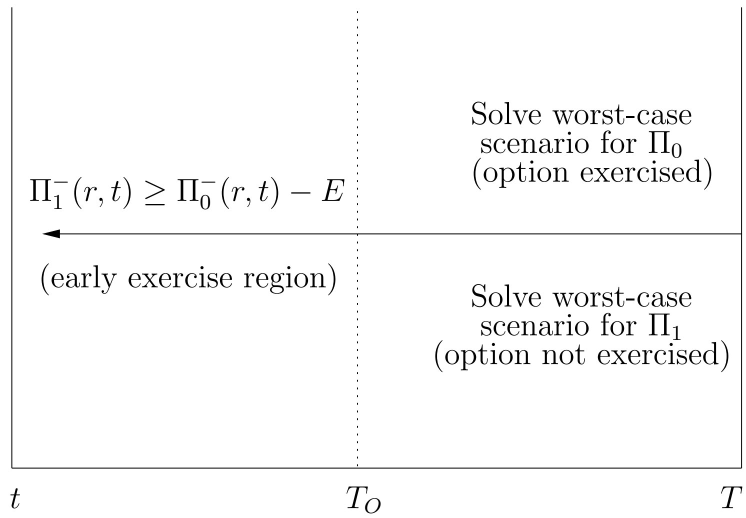


Figure 5.6: Pricing American options

We can also find the best-case scenario value of the option. We solve Equation (4.1) with the appropriate final data and jump conditions to find the best-case value of the portfolio in which we still hold the option, $\Pi_1^+(r,t)$, with a suitable constraint whilst we are allowed to exercise the option (e.g. for a call option,

$$\Pi_1^+(r,t) \geq \Pi_0^+(r,t) - E,$$

where $\Pi_0^+(r,t)$ is the solution of Equation (4.1) for the portfolio value when we exercise the option at time t).

Example: We price the vanilla American put option with expiry in 1 year and exercise price E, on a zero-coupon bond with principal 1 and maturity in 5 years. The current market price of the bond is 0.687. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

Initially, we solely hedge with the underlying zero-coupon bond. The results for the option valuation, without hedging and with the optimal static hedges for both worst- and best-case valuations, are shown in Table 5.7.

$E = 0.8$	Worst case	Best case	Hedge quantity
No hedge	0.000	0.328	-
Optimal hedge on worst-case	0.112	0.164	0.998
Optimal hedge on best-case	0.090	0.142	0.862

$E = 0.9$	Worst case	Best case	Hedge quantity
No hedge	0.048	0.428	-
Optimal hedge on worst-case	0.212	0.213	0.998
Optimal hedge on best-case	0.211	0.213	1.000

$E = 1.0$	Worst case	Best case	Hedge quantity
No hedge	0.148	0.528	-
Optimal hedge on worst-case	0.312	0.313	0.998
Optimal hedge on best-case	0.311	0.313	1.000

Table 5.7: Value of an American put option hedged with the underlying

We then hedge the option with the bonds of Table 4.1. The results when we price a put option, with exercise price 0.9, are shown in Tables 5.8 and 5.9. Figure 5.7 shows the value of the unhedged option in a worst-case scenario.

Put, $E = 0.9$	Worst case	Best case
No hedge	0.048	0.428
Optimal hedge on worst-case	0.212	0.213
Optimal hedge on best-case	0.212	0.213

Table 5.8: Value of the optimally-hedged American put option

Hedging bond	Maturity (yrs)	Worst case hedge quantity	Best case hedge quantity
Z_1	0.5	-0.002	-0.036
Z_2	1	0.002	-0.001
Z_3	2	-0.002	-0.003
Z_4	3	0.008	0.009
Z_5	5	0.994	0.994
Z_6	7	0.000	-0.001
Z_7	10	0.000	0.000

Table 5.9: The optimal static hedges for the American put option

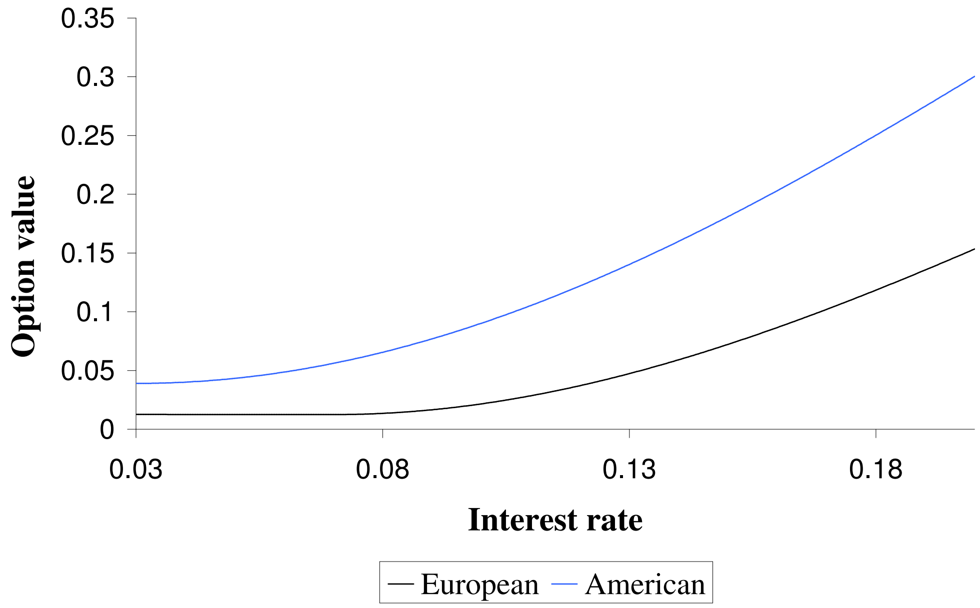


Figure 5.7: American put option value in a worst-case scenario

The further an option is ‘in the money’ (whether American or European) the more likely it is to be exercised and the nearer the quantity of the underlying, in the static hedge, is to unity. The spread in price also decreases because we are effectively valuing the exercised option and we can hedge this very efficiently with the underlying bond.

The spreads for the American option are therefore smaller than those for the European option and the hedge quantities of the underlying are larger. This is because the American option has more exercise opportunities and is consequently more likely to be exercised. We note that if we were to immediately exercise the option (with exercise price 0.9, for instance), it would be worth,

$$0.9 - 0.687 = 0.213,$$

which is contained within our spread for prices and is the optimally-hedged best-case price. (In this best-case scenario, the optimal strategy is therefore to exercise immediately).

If we were to price the American call option then we would find that it had the same value as the European call option. This is because the underlying is a zero-coupon bond. It is an equivalent result to the equality of American and European call options on equities which do not pay a dividend [64]. If we were to price a call option on a coupon bond (where the coupon was paid before expiry of the option) then the American option would be worth more than the European, since the holder of the option would not receive the coupon, whereas the holder of the bond would.

We remark that we can also use this approach to value Bermudan options (options with exercise allowed only on or between specified dates). To value such an option, we proceed as for the American option. However, when we come to solve our partial differential equation for Π_1, we only include the relevant constraint at (or between) times when exercise is allowed.

5.1.5 Generalisation of the option pricing methodology

We have developed two different approaches to pricing options. The former is simple to implement but only appropriate for European options hedged with no more than the underlying. The latter approach is more general and can be used to value European and American options with no such constraint on the choice of hedging instrument. Unfortunately, there is a drawback to this approach. For each instance of either/or optionality, we must double the number of cases to be considered. This means that to value a portfolio which includes n instances of optionality (n vanilla options, for example), we must price 2^n separate portfolios. This can quickly become computationally intensive.

So far, we have only discussed the pricing of options on zero-coupon bonds. However, both methodologies are still valid for other underlying contracts. The former approach is still appropriate for any underlying contract that we can price using our model, as long as all of the cashflows of the contract are after the expiry date of the option (since all we have to do is to find the spread in price for the underlying at this date).

However, if we want to hedge the option with anything but the underlying, or with an underlying which has cashflows before expiry of the option, then we must use the latter approach. This methodology is still valid for any underlying contract, as long as the contract can be expressed as a set of cashflows which are either fixed or only dependent on our short-term interest rate, r. We can then include them as jump conditions when we solve the partial differential equation for the portfolio value when we exercise the option, Π_0.

Section 7.2.2 provides some details of a possible approach that could price more exotic options and may also allow us to price hedged European and American options in a more efficient and less computationally expensive fashion.

5.2 Multi-choice swaps (contracts with embedded decisions)

The multi-choice swap requires the holder to make a series of decisions during the life of the contract. The swap has M possible cashflow dates, on each of which, the holder may choose to exchange a floating rate payment for a fixed rate payment. Moreover, the holder must do so on exactly m occasions (where $m \leq M$).

The fixed rate payments are the interest that would be due on some principal due to a predetermined fixed rate. The floating rate payments are the interest that would be due on the principal due to some designated interest rate, the reference rate.

We solve the problem, in a worst-case scenario, by approximating this designated interest rate using the short-term interest rate, r. This is because we must make a decision on each cashflow date and we cannot do this without knowledge of the value of the entire cashflow. We will then hedge the contract with vanilla swaps, and to be consistent, we will also value these using the approximation approach. As we shall note later, it is possible to value the contract using the swap decomposition approach, but it can be impractical to do so.

We assume that the M cashflow dates are on the τ period interest rate and are τ apart. We designate these dates by T_j for $j = 1, 2, \ldots, M$. If the holder chooses to exchange interest payments on one of these dates, then he receives a cashflow of the form,

$$P(r - r_f)/\tau,$$

where P is the principal and r_f is the fixed rate.

To price the contract, we must consider separately the cases when there are i cashflows still to be taken, for $i = 0, 1, \ldots, m$. We therefore introduce $m+1$ functions,

$V(r, t, i)$. The index i represents the number of cashflows that the holder has left to take before maturity and V is the subsequent value of the contract.

We solve Equation (2.6) with suitable final, jump conditions and constraints, to find the value of each of these functions today. The value of our contract in a worst-case scenario is then $V(r, t; m)$. (We can also find the best-case scenario value by solving Equation (4.1) instead).

Clearly

$$V(r, t; 0) = 0,$$

since there are no cashflows left to be taken.

At each possible swaplet date, we must apply either a jump condition, if we are forced to take the cashflow, or a constraint, if we are not obligated to take the cashflow and only do so in the case that it is the optimal decision.

If there are i cashflows left to choose between, and we still have to take a total of i cashflows, then we must take every cashflow left. Mathematically, we represent this as,

$$V(r, T_{M-i+1}^{-}; i) = V(r, T_{M-i+1}^{+}; i-1) + P(r - r_f)/\tau,$$

for $i = 1, 2, \dots, m$.

On the other hand, if we are at a cashflow date, and there are more cashflow dates left than the number of cashflows that we are obligated to take, then we only take the cashflow if it is optimal to do so. This is the case when the value of the contract is less than the value of the contract with one less cashflow left to take plus the value of the cashflow. We therefore have the constraint,

$$V(r, T_j^{-}; i) \geq V(r, T_j^{+}; i-1) + P(r - r_f)/\tau,$$

for $M - j + 1 > i$.

To hedge the contract, we just include the appropriate jump conditions for the hedging cashflows, price the overall portfolios $\Pi(r, t; i)$, for $i = 1, 2, \dots, m$ and then subtract the cost of the static hedge from $\Pi(r, t; m)$ to find the marginal contract value, $V(r, t; m)$. We remark that in this case, $\Pi(r, t; 0)$ no longer has zero value, but represents the value of the hedging instruments when there is no swap contract.

Example: We price an eight-choice swap. The swap has a principal of \$1,000,000 and a fixed rate of 7%. The holder must take four of the eight possible cashflows, which are every six months, with the first cashflow in one year's time.

The spot short-term interest rate is 7.4% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

We hedge the contract with the market-traded swaps from Table 5.10. They are par swaps with a principal of $100,000 and payment dates bi-annually until their maturity.

The results for the worst- and best-case valuations are shown in Tables 5.11 and 5.12. We remark that the time taken to price the contract will be approximately $m + 1$ times the size of that to price the vanilla swap.

Hedging Swap	Maturity (yrs)	Fixed Rate (%)
S_1	2	7.10
S_2	3	6.85
S_3	4	6.69
S_4	5	6.58
S_5	6	6.49
S_6	7	6.42
S_7	8	6.39
S_8	9	6.36
S_9	10	6.34

Table 5.10: The par-value hedging swaps

	Worst case	Best case
No hedge	-68784.4	108726.3
Optimal hedge on worst-case	-26655.1	14545.1
Optimal hedge on best-case	-22647.3	3774.1

Table 5.11: Value of an eight-choice swap

Hedging swap	Worst case hedge quantity	Best case hedge quantity
S_1	0.302	1.725
S_2	0.859	4.968
S_3	-6.215	-10.009
S_4	0.000	0.000
S_5	0.000	0.000
S_6	0.000	0.000
S_7	0.000	0.000
S_8	0.000	0.000
S_9	0.000	0.000

Table 5.12: The optimal static hedges for the eight-choice swap

It is also possible to price this contract using the swap decomposition approach. However, if we decompose the swap cash flows into sets of zero-coupon bonds, then we will not be able to make the decision on whether or not to take a cashflow at the cashflow date. This is because one of the bond cashflows will be before this date. To price the swap, we must use the same approach to optionality as for the general bond option problem. At each possible cashflow date, we consider the cases when we do and do not choose to take the cashflow separately.

Fortunately, we will not have to consider the 2^M separate possible cases, because we are constrained to take exactly m cashflows. We therefore only have to consider the ${}^{M}C_{m}$ possible sets of cashflows and then pick the best of these. This latter approach will take of the order of ${}^{M}C_{m}$ times the length of time taken to price the vanilla swap. Since we will, in general, have a large number of potential cashflow dates (M), and a significant number of cashflows to take (m), the time taken to price the swap is likely to be too high for the approach to be of practical value. The suggested approach detailed in Section 7.2.2 may lead to a more applicable method of solution.

5.3 Index amortising rate swaps

The index amortising rate swap is an agreement between two parties to exchange payments of interest, one at a fixed rate and one at a floating rate, on a principal that decreases (amortises) at a rate dependent on some index, generally, the floating rate. The principal only amortises on a payment date, at a rate determined by an amortising schedule.

Given the multitude of possible amortising schedules, a particular contract will generally be 'over the counter' and illiquid. In such a case, where it is difficult to move in and out of a position, it is particularly important to have an idea of the extreme possible outcomes and, especially, what the worst possible outcome could be. A worst-case scenario valuation is therefore clearly an appropriate measure for the contract value. We will hedge the contract with liquid, market-traded swaps. This will provide us with the necessary information to set up an optimal static hedge for the contract and also leads to an improved worst-case scenario valuation.

In practice, the swap will depend on some τ period interest rate, and not the short-term interest rate. However, we must use the approximation approach and substitute the short-term interest rate for our τ period rate because the problem is path-dependent. At a cashflow date, we must amortise the principal. We will not be able to do this if we do not know the floating rate at that time. Since the true τ

period rate depends on the value of a cashflow at an earlier time, we cannot find its value directly at the cashflow date.

As an alternative, we could try to keep track of the floating rate and adjust the principal at an earlier time, when the relevant cashflow were known. However, to accomplish this, we would have to value the cashflow separately from the swap. This would lead to the same problem as we encountered when we tried to price a European bond option hedged with an instrument that was not the underlying. We would again be trying to perform two separate worst-case valuations, with no guarantee that the two solutions would have the same interest rate path. The values calculated using this approach would then not be optimal. A possible area for further research which may allow us to perform this calculation, by keeping track of the relevant floating rate in a consistent fashion, is described in Section 7.2.2.

The value of our swap depends on the current short-term interest rate, r, and the current principal, P, i.e.

$$V = V(r, P, t).$$

However, our governing equation does not change. V still satisfies Equation (2.6). The effect of P is only seen in the jump and final conditions.

Over a cash flow date, the value of the swap jumps by

$$P(r - r_f)/\tau,$$

where r_f is the fixed rate, and the principal changes to

$$g(r)P,$$

where $g(r)$ is the amortising schedule.

Over the payment date t_i, we therefore find the jump condition

$$V(r, P, t_i^-) = V(r, g(r)P, t_i^+) + P(r - r_f)/\tau.$$

At maturity, the last exchange of payments occurs and we have the final condition

$$V(r, P, T) = P(r - r_f)/\tau.$$

Although the problem is nonlinear, the amortisation is linear in P. We can therefore find a similarity reduction which reduces the problem from three independent variables to two [63]. We set

$$V(r, P, t) = PH(r, t),$$

to find that the problem for H is

$$H_t + c\,(r, H_r)\, H_r - rH = 0,$$

i.e. H also satisfies Equation (2.6), with the jump condition

$$H(r, t_i^-) = g(r)H(r, t_i^+) + (r - r_f)/\tau,$$

over a payment date, and the final condition

$$H(r, T) = (r - r_f)/\tau.$$

Example: We value an index amortising rate swap with a maturity of $5\frac{1}{2}$years, fixed interest rate of 6.5% and initial principal \$1,000,000. Payment dates occur once a year until maturity, with the first payment $3\frac{1}{2}$years from today. The amortising schedule is shown in Figure 5.8.

The spot short-term interest rate is 7.40%. We set the parameters of our model to be

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

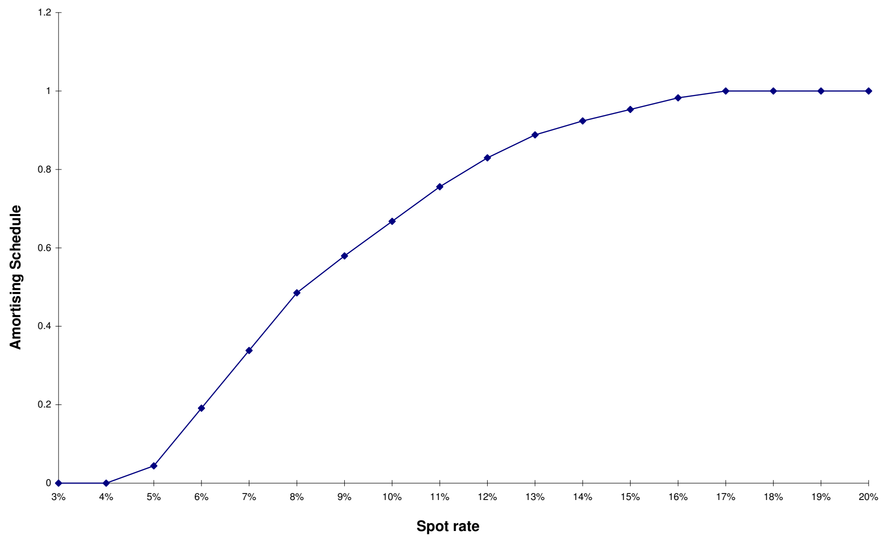


Figure 5.8: The amortising schedule

We hedge the contract with the market-traded swaps from Table 5.10 (these were the actual market prices on 14th May 1998). They are par swaps with a principal of \$100,000 and payment dates biannually until their maturity. The yield curve for these swaps is shown in Figure 5.9.

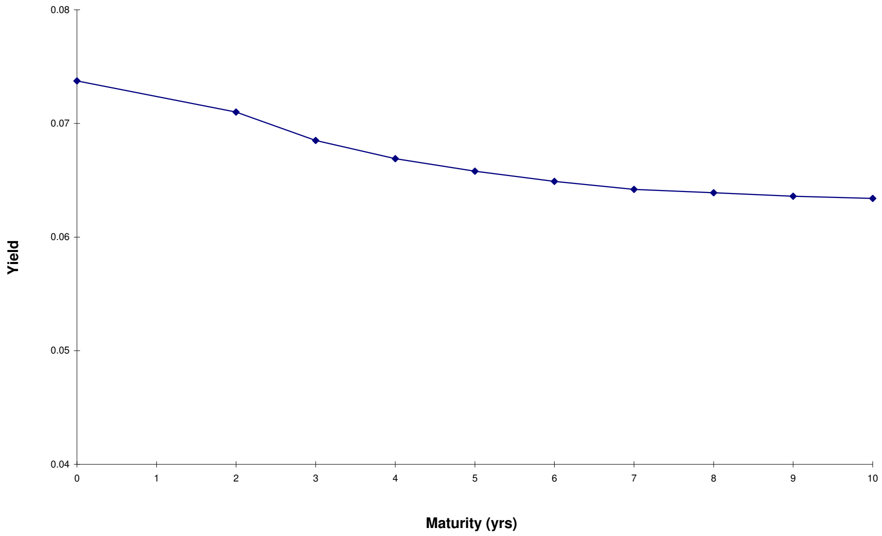


Figure 5.9: The swap yield curve

	Worst case	Best case
No hedge	-30820.2	216336.6
Optimal hedge	-11753.2	143129.6

Table 5.13: Value of the index amortising rate swap

Hedging swap	Worst case hedge quantity
S_1	-0.008
S_2	9.433
S_3	-9.528
S_4	-0.009
S_5	0.001
S_6	0.000
S_7	-0.001
S_8	0.001
S_9	0.000

Table 5.14: The optimal static hedge for the index amortising rate swap

The results of the valuation for the index amortising rate swap, without hedging and with the optimal static hedge for the worst-case scenario, are shown in Table 5.13. The optimal static hedge for this valuation is shown in Table 5.14.

The use of a static hedge has significantly improved the value of the index amortising rate swap in a worst-case scenario. The optimal hedge does not consist of all of the possible contracts, and is essentially composed of the few contracts that are most similar in form to the cashflows of our swap.

We can also price the index amortising rate swap directly from the yield curve of Figure 5.9, using linear interpolation between points. In this case, we find that the swap is worth -2440.3. We can then test the sensitivity of the valuation to shifts in the yield curve. The results are shown in Table 5.15. We note that the hedged worst-case scenario price is similar in value to the swap price with a -1% yield shift (a comparable result to that found for the leasing portfolio in Section 4.9).

Yield Shift (%)	Swap Value
+2	21552.8
+1	8297.5
0	-2440.3
-1	-10908.0
-2	-18786.6
-3	-28219.6

Table 5.15: Sensitivity of the index amortising rate swap value to parallel shifts in the yield curve

5.4 Convertible bonds

The convertible bond contract is similar to the coupon-bearing bond in that the holder receives coupon payments at specified payment dates. It also has equity characteristics since the holder can, at specified times, exchange the bond for a quantity of some underlying asset [17]. This exchange is called conversion. When the stock price is low, there is little reason to convert the bond and so it behaves like a simple, non-convertible, coupon-bearing bond. When the stock price is high, the option to convert gives the bond a value closer to the value of the relevant quantity of the underlying asset. In some cases, it is optimal to convert the bond before maturity. Mathematically, the conversion problem is similar to the early exercise of an American option, and can be thought of as a free boundary problem.

We use a lognormal random walk for the share price,

$$dS = \mu S dt + \sigma S dX,$$

and form a Black–Scholes hedged portfolio, $\Pi = V - \Delta S$, where $V = V(r, S, t)$. Itô's Lemma gives us that

$$d\Pi = V_t dt + V_S dS + V_r dr + \tfrac{1}{2}\sigma^2 S^2 V_{SS} dt - \Delta dS - D\Delta S dt.$$

We choose $\Delta = V_S$ to eliminate the leading order randomness from the share price movements and value the portfolio in a worst case scenario. Under this worst case assumption, the change in the value of our portfolio is

$$\min_{dr} d\Pi = \min_{dr} \left(V_t dt + V_r dr + \tfrac{1}{2}\sigma^2 S^2 V_{SS} dt - DSV_S dt \right)$$

We require that, in the worst case, our portfolio always earns the risk-free rate. This gives us that

$$\min_{dr} d\Pi = r\Pi dt,$$

and so the pricing equation for the bond is

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + (r - D)SV_S + c\left(r, V_r\right) V_r - rV = 0, \tag{5.1}$$

where $V(S, r, t)$ is the bond price,

$$c(r, X) = \begin{cases} c^+ & \text{if } X < 0 \\ c^- & \text{if } X > 0, \end{cases} \tag{5.2}$$

and D is the dividend yield on the asset, here assumed constant and continuously paid. This is the worst-case scenario value for the contract. The best value would be given by the solution of Equation (5.1) with reversal of the inequalities in Equation (5.2).

The derivation of the equation assumes that the random movements in the underlying asset are delta hedged away by holding Δ of the underlying asset where

$$\Delta = V_S.$$

In contrast, the interest rate risk is not hedged, we are assuming the worst outcome are far as the short rate path is concerned.

Optimal conversion into n of the stock is assured by insisting that

$$V(S, r, t) \geq nS,$$

at all times that conversion is permitted along with continuity of V and V_S.

The final condition at maturity of the bond, T, is that the bond value is equal to the principal, assumed to be 1, plus the last coupon

$$V(S, r, T) = 1 + c_T.$$

Across each coupon date, t_i, the bond falls by the amount of the coupon, and,

$$V(S, r, t_i^-) = V(S, r, t_i^+) + c_i.$$

This completes the specification of the convertible bond model under the risk-neutral measure for the asset and the worst-case scenario for the interest rate.

Example: We consider the pricing of a convertible bond using several popular models for interest rates and examine the robustness of the prices to variations in parameters. We then price the convertible bond using the non-probabilistic model.

The underlying asset has current value 100, the volatility is 15% and the dividend yield is 4%. Note that we are not questioning the accuracy of these asset price parameters.

We value a convertible bond with a maturity of 25th November 2001 (where today is 14th May 1998). The bond has principal 1, can be converted into 0.01 of the asset and pays a coupon of 3% every six months until expiry. The spot short-term interest rate is 7%.

1. Constant interest rate of 7%

   In the first example the interest rate is a constant, there are no dynamics either deterministic or stochastic. Figure 5.10 shows the convertible bond value under this assumption.

   Convertible bond value: 1.131

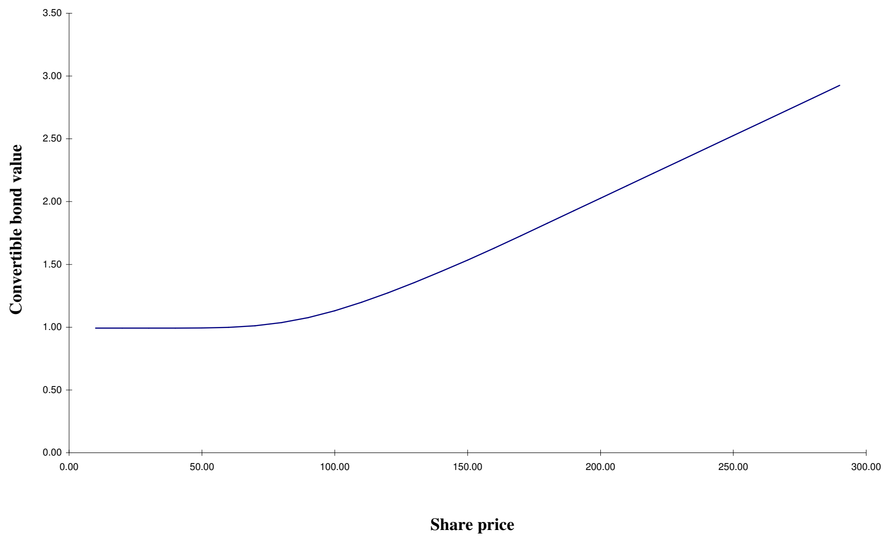


Figure 5.10: Convertible bond value with constant interest rate

2. Deterministic forward rate given by a linear interpolation of rates determined from the zero-coupon bond yield curve shown in Table 5.16.

   Convertible bond value 1.147

   The sensitivity to parallel shifts in the yield curve is shown in Table 5.17.

Maturity	Yield
1 m	0.07000
6 m	0.07447
1 yr	0.07016
2 yr	0.06631
5 yr	0.06224
7 yr	0.06121
10 yr	0.06037
30 yr	0.05990

Table 5.16: The yields for the zero-coupon bonds used to price/hedge the convertible bond

Yield curve shift	Convertible bond value
+2%	1.118
+1%	1.132
0%	1.147
-1%	1.165
-2%	1.185

Table 5.17: Sensitivity of the convertible bond value to parallel shifts in the yield curve

3. The Vasicek model of the form

$$dr = (a - br)dt + \nu dX2$$

with $a = 0.007$, $b = 0.1$, $\nu = 0.02$ and a correlation of 0.1.

Convertible bond value 1.137

The sensitivities to a, b and ν are shown in Tables 5.18, 5.19 and 5.20, respectively.

a	Convertible bond value
0.009	1.132
0.008	1.134
0.007	1.137
0.006	1.139
0.005	1.141

Table 5.18: Sensitivity of the Vasicek model to shifts in a

b	Convertible bond value
0.12	1.139
0.11	1.138
0.10	1.137
0.09	1.135
0.08	1.134

Table 5.19: Sensitivity of the Vasicek model to shifts in b

ν	Convertible bond value
0.030	1.142
0.025	1.139
0.020	1.137
0.015	1.135
0.010	1.133

Table 5.20: Sensitivity of the Vasicek model to shifts in ν

4. The Extended Vasicek model fitted to the yield curve of Table 5.16, with $b = 0.1$, $\nu = 0.02$ and a correlation of 0.1.

   Convertible bond value 1.161

   The sensitivities to b and ν are shown in Tables 5.21 and 5.22 respectively.

b	Convertible bond value
0.12	1.164
0.11	1.163
0.10	1.161
0.09	1.160
0.08	1.158

Table 5.21: Sensitivity of the Vasicek model to shifts in b

ν	Convertible bond value
0.030	1.166
0.025	1.163
0.020	1.161
0.015	1.159
0.010	1.157

Table 5.22: Sensitivity of the Vasicek model to shifts in ν

5. ACKW model

   The Apabhai, Choe, Khennach, Wilmott (1995) one-factor model is based on a statistical analysis of the US short-term interest rate and the yield curve slope at the short end over 20 years. The model fits the average dynamics of the yield curve.

   Convertible bond value 1.130

6. Epstein-Wilmott model

   We set the parameters of our model to be

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

   We find that the worst price attainable is 1.072 and the best price is 1.191.

5.4.1 Optimal static hedging

To price the hedged convertible bond, we must use the same approach to optionality as for the American bond option. This is because we must again keep track of two separate quantities. This time, they are the value of the hedging instruments and the overall value of the portfolio. To make sure that we find the true worst-case price for each of these, with a consistent interest rate path valid for both, we must consider the two cases when we choose to convert or not separately.

We consider two functions, Π_0 and Π_1. $\Pi_0(S, r, t)$ is the value of the hedging instruments and $\Pi_1(S, r, t)$ is the value of the portfolio consisting of the hedging instruments and the convertible bond.

To price Π_0, we solve Equation (5.1) with the appropriate final and jump conditions to represent all of the cashflows for the hedging instruments. Since there is no S dependence in these hedging cashflows, we are effectively just solving our usual worst-case problem for the hedging instruments, i.e. Equation (2.6).

To price Π_1, we then solve Equation (5.1) with the appropriate final and jump conditions to represent all of the cashflows for the hedging instruments and the convertible bond (before conversion). Since we will only convert if it is optimal to do so, we have the additional constraint that

$$\Pi_1(S, r, t) \geq nS + \Pi_0(S, r, t),$$

i.e. we convert when the value of the portfolio including the convertible bond is less than the value of the portfolio of hedging instruments plus the assets that we would receive if we were to convert.

Example: We hedge the convertible with the 6 month and the 1, 2 and 5 year zero-coupon bonds from Table 5.16. (These were the bonds that we used to calibrate the classical models, where appropriate). The optimal worst-case hedge for these bonds is shown in Table 5.23.

Bond Maturity	Bond Yield	Worst-case hedge
6m	0.07447	0.283
1yr	0.07016	0.053
2yr	0.06631	-0.421
5yr	0.06224	-0.290

Table 5.23: Optimal static hedge for the convertible bond

With this static hedge in place, we find that the new worst-case price is 1.112 (unhedged was 1.072) and the best-case price is 1.193 (was 1.191).

The present value of all the coupons and the principal, valued off an interpolated yield curve, is approximately 1.026. The added value due to the convertability in the bond is thus the difference between the bond value and the present value of all the coupons plus principal. The bond values and the added value according to each of the models are summarised in Table 5.24.

Model	CB Value	Added Value due to Conversion
Constant interest rate	1.131	0.105
Deterministic yield curve	1.147	0.121
Vasicek, unfitted	1.137	0.111
Vasicek, fitted	1.161	0.135
ACKW	1.130	0.104
Worst-case, unhedged	1.072	0.046
Best-case, unhedged	1.191	0.165
Worst-case, optimally hedged	**1.112**	**0.086**
Best-case, optimally hedged	**1.193**	**0.167**

Table 5.24: Added value due to convertability

We have applied our nonlinear, non-probabilistic interest rate model to the pricing of a convertible bond. The resulting worst-case scenario valuation produced results which were far lower in price than those found using typical interest rate models. However, unlike the worst-case scenario approach, these models were found to be quite sensitive to their parameters, the values of which can often be uncertain.

The theoretical prices and particularly the added value are significantly different across different models. The optimally hedged worst-case added value is 0.086 which can be seen as a benchmark against which to compare the other values. The historically accurate ACKW gives a value closest to this worst case (0.104) and the other models can be considerably higher. It is particularly interesting to note that in this example the more 'fitting' that is done, the higher the price, and the more empirically justified the model, the lower the price. Of course, this would be reversed for a yield curve sloping the other direction.

If we believe that the non-probabilistic model gives a conservative price range then we would hope that other models give prices within this range. The ability to find such definitive bounds for the value of the convertible bond may prove invaluable in the task of validating some of the more complex stochastic models.

Through the process of static hedging, we were able to significantly increase the worst-case scenario price for the convertible bond. In addition, the process found an optimal static hedge. This hedge could be applied to a convertible bond portfolio to reduce the inherent interest rate risk, regardless of which model were chosen to price the resulting portfolio.

Chapter 6

Extensions to the model

In this chapter we consider various extensions to our non-probabilistic model. First of all, we introduce the uncertainty band. This allows us to more accurately model real interest rate movements. We then examine past data to chooses a sensible width for the band. We further generalise the model by allowing for the possibility of jumps and crashes and re-examine the data under this new light. Finally, we consider the impact of liquidity on our work.

6.1 Uncertainty bands

Real interest rates have a stochastic nature, with unbounded short term fluctuations, a property not yet captured by our model. However, using the concept of the uncertainty band, we can address this potential criticism. The uncertainty band allows interest rate movements that are practically indistinguishable from the real short-term interest rate behaviour.

We consider the quantity that we have modelled, r, to be some estimate of the short-term interest rate that is always within a distance ϵ of the real short-term rate, i.e.

$$|r - r'| \leq \epsilon, \tag{6.1}$$

where r' represents the real rate. Figure 6.1 shows possible paths for r and r'.

We re-derive our pricing equation under this new model to find the worst-case value of a contract. This will still be a function of r and t. However, the risk-free rate will now be r'. Since this is the actual short-term rate, all cashflows will also depend on this quantity rather than r, as will the actual value of the contract today.

Let $V(r,t)$ be the value of our contract, when the estimated short-term interest rate is r at time t. We consider the movement in the value of the contract over a

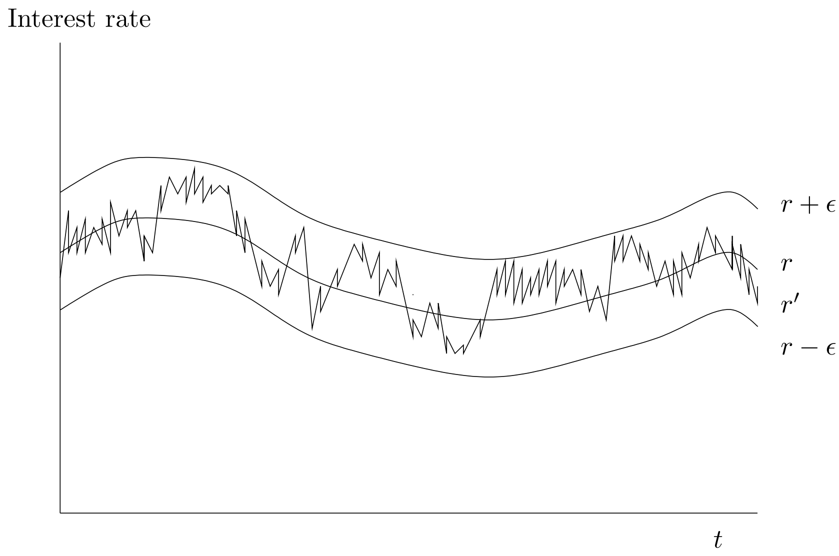


Figure 6.1: A possible evolution of r and r', the 'real' rate

time step dt. As in Section 2.3.1, we find that the minimum possible increase in the contract value should be equivalent to the risk-free increase. This is now $r'V\,dt$. Therefore, we have

$$\min(dV) = dV_{\text{worst case}} = r'V\,dt.$$

We know that r' lies within a certain distance of r. Since we are in a worst-case scenario and are not investing in the risk-free rate, we assume that the risk-free account grows at the highest possible rate. We can therefore expand dV and substitute for r' in terms of r, to find

$$\min_{dr}\left(V_r dr + V_t dt\right) = \max_{e(V)}((r + e(V))V\,dt).$$

where dr is bounded by Equation (2.5), and $e(V)$ is bounded by

$$-\epsilon \leq e(V) \leq \epsilon.$$

We can then take the optimisations inside the appropriate brackets, to give

$$V_t + c\,(r, V_r)\,V_r - (r + e(V))V = 0, \tag{6.2}$$

where

$$c(r, X) = \begin{cases} c^+ & \text{if } X < 0 \\ c^- & \text{if } X > 0, \end{cases} \tag{6.3}$$

and

$$e(X) = \begin{cases} \epsilon & \text{if } X \geq 0 \\ -\epsilon & \text{if } X < 0. \end{cases} \tag{6.4}$$

Since the cashflows depend on r' rather than r, we minimise their value to the holder (as we are in a worst-case scenario), when we write them in terms of r.

We apply the last cashflow as final data for the equation,

$$V(r,T) = C_N(r'),$$

which gives us that, in the worst-case,

$$V(r,T) = \min_{-\epsilon \le e \le \epsilon}(C_N(r+e)). \tag{6.5}$$

Similarly, a cashflow of $C_i(r')$ at time T_i gives the jump condition that, in the worst-case,

$$V(r,T_i^-) = V(r,T_i^+) + \min_{-\epsilon \le e \le \epsilon}(C_i(r+e)). \tag{6.6}$$

The characteristics of the equation are still $dr/dt = c$. We can therefore solve the pde, as before, to find $V(r,t)$. The current value of the portfolio in a worst-case scenario is then,

$$V(r',t) = \min_{-\epsilon \le e \le \epsilon}(V(r+e,t)). \tag{6.7}$$

We remark that to perform a best-case scenario valuation, we must reverse the inequalities in Equations (6.3) and (6.4) and maximise where we had minimised in Equations (6.5) - (6.7).

Example: We price and hedge a 4 year zero-coupon bond with principal 1. The hedging instruments are the zero-coupon bonds from Table 4.1. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

Table 6.1 shows the results for the zero-coupon bond value in a worst-case scenario with various values for ϵ without hedging and with the optimal static hedge for the worst-case scenario. Table 6.2 shows the hedge quantities for these static hedges. Figure 6.2 shows the value of the zero-coupon bond with varying ϵ.

It is clear that the wider the uncertainty band, the lower the worst-case price. As ϵ increases, we also see that the quantity of lower maturity hedging instruments in the portfolio decreases and the quantity of higher maturity instruments increases. The effect of this hedging strategy is that the worst-case price decreases less quickly with increasing ϵ.

ϵ	0.00	0.01	0.02	0.03
No hedge	0.575	0.534	0.497	0.463
Optimal hedge on worst-case	0.730	0.722	0.714	0.707

Table 6.1: Worst-case value of a 4 year zero-coupon bond

Hedging bond	$\epsilon = 0.00$ hedge quantity	$\epsilon = 0.01$ hedge quantity	$\epsilon = 0.02$ hedge quantity	$\epsilon = 0.03$ hedge quantity
Z_1	0.000	0.002	0.002	0.002
Z_2	-0.004	-0.004	-0.004	-0.001
Z_3	0.169	0.091	0.091	0.010
Z_4	-0.699	-0.569	-0.563	-0.439
Z_5	-0.468	-0.511	-0.506	-0.527
Z_6	0.020	0.018	0.018	0.000
Z_7	0.000	0.000	0.000	0.000

Table 6.2: The optimal static hedges for a 4 year zero-coupon bond

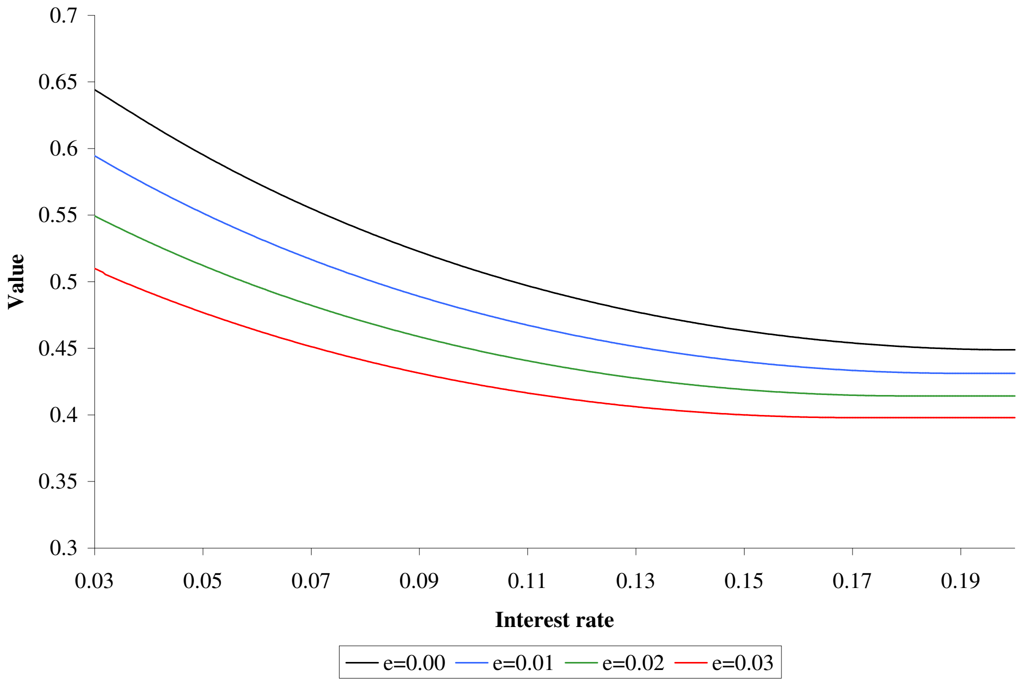


Figure 6.2: Value of a 4 year zero-coupon bond with varying ϵ

6.1.1 Estimating ϵ from past data

We can use past interest rate data to choose a sensible value for this parameter. We use data for a longer period rate, the t_d period rate say, which is more readily available, and relate this to our short-term interest rate. We can then examine the data and find the lowest value of ϵ for which the actual interest rate movements are consistent with our model.

First of all, we must relate the level of the t_d period rate at some specified time to the level of our estimated short-term rate, r, at that time. In absence of arbitrage, the t_d period rate, $r_d(t)$, satisfies

$$r_d(t) = \frac{1}{t_d} \int_t^{t+t_d} r'(s) ds.$$

From Equation (6.1), we have bounds on the size of $r'(t)$,

$$r(t) - \epsilon \leq r'(t) \leq r(t) + \epsilon,$$

and from Equation (2.3), we can find bounds for $r(t+\tau)$, for $\tau > 0$,

$$r(t) + c^- \tau \leq r(t+\tau) \leq r(t) + c^+ \tau,$$

where we have assumed that c^- and c^+ are constants, and hence for $r'(t+\tau)$,

$$r(t) + c^- \tau - \epsilon \leq r'(t+\tau) \leq r(t) + c^+ \tau + \epsilon.$$

Using these bounds, we can bound the integral in question,

$$t_d \left(r(t) - \epsilon + \tfrac{1}{2} c^- t_d \right) \leq \int_t^{t+t_d} r'(s) ds \leq t_d \left(r(t) + \epsilon + \tfrac{1}{2} c^+ t_d \right).$$

We can therefore calculate bounds for the value of $r_d(t)$ in terms of $r(t)$, and find

$$r(t) - \epsilon + \tfrac{1}{2} c^- t_d \leq r_d(t) \leq r(t) + \epsilon + \tfrac{1}{2} c^+ t_d.$$

Inverting these inequalities. we find the bounds on r in terms of r_d at time t, are

$$r_d(t) - \epsilon - \tfrac{1}{2} c^+ t_d \leq r(t) \leq r_d(t) + \epsilon - \tfrac{1}{2} c^- t_d.$$

Our model for r is then consistent with the data, if r can evolve such that these inequalities always hold at each data point. This is effectively a set of uncertainty bars for the estimated short-term interest rate and the model is consistent with the data if we can fit an allowed evolution of the interest rate through them.

If we choose the parameters for our model (r^-, r^+, c^- and c^+), then we can use the uncertainty bars to find the minimum value of ϵ for which the interest rate can evolve, consistent with both Equations (2.3) and (2.4) and these bars. To illustrate the methodology, we consider just two data points, as shown in Figure 6.3.

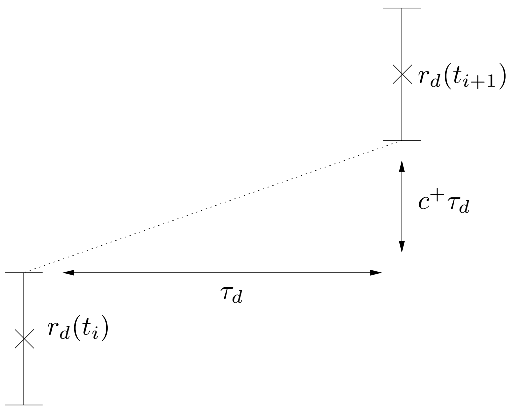


Figure 6.3: Examining data to choose a sensible value for ϵ

We must calculate the minimum value of ϵ required so that r can evolve, consistent with our bounds, such that r passes through both bars. To minimise the value of ϵ, we use the ends of the two uncertainty bars that are nearest to each other and let r grow as fast as possible in the required direction.

However, when we have further data points, we must find the value of ϵ that allows us to move from one bar to the next, throughout the data set, in a consistent fashion. If we believe that the real rate is at the bottom of the uncertainty bar for a particular data point when we are considering the interest rate evolution between this and the previous point, then the real rate must still be at the bottom when we consider the evolution between this and the next point. This concept is demonstrated in Figure 6.4, where there is an inconsistency in the measurements performed in (a). When we are consistent, the real rate is at the same point in the uncertainty bar for both measurements, as shown in (b).

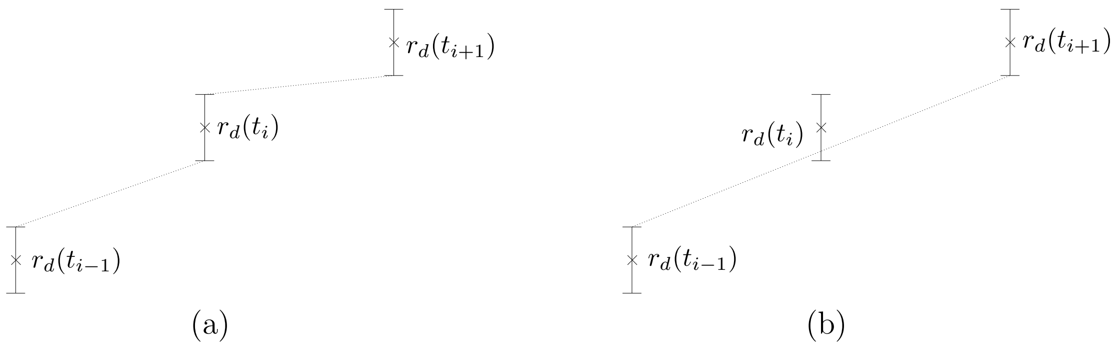


Figure 6.4: (a) An inconsistent use of the uncertainty bars, (b) The consistent picture

Example: We choose our parameters to be

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

and examine daily, 1 month US interest rate data from 21st October 1986 to 25th April 1995. This data is shown in Figure 6.5.

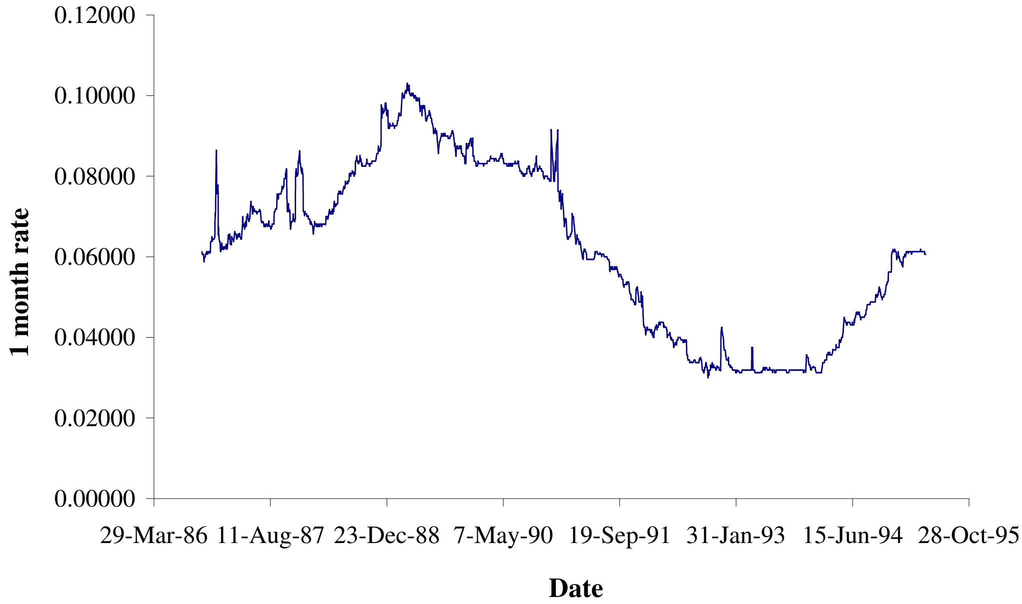


Figure 6.5: 1 month US interest rate data

We find that the minimum value for ϵ for which our model is consistent with the data is

$$\epsilon = 0.005754.$$

If we use a value of ϵ slightly larger than this, then we can be fairly certain that any interest rate movements seen in the market would be allowed under our model. The spreads for prices predicted by our model, as well as all the other applications that we have discussed, would then be realistic in practice.

6.2 Crash modelling

To further model the precise nature of actual interest rate movements, we may wish to include the possibility of jumps or crashes in the interest rate. With these in place, we would find an even smaller minimum value for ϵ when we examine interest rate data.

We present two different approaches to the modelling of a crash. Both are instantaneous changes in the interest rate. In the first approach, the interest rate can crash on, at most, a specified number of occasions over the time horizon. In the second approach, the interest rate can crash an unlimited number of times, but can only do so when a specified length of time has passed since the previous crash. In both cases, we assume that the crashes occur at the worst possible time, since we are pricing the contract in a worst-case scenario. We remark that, under these circumstances, it may be optimal for there to be no crash at all [41].

6.2.1 A maximum number of crashes

We first consider the situation when there can be at most one crash before the maturity of our contract. We value the contract, V, in a worst-case scenario and model the crash as an instantaneous movement of the short-term interest rate from r to $(1-k)r$, for some specified k.

We introduce the subscript 0 to denote the value of the portfolio when there is no crash allowed and 1 to denote the value when the interest rate is allowed to crash once. Thus V_0 is the usual worst-case value and is the solution of Equation (2.6) with suitable final and jump conditions for the contract in question.

To value V_1, we also solve Equation (2.6), with the same final and jump conditions as before. However, the interest rate is now allowed to crash if that would lower the value of the contract. If the interest rate does not crash, then the contract is worth $V_1(r,t)$ since a crash is still allowed in the future. On the other hand, if the interest rate does crash from r to $(1-k)r$, then the contract is worth $V_0((1-k)r,t)$, since the interest rate cannot crash again. In a worst-case scenario, a crash will only occur if that would give the contract a lower value. We therefore have the constraint,

$$V_1(r,t) \leq V_0((1-k)r,t).$$

The value of the contract today is then $V_1(r,t)$.

We can generalise the model by allowing a range for the size of any possible crash. We model the crash as an instantaneous movement from r to $(1-k)r$, where

$$k^- \leq k \leq k^+,$$

for some specified k^- and k^+. The effect of this generalisation is seen in the constraint. In a worst-case scenario, the interest rate will always crash to the level that would minimise the value of the contract. The constraint therefore becomes

$$V_1(r,t) \leq \min_{k^- \leq k \leq k^+} \left(V_0((1-k)r,t)\right).$$

We can also generalise the model to allow a number of crashes to occur before the maturity of the contract. We consider the situation where there are a maximum number of crashes, N say, before maturity. To price the contract under these circumstances, we must introduce $N+1$ functions, V_i, for $i = 0, 1, \ldots, N$. V_i is the value of the contract when the total number of crashes still allowed is i. As before, each of the V_i satisfies Equation (2.6) with the appropriate final and jump conditions for the contract in question. Rather than a single constraint, we now have a set of constraints linking the $N+1$ functions, of the form,

$$V_i(r,t) \leq \min_{k^- \leq k \leq k^+} \left(V_{i-1}((1-k)r, t) \right),$$

for $i = 1, 2, \ldots, N$. The value of the contract, today, in a worst-case scenario is then $V_N(r,t)$.

We note that to value the contract in a best-case scenario, we would solve Equation (4.1) for each of the V_i, rather than Equation (2.6). Crashes would now occur if they were to raise the value of the contract, and so our constraints would become

$$V_i(r,t) \geq \max_{k^- \leq k \leq k^+} \left(V_{i-1}((1-k)r, t) \right).$$

Example: We price and hedge a 4 year zero-coupon bond with principal 1. The hedging instruments are the zero-coupon bonds from Table 4.1. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

Table 6.3 shows the results for the zero-coupon bond value in a worst-case scenario with various numbers of crashes allowed. These crashes have a maximum magnitude of 1% (i.e. $k^- = -1\%$ and $k^+ = 1\%$). Results are shown for the unhedged bond and for the bond optimally hedged in a worst-case scenario. Table 6.4 shows the relevant hedge quantities for some of these scenarios. Figure 6.6 shows the value of the zero-coupon bond when various numbers of crashes are allowed.

No of crashes	0	1	2	3	4
No hedge	0.575	0.558	0.542	0.528	0.516
Optimal hedge on worst-case	0.730	0.728	0.726	0.724	0.722

Table 6.3: Worst-case value of a 4 year zero-coupon bond with crashes allowed

Hedging bond	no crashes hedge quantity	1 crash hedge quantity	2 crashes hedge quantity
Z_1	0.000	0.000	-0.002
Z_2	-0.004	-0.004	0.001
Z_3	0.169	0.204	0.239
Z_4	-0.699	-0.753	-0.816
Z_5	-0.468	-0.453	-0.425
Z_6	0.020	0.027	0.028
Z_7	0.000	0.000	0.000

Table 6.4: The optimal static hedges for a 4 year zero-coupon bond with crashes allowed

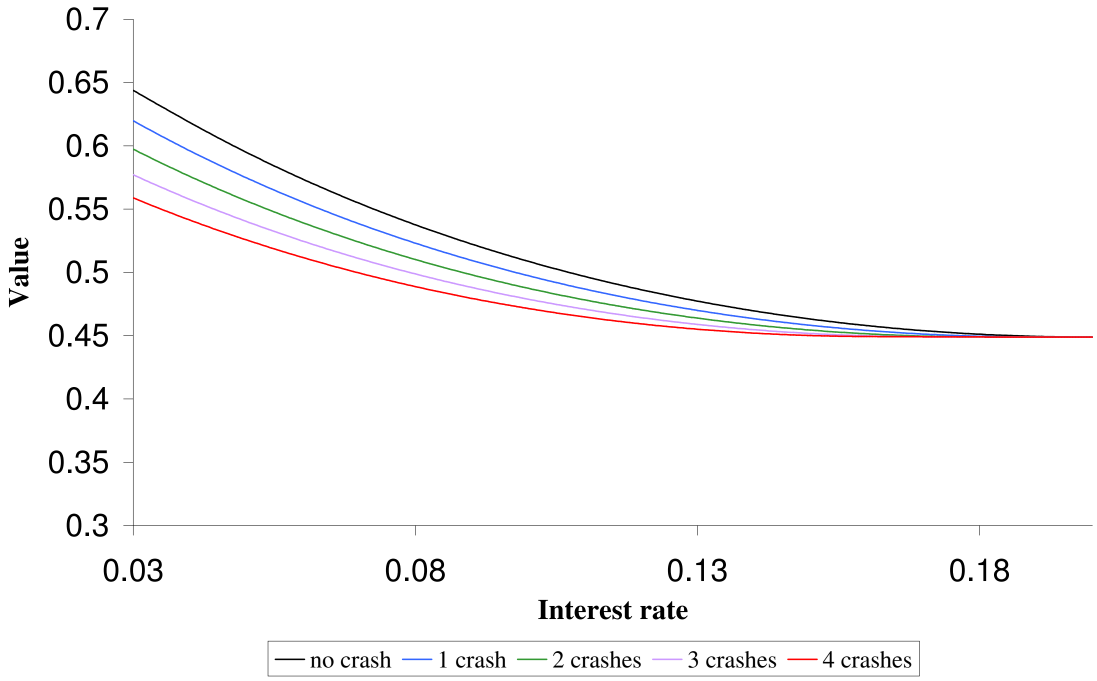


Figure 6.6: Value of a 4 year zero-coupon bond with a crash allowed

The addition of crashes lowers the value of the unhedged zero-coupon bond in a worst-case scenario. However, it has less effect on the value of the optimally-hedged bond because almost the entire interest rate risk of the bond can be hedged away. However, to hedge away this risk, increasing amounts of the hedging instruments are required as the number of crashes allowed increases.

We remark that for the long zero-coupon bond, the worst-case scenario interest rate path will always be as high as possible. Since there is an upper bound for the short-term interest rate, given by Equation (2.3), the addition of a crash does not have any effect at high interest rates, as the rate cannot crash to a higher value than the upper bound. This is illustrated in Figure 6.6 where the values converge at high interest rates.

6.2.2 A maximum frequency of crashes

As an alternative, we can constrain the frequency of the crashes, instead of the total number allowed. We model crashes from r to $(1-k)r$, as before. However, the interest rate can only crash again when a specified time ω has passed since the previous crash.

To value a contract under this model, we must introduce another variable, τ, the time since the last crash. We also introduce two functions. $V_0(r,t;\tau)$ is the value of the contract when the last crash was τ ago. $V_1(r,t)$ is the value of the contract when a crash is allowed. The value of the contract today is then $V_1(r,t)$.

Since τ and t increase at the same rate when a crash is not allowed, the pricing equation for V_0 is

$$V_{0\tau} + V_{0t} + c\,(r, V_{0r})\,V_{0r} - rV_0 = 0.$$

When $\tau = \omega$, another crash is allowed. We therefore have the final condition in τ,

$$V_0(r,t;\omega) = V_1(r,t).$$

The final condition in t, and any jump conditions are dependent on the cashflows of the contract in question.

V_1 satisfies Equation (2.6), with suitable final and jump conditions dependent on the contract. In a worst-case scenario, a crash occurs if that would lower the value of the contract, and so we have the constraint,

$$V_1(r,t) \leq \min_{k^- \leq k \leq k^+} \left(V_0((1-k)r,t;0)\right).$$

With these two approaches to crash modelling, we can formulate a wide variety of crash events. For instance, we could combine the two models to allow a limited number of large crashes (1% say) plus a larger number of smaller jumps (0.1% say) to which we could assign a frequency. We must examine interest rate data to choose a sensible option for these parameters.

Example: We price and hedge a 4 year zero-coupon bond with principal 1. The hedging instruments are the zero-coupon bonds from Table 4.1. The spot short-term interest rate is 6% and the parameters of our model are

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

One crash of maximum size 1% is allowed over the time horizon. Smaller crashes of at most 0.1% are also allowed, but can occur, at most, once a month.

The worst-case scenario value of the 4 year bond without hedging is 0.558. When we optimally hedge the bond, the worst-case value increases to 0.728. The static hedge for this valuation is shown in Table 6.5.

Hedging bond	Worst case hedge quantity
Z_1	0.000
Z_2	-0.001
Z_3	0.202
Z_4	-0.770
Z_5	-0.415
Z_6	0.004
Z_7	0.000

Table 6.5: The optimal static hedges for a 4 year zero-coupon bond with crashes allowed

6.2.3 Estimating ϵ from past data

When we performed the data analysis of Section 6.1.1, we effectively identified the point at which the largest jump in the short-term interest rate occurred. We then used this jump to find the smallest possible value for ϵ under our model. This is because any smaller movement would also be allowed if this movement was allowed by the model.

By including the possibility of a crash in our model, we can exclude this largest jump (by making it the point where the crash occurred). We can then calculate the lowest value of ϵ for the next largest jump. This should lead to a smaller value for ϵ. We therefore examine the data to choose a sensible number and size for crashes and then calculate the corresponding ϵ.

Example: We choose our parameters to be

$$r^- = 3\%,\ r^+ = 20\%,\ c^- = -4\% \text{ p.a. and } c^+ = 4\% \text{ p.a.}$$

and examine daily, 1 month US interest rate from 21st October 1986 to 25th April 1995, as before. The 8 largest changes in the 1 month rate are shown in Table 6.6.

When we exclude these points, we find that the new minimum value for ϵ, so that the data is consistent with our model, is

$$\epsilon = 0.001066.$$

Date	Size of crash
22 DEC 86	1.328%
30 DEC 86	-1.203%
27 NOV 87	1.047%
30 DEC 87	-0.813%
29 NOV 88	1.000%
29 NOV 90	1.125%
28 DEC 90	-1.500%
27 NOV 92	0.938%

Table 6.6: The largest changes in the 1 month daily interest rate data

We note that with this value for ϵ, the changes in question correspond to jumps of between 0.953% and 0.266% in the short-term interest rate, r.

We remark that all of these crashes are at very similar dates during the year. They are to some extent end-of-year effects and occur regularly. However, their exact size and timing is not predetermined, so it is open to question whether or not the data should be cleaned to remove them. This matter is further complicated by the fact that we are using the data to examine a rate of a different period, rather than the one month rate itself.

6.3 Liquidity

The liquidity of a market is a measure of the inherent difficulty encountered on entering or exiting the market. As a market becomes more illiquid, the spreads between bid and offer prices increase. These spreads are likely to increase further for trades of particularly large quantities. However, this feature is more often apparent on exit rather than entry into a trade [59].

We can gauge the effect of liquidity on our model by including a bid-offer spread in the price of a hedging instrument, as discussed in Section 4.4.2. We can increase this spread, simulating illiquidity in the market, and examine how the worst-case price of a hedged contract changes, as well as the effect of the spread on the make-up of the optimal static hedge.

To try and gain some insight into the effects concerned, we shall only make one particular instrument illiquid. All of the hedging instruments, however, will admit a bid-offer spread. As the instrument becomes more illiquid, we would expect the static hedge to adjust to include less of the illiquid instrument and larger quantities of more liquid instruments. We shall increase the size of the bid-offer spread for the illiquid instrument. However, we will not make this spread dependent on the quantity of the instrument being traded. In Section 7.2, we consider areas for further research and suggest a more realistic model for liquidity.

Example: We include a bid-offer spread for the hedging bonds of Table 4.1. The resulting prices are shown in Table 6.7.

Hedging bond	Maturity (yrs)	Bid price	Offer price
Z_1	0.5	0.966	0.974
Z_2	1	0.929	0.937
Z_3	2	0.864	0.872
Z_4	3	0.801	0.809
Z_5	5	0.683	0.691
Z_6	7	0.575	0.583
Z_7	10	0.445	0.453

Table 6.7: The hedging bonds with a bid-offer spread

We then consider different liquidities for the 5 year bond, Z_5, increasing the bid-offer spread from 0.683 - 0.691, to 0.647 - 0.727 and then to 0.607 - 0.767. The Yield Envelopes for these liquidity levels, are shown, in close-up, in Figure 6.7. It is clear that as the 5 year bond becomes less liquid, the yield spread increases near 5 years maturity. There is still a maximum spread possible under the model (that of the contract hedged with all but the 5 year bond) and the yield spread tends towards this as the liquidity decreases.

To examine the effect of the liquidity changes on the optimal static hedge, we consider the value of the 4 year zero-coupon bond. The results of this worst-case scenario valuation are shown in Tables 6.8 and 6.9.

5 yr spread	0.683 - 0.691	0.647 - 0.727	0.607 - 0.767
No hedge	0.575	0.575	0.575
Optimal hedge on worst-case	0.725	0.711	0.708

Table 6.8: Worst-case value of the 4 year zero-coupon bond with an illiquid hedge

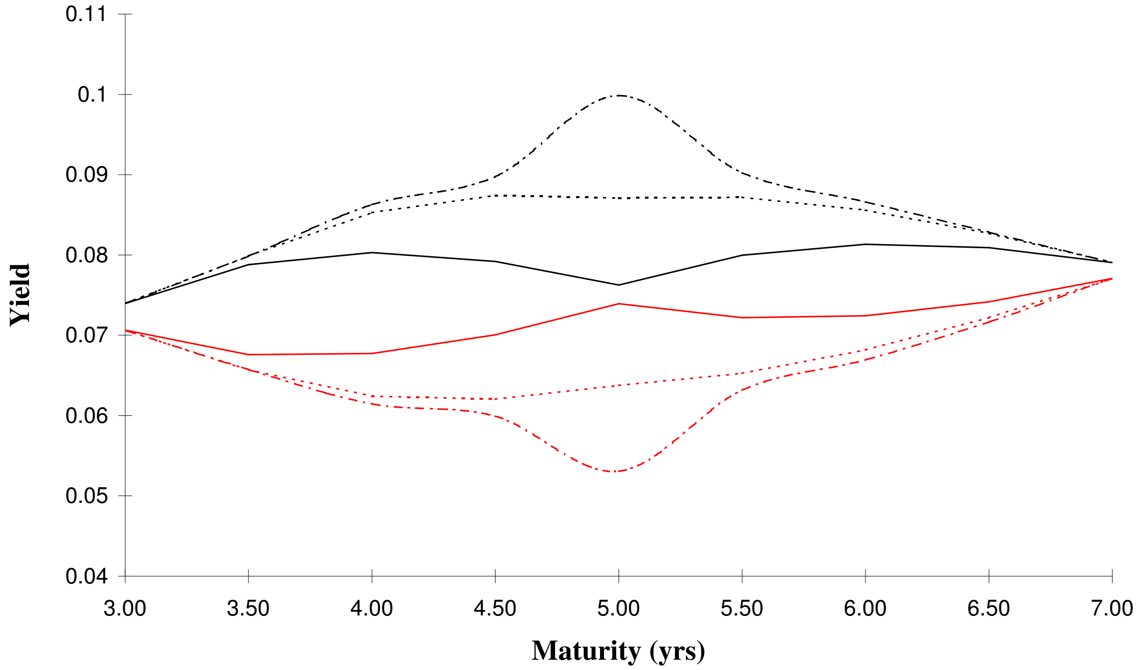


Figure 6.7: Illiquid Yield Envelope

Hedging bond	0.683 - 0.691 hedge quantity	0.647 - 0.727 hedge quantity	0.607 - 0.767 hedge quantity
Z_1	0.000	0.000	0.000
Z_2	0.000	0.000	0.000
Z_3	0.105	0.370	0.577
Z_4	-0.612	-1.047	-1.437
Z_5	-0.479	-0.301	0.000
Z_6	0.002	0.000	-0.105
Z_7	0.000	0.000	0.000

Table 6.9: The illiquid optimal static hedges for the 4 year zero-coupon bond

As we would expect, the value of the optimally-hedged 4 year bond decreases as the liquidity of the hedging instrument decreases. The quantity of the 5 year bond in the static hedge also decreases and the quantities of the bonds close in maturity to the 5 year bond increase significantly to make up for this shortfall.

Chapter 7

Conclusions

7.1 Summary of thesis

In Chapter 1, we reviewed common fixed-income contracts, and traditional approaches to interest rate modelling. We described the contract specifications for the bond, swap, cap, floor and bond option and demonstrated the processes by which we could decompose the swap into a set of zero-coupon bonds and the cap or floor into a set of bond options. We then summarised traditional approaches to interest rate modelling. We discussed the concepts of arbitrage, present value and yield to maturity before explaining how to price a contract off the yield curve. This motivated the concept of the forward rate and we described how to generate forward rates from a set of zero-coupon bond prices, by the process of bootstrapping, and how to price a contract using this forward rate curve. We then moved on to stochastic models, discussing one-factor and multi-factor models for the short-term interest rate. Particular attention was paid to the Vasicek and ACKW one-factor models, which we would use later on in the thesis. We concluded our review by describing the Heath, Jarrow & Morton model for the movement of the forward rate curve. Finally, we summarised the uncertain volatility model for equity derivatives, proposed by Avellaneda, Levy & Paras - in some sense the inspiration for our uncertain interest rate model.

In Chapter 2, we discussed the concept of a worst-case scenario valuation and illustrated the idea with a simple uncertain model for the short-term interest rate. The worst-case scenario value was the lowest possible value for a contract under the given model for the interest rate. We then described our general uncertain, non-probabilistic model, in which we bounded both the short-term interest rate and its growth rate. We formulated the pricing equation for the worst-case scenario value of a contract under this model. This was a first-order, nonlinear, hyperbolic partial differential equation with final data and jump conditions representing the contract cashflows. We examined the solution of this equation via the method of characteristics and found that particular attention had to be paid to points where the derivative of the contract value, with respect to the interest rate, was zero. This was because the direction of the characteristics was ambiguous at these points. We examined the effect of an internal maximum in the contract value and found that we had to introduce a shock into the problem to find a unique solution. The maximum evolved along the path of the shock which was positioned so that the contract value would be continuous (to prevent arbitrage opportunities). We then examined the effect of an internal minimum in the contract value and found that the minimum evolved along a characteristic. Finally, we remarked on the possibility of multiple maxima or minima in the contract value before discussing other possible occurrences of a zero interest rate derivative and the behaviour at the interest rate boundaries.

We illustrated the solution by the method of characteristics in Chapter 3. We first discussed the general methodology and then considered various keynote examples. These were final data with a solely positive gradient, final data with a solely negative gradient, final data with an internal maximum and final data with an internal minimum. In each case, we considered all of the possible characteristic pictures that could occur and found the solution of the equation for each of these situations.

In Chapter 4, we applied the model and the associated partial differential equation to the pricing of simple fixed-income products. We discussed the consequences of the nonlinearity of our pricing equation and considered the problem of the contract value in a best-case scenario. We found that our model therefore predicted a spread for the possible price of a contract. This spread could then be reduced by the process of static hedging. Using the zero-coupon bond to illustrate the procedure, we considered in detail the pricing and hedging of a contract. We found a spread for the possible price of this contract and then hedged with first a single contract and then a number of market-traded zero-coupon bonds to reduce the spread. We found that there was an optimal static hedge for which the worst-case scenario value of the contract reached a maximum level. Similarly, there was another optimal static hedge for which the best-case scenario value of the contract reached a minimum level. The spread between these two levels was significantly smaller than that between the unhedged worst- and best-case prices. Associated with these results was the Yield Envelope. This was similar in form to the yield curve, however, at a maturity where no traded contract existed, we found a yield spread.

We then applied the model to the pricing and hedging of swaps, caps and floors, describing the appropriate jump and final conditions for the pricing equation in each case. Finally, in the light of these results, we discussed possible applications for the model and presented a real world problem. The model could be used to identify arbitrage opportunities in the market, to establish prices for the market maker, to find the optimal static hedge for a contract, thereby reducing the interest rate risk, and as a risk management tool, to find an absolute measure of loss. To demonstrate the latter of these applications, we priced and hedged a real-world leasing portfolio and compared the results to the method of yield curve pricing. We found that the optimally-hedged worst-case value was equivalent in magnitude to a 1% downwards shift in the yield curve. However, the static hedge guaranteed that this was the maximum possible loss, whilst still preserving some upside potential.

In Chapter 5, we applied our model to the pricing and hedging of more exotic fixed-income contracts. We began with the European bond option. We found that if we only hedged the option with the underlying, there was a simple valuation methodology to follow. However, to price the American option, or either option hedged with other market-traded instruments, we had to develop a new approach to ensure that we found a consistent interest rate path. In this approach, we considered the various choices available to us (i.e. to exercise or not to exercise) separately and then chose the optimal exercise strategy. Second, we priced the multi-choice swap, a contract with embedded decisions. This swap allowed the holder to choose on which m of the M possible cashflow dates to exchange interest rate payments. To price this contract, we introduced a set of $m+1$ functions for the contract value, dependent on how many cashflows there were left to take. On a cashflow date, we found relationships between these functions to ensure that we followed the optimal choice strategy.

We then considered the pricing problem of the index amortising rate swap. In this contract, the principal amortised on cashflow dates, at a rate determined by an amortising schedule. We found the pricing equation for the worst-case scenario value of the contract and determined a similarity reduction to reduce the problem from three independent variables to two, comparing the results to the value of the swap off the yield curve. Again, we found that the optimally-hedged worst-case value was equivalent in magnitude to a 1% downwards shift. Finally, we examined the case of the convertible bond. This contract had coupon payments, of the same form as a vanilla coupon bond, but had the additional property that the holder could choose to exchange the bond for a specified number of an underlying asset. We described the partial differential equation for the worst-case value of the bond and compared the results of the pricing process to a number of more traditional approaches to interest rate modelling. Motivated by the results, we hedged the contract. In order to preserve a consistent worst-case interest rate path, we used the same approach to optionality as for the American bond option to price this hedged contract.

Finally, in Chapter 6, we presented extensions to our uncertain model. These allowed for interest rate paths that were indistinguishable from those seen in practice. We considered the concept of an uncertainty band, in which our model for the short-term interest rate became an estimate of the real short-term rate. We derived the new partial differential equation for the worst-case value of a contract under this assumption and illustrated with the example of the zero-coupon bond. We described how to relate the short-term interest rate to rates of a longer period and used this concept to examine past interest rate data. From this study, we were able to find a sensible width for the uncertainty band. As a further extension, we included the possibility for crashes in the short-term interest rate. These crashes could take one of two forms. There could either be a maximum possible total number or a maximum possible frequency for the crashes. We described the pricing equation framework for both cases and again illustrated by considering the zero-coupon bond. We then re-examined the data to find an adjusted uncertainty bandwidth, along with sensible parameters for these events. Finally, we studied the effect of illiquidity of the hedging instruments on the worst-case scenario valuation of a contract and presented an illiquid version of the Yield Envelope.

7.2 Areas for further research

There is considerable scope for further examination of the model and its applications in the marketplace. We now highlight some possible avenues for further work in this area.

- Market testing of the model applications

  There were a number of possible applications of the model, discussed in Section 4.8. However, no detailed market testing has yet been carried out to determine how viable these might be in practice. For example, the work of Section 4.9 on the hedging of the leasing portfolio could be extended significantly. We could rehedge the portfolio daily and compare the results to the current hedging strategy (which relies on the study of yield curve shifts) over a significant length of time. Alternatively, we could examine past market price data, to determine the feasibility of arbitrage spotting or predicting spreads for the market maker, for example.

- Parameter estimation

  In Chapter 6, we studied 1 month US data and calculated the uncertainty bandwidth for a given set of short-term interest rate bounds. Further work could examine interest rate data of a different origin or period. We could also consider the effect of altering the underlying bounds on the bandwidth. There may be an optimal choice of these parameters for which we could minimise some desirable spread in price, for instance.

- Liquidity modelling

  We presented a simplistic model for the liquidity of a hedging instrument in the previous chapter. There is significant room for improvement in this model. For instance, the model could be adjusted to include an increase in the bid-offer spread as the size of the trade increases. However, in the absence of arbitrage opportunities, we would have to ensure that any liquidity model proposed did not predict prices for the instrument that would lie outside of the spread of values predicted by our uncertain interest rate model.

On a more esoteric level, we present some possible future directions for the development of the uncertain interest rate model.

7.2.1 Economic cycles

The current model is, to some extent, ill-suited to long-term risk management. In this field, we require a model which takes into account the long-term economic cycles underlying interest rate movements. We would use such a model for pricing guaranteed annuity options, for instance, or in the life insurance industry [20].

We propose an extension of our uncertain model which could model economic cycles. To accomplish this, we think of an economic cycle (with a period of between five and ten years, say) as a form of simple harmonic motion, i.e.

$$\frac{d^2r}{dt^2} = a - \omega^2 r,$$

for some a and ω. This motion has period π/ω and is centred on $r = a/\omega^2$.

This motivates the addition of a constraint on the second derivative of the short-term interest rate in our model. We therefore propose a constraint of the form,

$$a^-(r,s) \leq \frac{d^2r}{dt^2} \leq a^+(r,s), \tag{7.1}$$

where

$$s = \frac{dr}{dt}.$$

We can rederive our worst-case scenario pricing equation under this new assumption. We again consider the movement in the value of a contract, $V(r, s, t)$, over a time step dt. Using Taylor's theorem to expand about a small time step dt and space steps dr, ds:

$$\begin{aligned} V(r+dr, s+ds, t+dt) &= V(r,s,t) + V_r dr + V_s ds + V_t dt + O(dr^2) + O(ds^2) + O(dt^2) \\ &= V(r,t) + sV_r dt + aV_s dt + V_t dt + O(dr^2) + O(ds^2) + O(dt^2), \end{aligned}$$

where

$$a^-(r,s) \leq a \leq a^+(r,s),$$

since ds is bounded from Equation 7.1.

Hence, we approximate (to $O(dt)$)

$$dV = V(r+dr, s+ds, t+dt) - V(r,s,t) = sV_r dt + aV_s dt + V_t dt.$$

We want to investigate the value of this portfolio in a worst-case scenario. We require that, in this worst case, our portfolio always earns the risk-free rate of interest. We therefore have

$$\min(dV) = dV_{\text{worst case}} = rV dt.$$

Otherwise, we could make an arbitrage profit on our belief that this is a worst-case scenario. Hence, we have

$$\min_a(dV) = \min_a(sV_r dt + aV_s dt + V_t dt) = rV dt.$$

Thus,

$$\min_a(sV_r + aV_s + V_t) = rV.$$

We can then take the minimisation inside the brackets, to find that the value of the contract in a worst-case scenario is

$$V_t + sV_r + a\left(r, s, V_s\right) V_s - rv = 0,$$

where

$$a(r,s,X) = \begin{cases} a^-(r,s) & \text{if } X > 0 \\ a^+(r,s) & \text{if } X < 0. \end{cases}$$

This is again a first-order nonlinear hyperbolic partial differential equation for the portfolio value, with characteristics given by

$$\frac{dt}{1} = \frac{dr}{s} = \frac{ds}{a(r,s,V_s)} = \frac{dV}{rV}.$$

7.2.2 A model for the forward rate curve

In Chapter 5, we often found that we were required to keep track of the worst-case scenario values of two different quantities at the same time in a consistent manner. For instance, the value of the underlying and the value of the hedging portfolio for an option. We also found that keeping track of a reference rate of longer period than the short-term rate was a complex, if not impossible, task (for the index amortising rate swap, for example).

We now comment on a potential solution to these problems. We propose a method that will automatically track the underlying bond price, leaving us with only one quantity left to consider. Similarly, the method will track interest rates of non-instantaneous period, allowing us to price contracts dependent on these rates more effectively.

To accomplish these objectives, we suggest an uncertain model for the entire forward rate curve. Where our previous model was the uncertain version of a short-term interest rate model, this new approach can be seen as synonymous to HJM in our uncertain world. We propose an uncertain model for the forward rate curve, $F(t,\tau)$, where

$$F^- \leq F \leq F^+,$$

with constraints on the derivatives of the form,

$$c^- \leq F_t \leq c^+,$$

$$d^- \leq F_\tau \leq d^+,$$

$$e^- \leq F_{\tau\tau} \leq e^+.$$

We have bounded the second derivative to ensure that points in the forward rate curve remain 'close' to each other. We could then price a contract as a function of this curve, $V(F,t)$.

It may be more convenient to work with a discrete set of points, $F_i(t)$, with separately specified bounds for each point (possibly with an approach synonymous to BGM in our uncertain world). In the extreme, it may even be possible to solely model the point on the forward rate curve that we are interested in, the reference rate for our contract for instance, as opposed to the entire curve.

7.3 Discussion

In this thesis, we have presented an uncertain, but non-probabilistic, model for the short-term interest rate and determined the pricing equation to find the value of a contract in a worst- or best-case scenario under this model. We have demonstrated how to price and hedge various fixed-income products using this approach and developed a number of extensions which allow for interest rate evolutions, that are indistinguishable from those seen in practice. Motivated by the results, we have proposed a number of plausible applications for the methodology. Since the model produces robust bounds for market prices, it could be used to spot potential arbitrage opportunities, by a market-maker to set bid-offer spreads or in risk management as a measure of absolute loss. The theory also generates a systematic and optimal approach to static hedging for the reduction of interest rate risk. It is our hope that through this work, we have laid the foundations for a new perspective in interest rate theory with practical application in the marketplace.

Appendix A

Numerical solution of the pde

In practice, the complexity of the characteristic picture requires us to solve the pde numerically. In this appendix, we consider two different approaches to the numerical solution of the problem. We first describe the explicit finite-difference scheme. This is a method by which we discretise the derivatives of the pde directly. Second, we construct a trinomial tree. This scheme is derived from the original worst-case assumption and the bounds on the interest rate. The latter approach is computationally faster, but less general in its application. In either case, we must first discretise the region in which we wish to find the solution.

A.1 Discretisation of the solution space

We discretise the solution space

$$0 \leq t \leq T \ , \ r^- \leq r \leq r^+,$$

using a grid of m space steps, Δr apart, and n time steps, Δt apart, where

$$\Delta r = \frac{(r^+ - r^-)}{m} \text{ and } \Delta t = T/n.$$

A general point on the grid has position

$$(r,t) = (r^- + i\Delta r, j\Delta t),$$

where

$$0 \leq i \leq m \text{ and } 0 \leq j \leq n.$$

This grid is shown in Figure A.1.

We approximate the solution V at a gridpoint by U, where,

$$V(r^- + i\Delta r, j\Delta t) \approx U_i^j.$$

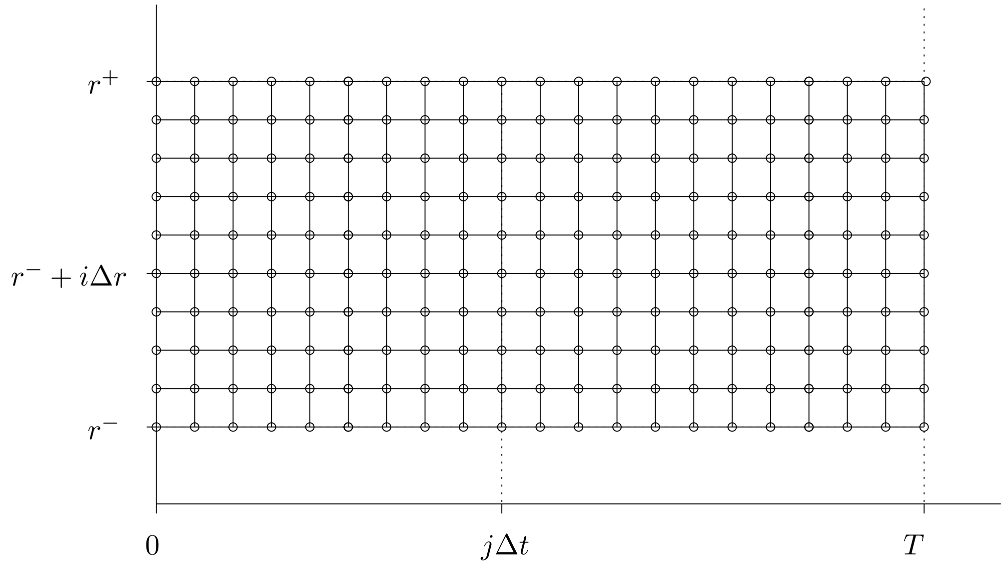


Figure A.1: The discretised solution space

We then solve backwards in time from expiry, using our chosen numerical scheme. We place final data at expiry, from Equation (2.8), of

$$U_i^n = C_N(r^- + i\Delta r).$$

At a cash flow date, Equation (2.9) tells us that

$$V(r, T_C^-) = V(r, T_C^+) + C(r),$$

which we discretise by including

$$U_i^{j_c} = U_i^{j_c} + C(r^- + i\Delta r),$$

in our scheme at the cashflow timestep, j_c, where $j_c = T_C/\Delta t$.

A.2 Explicit finite difference scheme

In the explicit finite-difference scheme, we approximate the partial derivatives in Equation (2.6), using Taylor series expansions near the points of interest [51].

We approximate V_t using a backwards difference,

$$V_t(r,t) \approx \frac{U_i^j - U_i^{j-1}}{\Delta t}.$$

We approximate V_r using an upwind scheme,

$$V_r(r,t) \approx \begin{cases} \frac{U_{i+1}^j - U_i^j}{\Delta r} & \text{if } c > 0 \\ \frac{U_i^j - U_{i-1}^j}{\Delta r} & \text{if } c < 0. \end{cases}$$

This gives us the numerical scheme:

$$\frac{U_i^j - U_i^{j-1}}{\Delta t} + c(r, V_r)\frac{1}{\Delta r}\left\{ \begin{array}{ll} U_{i+1}^j - U_i^j & \text{if } c > 0 \\ U_i^j - U_{i-1}^j & \text{if } c < 0 \end{array} \right\} - (r^- + i\Delta r)U_i^j = 0.$$

Rearranging this equation, we have

$$U_i^{j-1} = (1 - (r^- + i\Delta r)\Delta t)U_i^j + c(r, V_r)\frac{\Delta t}{\Delta r}\left\{ \begin{array}{ll} U_{i+1}^j - U_i^j & \text{if } c > 0 \\ U_i^j - U_{i-1}^j & \text{if } c \leq 0. \end{array} \right. \tag{A.1}$$

The scheme is shown in Figure A.2.

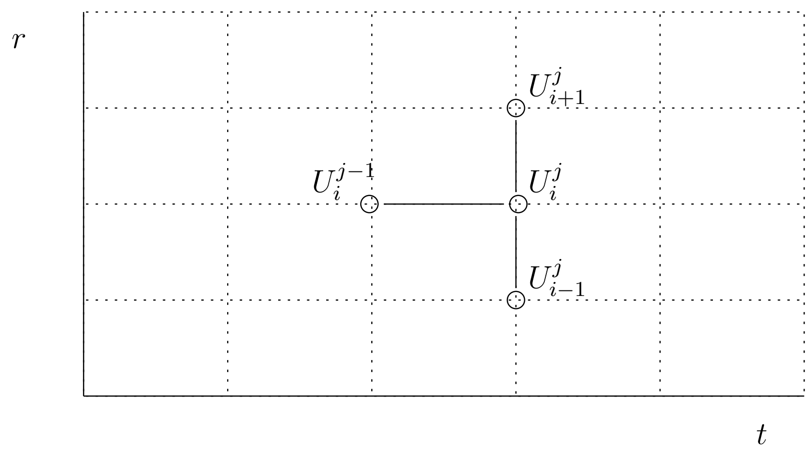


Figure A.2: The explicit finite difference scheme

We must satisfy the CFL condition. This is a necessary condition for the convergence of any finite difference scheme for a first-order hyperbolic pde [57]. It states that for a convergent scheme, the domain of dependence of the pde must lie within the domain of dependence of the numerical scheme. Otherwise, it would be possible to alter the problem in such a way that the analytical solution would change, but the numerical solution would remain the same. Our scheme satisfies this condition as long as

$$|c(r, V_r)|\Delta t \leq \Delta r.$$

We have not yet described the discretisation of $c(r, V_r)$ in our equation. There is no obvious definition, since the term is dependent on V_r and there are a number of possible discretised forms for this derivative. For instance, we could employ a forwards, backwards or central difference. Unfortunately, none of these options quite captures the qualitative behaviour of our model near a maximum or minimum. Instead we must construct a 'lower level' definition of c, dependent on U_{i+1}^j, U_i^j and U_{i-1}^j, so that we minimise $c(r, V_r)V_r$ at each point, given that we will use the upwind scheme for the latter V_r term.

If we are at a minimum, i.e.

$$U_i^j \leq U_{i-1}^j \text{ and } U_i^j \leq U_{i+1}^j,$$

then to minimise $c(r, V_r)V_r$, we choose $c(r, V_r) = 0$.

If we are at a maximum, i.e.

$$U_i^j \geq U_{i-1}^j \text{ and } U_i^j \geq U_{i+1}^j,$$

then to minimise cV_r, we choose

$$c(r, V_r) = \begin{cases} c^+(r^- + i\Delta r) & \text{if } c^+(r^- + i\Delta r)(U_{i+1}^j - U_i^j) \leq c^-(r^- + i\Delta r)(U_i^j - U_{i-1}^j) \\ c^-(r^- + i\Delta r) & \text{if } c^+(r^- + i\Delta r)(U_{i+1}^j - U_i^j) > c^-(r^- + i\Delta r)(U_i^j - U_{i-1}^j). \end{cases}$$

Otherwise we choose

$$c(r, V_r) = \begin{cases} c^+(r^- + i\Delta r) & \text{if } U_{i+1}^j - U_{i-1}^j < 0 \\ c^-(r^- + i\Delta r) & \text{if } U_{i+1}^j - U_{i-1}^j > 0. \end{cases}$$

These last conditions mirror the choice of c^+ if $V_r < 0$ and c^- if $V_r > 0$ when we use a central difference to approximate V_r in $c(r, V_r)$.

At the boundaries we must use a one-sided scheme for the space derivative. At the lower boundary, we approximate

$$V_r(r^-, t) \approx \frac{U_1^j - U_0^j}{\Delta r},$$

and choose

$$c(r^-, V_r) = \begin{cases} c^+(r^-) & \text{if } V_r(r^-, t) < 0 \\ 0 & \text{if } V_r(r^-, t) \geq 0. \end{cases}$$

At the upper boundary, we approximate

$$V_r(r^+, t) \approx \frac{U_m^j - U_{m-1}^j}{\Delta r},$$

and choose

$$c(r^+, V_r) = \begin{cases} 0 & \text{if } V_r(r^+, t) \leq 0 \\ c^-(r^+) & \text{if } V_r(r^+, t) > 0. \end{cases}$$

A.3 Trinomial scheme

A second approach to the numerical solution of the pde is to use a lattice (or tree) method to solve the equation numerically. Here, we construct a trinomial tree scheme. This is only valid when $c^+ = -c^-$ is a constant, for all r. However, the lack of generality in the scheme is made up for by the fact that it is significantly faster to compute than the explicit finite difference scheme.

We construct a grid for the solution space such that

$$c^+\Delta t = \Delta r.$$

In this case, at each time-step, the interest rate, r, can evolve to one of three possible values, $r - \Delta r$, r, or $r + \Delta r$. This gives us a tree-like structure for the possible paths of the interest rate, as shown in Figure A.3.

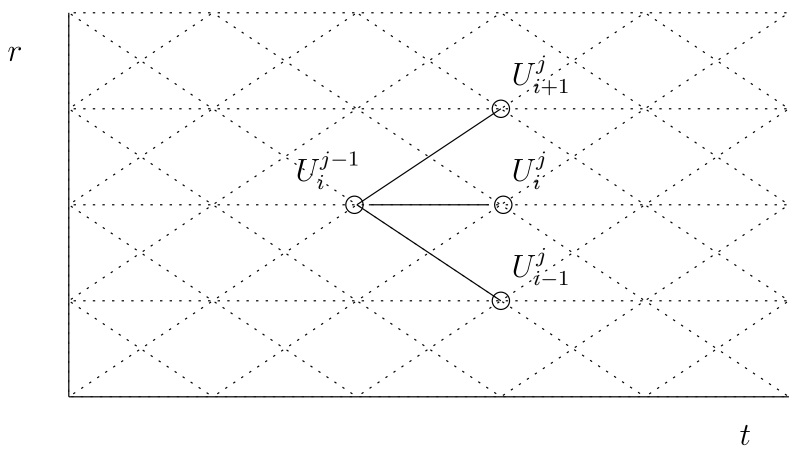


Figure A.3: The trinomial scheme

If we have found the solution for the value of the contract at time step j, then we know U_i^j for all i. The information for the point U_i^{j-1} must come from one of the points U_{i-1}^j, U_i^j and U_{i+1}^j, as the interest rate can jump up or down one step at most, over a time step.

We are interested in the worst-case scenario valuation, that is, the lowest possible value that U_i^{j-1} can have. We calculate the discounted value that we would find at U_i^{j-1} if we started at each of the three points, U_{i-1}^j, U_i^j and U_{i+1}^j, at time step j, where we discount at the average interest rate over the time step. We then set U_i^{j-1} to be equal to the lowest of these three values.

This gives us the scheme:

$$U_i^{j-1} = \min \begin{pmatrix} U_{i-1}^j(1-(r^- + (i-0.5)\Delta r)\Delta t), \\ U_i^j(1-(r^- + i\Delta r)\Delta t), \\ U_{i+1}^j(1-(r^- + (i+0.5)\Delta r)\Delta t). \end{pmatrix}$$

At the lower boundary, there is no 'down' step, and the scheme is

$$U_0^{j-1} = \min \begin{pmatrix} U_0^j(1-r^-\Delta t), \\ U_1^j(1-(r^- + 0.5\Delta r)\Delta t). \end{pmatrix}$$

At the upper boundary, there is no 'up' step, and the scheme is

$$U_m^{j-1} = \min \begin{pmatrix} U_{m-1}^j(1-(r^+ - 0.5\Delta r)\Delta t), \\ U_m^j(1-r^+\Delta t). \end{pmatrix}$$

A.4 A note on the optimisation routine

We use Microsoft Excel Solver to perform the optimisation routines for our numerical solutions (to find the optimal static hedges) [9]. Microsoft Excel Solver uses the Generalized Reduced Gradient (GRG2) nonlinear optimisation code developed by Leon Lasdon, University of Texas at Austin, and Allan Waren, Cleveland State University.

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