INTEREST RATE DYNAMICS AND THE PRICING OF CONTINGENT CLAIMS: A REVIEW OF THE MODELS AND A PROPOSAL

ANTONIO MANNOLINI

Abstract. The aim of this paper is twofold. In the first part I review some of the main theories of the term structure dynamics. These represent the starting point for the pricing of widely used derivatives. In the second part the focus of the article is shifted on a generation of new models introduced in the recent literature in order to reconcile the theory with real financial data. In the end an extension of the CIR model is proposed as a basis for pricing derivatives on the term structure as a possible direction for further research.

1. Introduction

Since the seminal paper of Black and Scholes, (1973), there has been a flourishing of works on the pricing of derivative instruments. The academic world has begun to work intensively on stochastic models of derivative products in the attempt to provide a robust, scientific approach to deal in a more consistent way with the uncertainty of business and trade. Pricing derivatives has indeed mainly practical purposes, and I can not deny that in recent years the intense use of derivatives has just confirmed the practical relevance of the mathematical models of Finance. On top of that, as recent models are becoming more and more technical, many traditionally trained economists leave this field to researchers with a different background, implicitly assessing that this way of doing finance is no more economics. In my opinion this a big misunderstanding of the issue. Financial Mathematics, in fact, is not to be considered merely a technical device to help practitioners. It is a new technical approach to deal with uncertainty, borrowed mainly from Mathematics and Physics. Economics is a relatively new Science, and it is not the first time it uses the instruments of other disciplines. It is true that up to now Finance has remained the main field of application of stochastic processes, but things could change as the recent use of these instruments in other economic fields [1] might suggest. It is my view that the use of these

[1] see, e.g., Aoki, (2002) modeling of industry dynamics

instruments will be more widespread in the economic profession in the next years because of their flexibility. This is only a point of view but a overwhelming fact still remains true: the theory of stochastic processes has proved very fruitful in finance, and one of the main reasons is that, especially in the recent characterizations of interest rate dynamics, stochastic explanations are becoming steadily more effective in taking into account what the real world data say. This should be indeed one of the main goals of economics.

Starting from this consideration, in this paper I review some of the main developments of the stochastic theories of term structures dynamics together with the classical arbitrage approach which so pervasively underlies the pricing of the most common contingent claims.

The term structure of interest rates is defined as the relationship between default free bonds of all maturities. The literature on the term structure can be traced back to Hicks, (1939) Modigliani Sutch (1966), but these approach were radically different from today's approach as they didn't take into account the stochastic dynamics of the short rate, which is indeed one of the main determinants of the term structure evolution over time, usually called term structure dynamics. After Black and Scholes (1976) model for options on stocks practitioners began to use the Black (1976) model to price derivatives on interest rates. The main assumption underlying this model is that the probability distribution of an interest rate, a bond price or another variable is lognormal at a well defined point in time. This implies that this model don't take into account the stochastic behavior of the variable as time elapses. This could undermine the consistency of the model every time one wants to price american style instruments, callable or puttable bonds, and other instruments in which price may be history dependent. This has been one of the main reasons of the development of the more recent literature on interest rate dynamics. The first explicit characterization of the short rate as a stochastic process has been made by Vasicek (1977) only twenty five years ago, but the recent years have seen a rapid development of this branch of economics. The outline of the paper is as follows. In section 2 I introduce in a very intuitive way the most common derivatives on interest rates, as they are the objects the theory wants to price. Section 3 is devoted to the explanation of two models, Vasicek (1977), CIR (1985), [2] belonging to the first generation of stochastic approaches to the term structure, known as

[2]still used used by and large by practitioners

equilibrium models. [3] Section 4 presents the Hull and White model which actually represent a change of perspective in the attempt to look more closely at what the real market data say. In section 5 I summarize the recent advances in the attempt to look at real world data in a more effective way (1992): HJM focus on forward rates, the analytical conditions on volatility structures proposed by Jeffrey (1995) to retain markovness of the spot rate process in the HJM approach, and a recent attempt to extend the CIR model to allow for arbitrary initial term structures without changing the volatility structure in order to preserve the affine structure, which guarantees analytical tractability. (Mari, 2001). Section 6 is a summing up of these theories and contains a research proposal: pricing derivatives according to the Extended CIR model and confronting these prices with the actual theoretical prices given by other methods to assess the relevance and usefulness of this extension from a practitioner's point of view.

2. Derivatives on Interest rates: main features

This section is a brief review of the main contingent claims whose final payoff depends on the movements of the term structure. This section might seem a bit boring, but it allows me to make clearer the purposes of this work.

2.1. **European Bond Options.** A European-type Call Option on a bond $B(t,T)$ is the right to buy the bond at a preset strike price K. This may be exercised only at the expiration date s of the option. The call option can be purchased for a price of C_t euros at time $t < s < T$ This is the standard and simplest example of interest rate contingent claim. In general, we have a formula for the value of the option at the expiration date s. In fact, under the hypothesis of absence of fees, commissions and bid ask spreads, the value of the option is given by

$$C_s = \max[P(s,T) - K, 0] \tag{2.1}$$

This is the main information we have we in advance; the determination of the price of the option at times $t < s$, is the aim of the theories we are going to present, and the previous expression is usually used as a boundary condition of some Cauchy problem.

[3] They are so called because they usually start with assumptions on some underlying variables to derive the equilibrium dynamics of the short rate

2.2. **Embedded Bond Options.** These are options which are embedded in Bonds to make them more attractive. A bond with an embedded call option is called Callable Bond: this allows the issuing firm to buy back it at a certain date at an ex ante fixed strike price; it is called callable because the holder of the bond has sold a call option to the issuer. Conversely, the case in which the holder is allowed to demand early redemption of a bond, at a predetermined prices at fixed date in the future, is an istance of puttable bond, because it is as if the issuer had sold a put option to the holder.

2.3. **Interest Rate Caps, (Strip of *Caplets*).** Imagine that you are borrowing money at an interest rate which is periodically reset equal to the Libor rate, and you want to avoid the risk of paying more than a certain amount of interests on your borrowing. The time elapsing between two close resets is called tenor and it lasts three months. It is clear that, for the first three months, you pay an interest rate on the capital you borrowed equal to the initial Libor rate, which is known. Unfortunately, for the next three months no one will tell you how much you'll have to pay. The cap is designed to insure against the rise of interest rate above a certain level. This level is known as Cap rate. The Cap is a derivative whose payoff is given by the following expression:

$$M\delta_k \max[r_k - r_x, 0] \tag{2.2}$$

where M is the capital on which the interest is calculated, $\delta_k = t_{k-1} - t_k$, the time elapsing, r_x is the interest rate cap, and r_k is the interest rate prevailing in the period of time under consideration. So, if the interest rate goes over the threshold of the cap, the buyer of the cap will get the difference between interest rate payment actually due and the cap interest rate payment from the issuer of the bond. This is indeed an insurance, whose fee is given by the option price, and the holder is insured every time there is a resetting; this motivates indeed the other two other alternative ways in which we may describe Caps: they can be seen as a portfolio of interest rate call options, each of which is called Caplet [4], and as a portfolio of put options on bonds. This can be easily seen with a slight modification of the previous

[4]This can be seen directly seen from the formula above; a Cap is indeed a strip of Caplets!

expression for the cap evaluated [5] at time t_k:

$$\frac{M\delta_k}{1 + r_k\delta_k} \max[r_k - r_x, 0] \tag{2.3}$$

with a few calculations this becomes:

$$\max[M - \frac{M(1 + r_x\delta_k)}{1 + \delta_k r_k}, 0] \tag{2.4}$$

The expression which is subtracted to M in previous formula is the value at t_k of a zero coupon bond that pays off $M(1 + r_x\delta_k)$ at time t_{k+1}. It is straightforward to see that the previous formula is the expression of a put option on that bond. A cap is simply a strip of such options.

2.4. **Interest Rate Floors, (strip of *Floorlets*).** Interest rate floors are the equivalent to caps except for the direction of the interest rate movements they want to cover. A floor provides a payoff to the holder when the interest rate on a loan falls below a certain threshold. In this case is the lender who wants an insurance for the case of too low remuneration of his capital. A floor provides a payoff at the generic time t_{k+1}:

$$M\delta \max[r_x - r_k, 0] \tag{2.5}$$

With a reasoning similar to that for the caps, mutatis mutandis, we can define an interest rate floor as a portfolio of put options on interest rates or a portfolio of call options on zero coupon bonds.Each of these options forming this portfolio is called floorlet.

2.5. **a Useful Parity Relation.** As in the case of stock derivatives there exists a call put parity relationship between the prices of caps and floors

$$P_C = P_F + V_S \tag{2.6}$$

where P_C denotes the price of the Cap, P_F the price of the floor, and V_S is the value of the swap. This relationship is very useful as one needs only to find the price of a side and then gets the other from this relationship. As for the validity we have to require that cap and floor have the same strike price, [6] and all the tree instruments have same life and same frequency of payments. This relation is quite intuitive, as a long position in a cap and a short position

[5] we discount using r_k

[6] which is also the fixed rate involved in the swap.

in a floor provides the same cash flows of a swap with no exchange of payments on the first reset date. This last observation has to be made because swaps are usually structured with an exchange of payments at the first reset date, while caps and floor usually do not.

2.6. **Collars.** A collar is a combination of a long position in a cap and a short position in a floor; usually it is formed in a way such that the price of a the cap is initially equal to the price of the floor, so that the entering cost is zero. The aim of that instrument is to guarantee that the interest rate on the underlying loan lies always between two predetermined levels.

2.7. **FRA, (Forward Rate Agreements).** Forward rate agreements are one of the most diffused derivative instruments to hedge interest rate risks. In this contract the two subscribers agree to exchange the cash flow generated by a predetermined interest rate F_t on a certain amount of capital against the cash flow generated by the Libor rate L_{t_i} quoted during the same period and on the same amount of capital. There are two types of FRA: standard and paid in Arrears. The buyer of the FRA paid in arrears receives at time t_{i+1} the sum

$$M(F_t - L_{ti})\delta \tag{2.7}$$

if the difference $F_t - L_{ti}$ is bigger than zero, and pays

$$M(F_t - L_{ti})\delta \tag{2.8}$$

if that difference is negative. If the cash flow of the FRA is paid at the same time the Libor is observed, the payoff has clearly to be discounted by $(1 + L_{ti}\delta)$ The Fra rate is selected in such a way that the time t price of the FRA contract is zero.

2.8. **Interest Rate Swaps.** In this type of contracts the two subscribers exchange mutually the cash flows generated by a fixed predetermined rate on a notional amount of capital against cash flows generated by a floating rate[7]on the same amount of capital during a predetermined period of time, in which rates are reset one or more time. It easy to realize that a Swap is a more complex form of a sequence of FRAs.

[7]usually the Libor rate

2.9. **Accrual Swaps.** This is a special kind of Swap in which interest on one side accrues only when the floating reference rate lies within a certain range, which can be held fixed or varied during the entire life of the Swap.

2.10. **Spread Options.** These are particular instruments whose payoffs depend on the spread between two interest rates, which can or cannot be calculated from the same yield curve.

2.11. **Interest Rate Swap Options(Swaptions).** As we mentioned before an option gives the right, but not the obligation, to buy or sell an underlying asset at a certain price at [8] a given strike price. Options on Swaps are designed to allow the holder to enter in a Swap contract at a certain date. In a sense, we can say that they bring into Swaps and FRAs the option feature of allowing but not forcing subscribers to do something at a certain date. Indeed in this sense they are different from forward, or deferred swaps, which involve no no up front costs but tie the hands of subscribers.

3. Stochastic Approaches to the Term Structure part 1 : The Classics

The earliest generation of stochastic models of the term structure is distinguished by the fact that the discount curve is derived starting from the law of motion of the short rate and arbitrage arguments. The first step, in fact, consists in postulating a law of motion of the instantaneous interest rate; in the case of diffusions, this is given by the following SDE:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dw(t) \tag{3.1}$$

where μ can be seen as the infinitesimal mean, and σ the infinitesimal variance. This characterization implies that the diffusion processes can be seen as the extension in continuous time of the uniperiodal mean variance models (De Felice Moriconi, p.202). Bond Prices are functions of $(r(t), t, T)$; we assume them enough smooth for our purposes, in the sense of being once differentiable with respect to t and twice differentiable with respect to r. Thus, the bond dynamics is given by

$$dP = \mu_p(r(t), t; T)Pdt + \sigma_p(r(t), t; T)Pdw(t) \tag{3.2}$$

[8] Or before in case of American Options

At first sight this might seem a geometric brownian motion, but in reality it is not, as drift and diffusion parameters depend also on the P, and are not constant as in the geometric case. Now let us write

$$\mu_p = \frac{1}{P}\left(\partial_t P + \mu(r(t), t)\partial_r P + \frac{1}{2}\sigma^2(r(t), t)\partial_{rr} P\right) \tag{3.3}$$

and

$$\sigma_p = \sigma(r(t), t)\partial_r P \tag{3.4}$$

This follows from Ito's lemma. If one wanted a risk free portfolio, she could get it simply selecting an ad hoc combination of bond differing only for maturity. To be more specific assume that ϵ_1 units of a first bond are purchased and ϵ_2 units of another bond with different maturity are shorted, for a total portfolio value of

$$V = \epsilon_1 P_1 - \epsilon_2 P_2 \tag{3.5}$$

After straightforward calculations is possible to prove that choosing the following portfolio weights

$$\epsilon_1 = \frac{\sigma_2}{P_1(\sigma_2 - \sigma_1)} V \tag{3.6}$$

and

$$\epsilon_2 = \frac{\sigma_1}{P_2(\sigma_2 - \sigma_1)} V \tag{3.7}$$

and replacing them in the bonds SDEs one can get the following instantaneous risk free portfolio dynamics:

$$dV = (\epsilon_1 \mu_1 P_1 - \epsilon_2 \mu_2 P_2)dt \tag{3.8}$$

This increment does not have any stochastic component and consequently is perfectly predictable. Now with a further manipulation we can get the following:

$$dV = \frac{\sigma_2 \mu_1 - \sigma_1 \mu_2}{\sigma_2 - \sigma_1} V dt \tag{3.9}$$

At this point the crucial argument comes in: if the market is arbitrage free, then the instantaneous return of the new portfolio should be equal to

$$r(t)dt \tag{3.10}$$

Imposing the equality, and rearranging we get a well known condition:

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda(r, t) \tag{3.11}$$

This condition holds for n maturities and his economic meaning is straightforward: the excess return of bonds, (or risk premia of bonds) of different maturities must be the same, once normalized by the correspondent volatility parameters. Another way of expressing this is saying that risk premia per unit volatilities must be the same across bonds of different maturities driven by the same source of uncertainty. As pointed out by De Felice Moriconi, (1992) notice that that nothing is said on the sign of the risk premia, as they can be even negative. The fact of not imposing restrictions on the sign of risk premia allows the model to be very easy to reconcile with most of the theoretical explanations of the term structure. If we want to consider the case of the pure expectation hypothesis, we have to put $\lambda = 0$; if on the other hand we believe in the liquidity preference framework by Hicks,(1939), then we simply have to insert a non negativity constraint on the sign of λ. The model without any constraint on λ is consistent with Modigliani, Sutch (1966) theory of the preferred habitat: risk premia are not constrained neither by the sign, not by the monotoncity with respect to the maturity. There is a compelling economic reason for this: the risk premium can be seen as a differential in the returns determined by different time preferences of the agents: as a rule, in fact, it is not always the case that investors prefer the same time horizon for their investments. Indeed, it is a well known fact that in the real world the composition of assets and liabilities is often determined by institutional features: for instance, a pension fund will have a different time preference for investments with respect to a single consumer. The implication is quite straightforward: if one wants to shift the investment of the agents from their optimal one, she will have to compensate them with the risk premium. The preferred strategy, which determines the choice of the financial instruments, depends on the time horizon preferred by the investors; it is now clear that putting constraint on the sign of the market price of risk is equivalent to assess the preferences of the investors for a certain instrument with respect to another. Thus, the model without any constraint on the sign of λ, besides being consistent with the theories of the preferred habitat, it also avoids to make any a priori assumptions on the preferences of investors. These will have to be made by the user of the model in the specific applications.

Now if we use the market price of risk equation in the expression for μ and impose the equivalence of the drift and diffusion components, which must hold in an arbitrage free market, after some algebraic manipulation we get the relevant PDE:

$$P_r\big(\mu(r_t,t)-\sigma(r_t,t)\lambda\big)+P_t+\frac{1}{2}P_{rr}\sigma(r_t,t)^2-r_tP=0 \tag{3.12}$$

The Boundary condition to this problem is given by

$$P(T,T)=1 \tag{3.13}$$

The solution to this problem gives us a stochastic process which represents the discount [9] function of the model, function of r and t. This is indeed the term structure we were looking for.

Now a remark is due: in this setting, the reasoning of Black and Scholes (1973), can not be applied sic et simpliciter, as we need to specify an additional parameter: the market price of risk. In the Black and Scholes setting, after the formation of a riskless portfolio, we usually replace the drift coefficient with the constant spot rate, and after straightforward manipulation we get the famous PDE. In this case,unfortunately, on one hand the spot rate is not constant, and, on the other, the driving process is no more the price of a traded asset, but a pure number, $r(t)$. As a result, replacing μ with $r(t)$,would force us to switch the interest rate drift to his risk neutral equivalent; to do this we need the exact specification of the market price of risk. For the sake of interpretation, the solution to this problem gives us a stochastic process which represents the discount [10] function of the model, function of r and t. This is indeed the term structure we were looking for. Before closing this section, notice a striking difference with the Black and Scholes (1973) work: in the case of stock as underlying assets by means of arbitrage arguments we succeeded completely in eliminating the need to model the drift component; the only objects to specify were the volatilities. In the case of the spot rate dynamics, we need to specify both the drift and the market price of risk parameter. As we will see later, the HJM approach will overcome this difficulty. The next step consists now in extending this reasoning to the final goal of these models: the pricing of contingent claims.

[9] equilibrium

[10] equilibrium

3.1. **The Pricing of Derivatives.** In the classical approach to the term structure, Bonds can be viewed as the simplest case of contingent claims, as they are deterministic at the expiry date; as a result the term term structure equation can be seen as the first example of derivative pricing[11]. As far as the general case is concerned, in which the expiry outcome is no more deterministic, we have to provide a general frame. Let us call

$$C = C(t, r(t)) \tag{3.14}$$

the price at time t of a contingent claim with

$$C(T, r(T)) = \Psi(T, r(T) \tag{3.15}$$

as the correspondent expiry condition, with $\Psi(.)$ real valued function. It can be proved that the following lemma holds:

Proposition 1. *If the Market is arbitrage free, then C(t,r(t)) is the solution of the following PDE:*

$$\partial_t C + (\mu - \lambda\sigma)\partial_r C + \frac{1}{2}\sigma^2 \partial_{rr} C - rC = 0 \tag{3.16}$$

with $C(T, r) = \Psi(T, r)$ *as boundary condition*

As a result the solution of this PDE will resolve the most general pricing problem. The story does not end here: in fact, it is also possible to prove that the solution of this Cauchy problem admits a stochastic representation known in literature as the Feymann- Kac solution given by the following:

$$C(t, r(t)) = E_t^Q \left[e^{-\int_t^T r(u)du} \Psi(T, r(T)) \right] \tag{3.17}$$

This representation is very useful as it expresses the price of a general contingent claim as the expected value of the discounted expiry payoff of the contingent claim itself. It must be stressed that this result holds under the martingale measure Q which also governs the short rate dynamics given by:

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma dw^\star(t) \tag{3.18}$$

[11]this is true only in the classical approach

where

$$w^\star(t) = w(t) + \int_0^t \lambda(u)du \tag{3.19}$$

from Girsanov theorem. On top of that it can be also shown that the discounted prices of the contingent claim are martingales under the risk neutral probability measure. Now the best way to give an intuitive idea of how this machinery actually works, is to provide a simple example. In case of the Price of the bond as an interest rate derivative the feymann-Kac representation gives us the following

$$P(t, r(t); T) = E_t^Q\left[e^{-\int_t^T r(u)du}\right] \tag{3.20}$$

We can interpret this simply as the expected value (under the martingale measure!) of an euro paid in T, discounted at time t. If instead we suppose you want to price an european call option written on a zero coupon bond with maturity T and expiry date $s < T$, and strike price K. According to the Feyman-Kac representation the price at time t of this option is given by:

$$C(t, r(t)) = E_t^Q\left[e^{-\int_t^s r(u)du} \max\{(P(s, r(s); T) - K, 0)\}\right] \tag{3.21}$$

The striking difference with respect to the Bond as an interest sensitive instrument lies in the expiry payoff: the expression $\max\{(P(s, r(s); T) - K, 0)\}$ is a deterministic function of $r(s)$ which is stochastic; as a result it is itself stochastic.

3.2. **The Mean Reversion Property.** The next step now, is to specify a functional form for the short rate movements; as a result, in the following, we will analyze two classical one factor models which admit closed form solution. In literature there are several characterization of the short rate process, but there is a widespread unifying feature that is worth mentioning: the mean reversion property. Interest rate models, in fact, differ radically from the Black and Scholes (1973) frame because of the departure from the geometric SDE. In fact, if we postulate a geometric brownian motion for the short rate, as t converges to infinity, the spot rate converges to plus or minus infinity, depending on the sign of the drift parameter. In the real world, interest rate seem not to have trends. Last, but not least, the percentage volatility of the r_t will be constant, but in the real world interest rate volatility seems to be a complicated non linear function of the short rate. As a result, this behaviour is not consistent with actual interest rate movements and is surely not sensible from an economic perspective. To avoid this inconsistency most of term structure models use a mean reverting specification for the short rate. There is also an economic argument in favor of mean reversion: when rates are high, the economy tends to slow down and borrowers require less funds: this leads to a decline in interest rates. On the other hand, if interest rate are low[12] the demands for funds tends to grow, and this pushes rates higher. As for the volatility behaviour we will talk later of the device used to account for the stochastic volatility characterizations presented in the literature. Now we are almost ready to analyze specific models of the short rate. Before that, however, I need to present a class of models which contains the above mentioned ones as subcases. This class is very used because of his the analytical tractability.

3.3. **Affine Models.** This class of term structures was introduced by Duffie e Kan, (1996). Let us start with the following definition:

Definition 1. *If the term structure* $\{P(t,T), 0 \leqslant t \leqslant T, T > 0\}$ *has the form*

$$P(t,T) = F(t, r(t); T) \tag{3.22}$$

and F is of the following form:

$$F(t, r(t); T) = e^{A(t,T) - B(t,T)r} \tag{3.23}$$

with $A(.,.)$ *and* $B(.,.)$ *deterministic functions, then the model is said to possess an Affine Term Structure.*

The functions $A(.,.)$ and $B(.,.)$ will have a great role in the following, so at least a remark is due. They are deterministic functions of the two real variables t and T. If we assume the generic dynamic form for the instantaneous interest rate, and we impose that this model actually has an affine term structure, we can substitute $A(.,.)$ and $B(.,.)$, together

[12]notice that the argument is not completely symmetric: if interest rate has been lowered by the monetary authorities in the attempt to boost the economy, no one can assure that the economy will grow, nevertheless, it is still true that borrowers have an incentive to raise funds if they are cheaper

with their derivatives, in the PDE holding in an arbitrage free setting. This gives us the following result:

$$A_t(t,T) - \{1 + B_t(t,T)\}r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0 \tag{3.24}$$

it is easy to notice that the boundary condition

$$F(T,r;T) = 1 \tag{3.25}$$

implies

$$A(T,T) = 0 \tag{3.26}$$

$$B(T,T) = 0 \tag{3.27}$$

It is clear that we need to find conditions on the drift and diffusion terms to guarantee the existence of solutions to the affine PDE term structure. This is a very complicated task, but we can observe that if μ and σ are affine in $r(t)$, then the PDE becomes a separable differential equation for $A(.,.)$ and $B(.,.)$ I report an approach which, far from being complete from the mathematical point of view, [13] will nevertheless allow me to derive closed form solutions for the following models. Let us see the specific example reported in Bjork (1998). Assume that the drift and diffusion coefficients have the following form:

$$\mu(t,r) = \alpha(t)r + \beta(t) \tag{3.28}$$

and

$$\sigma(t,r) = \sqrt{\gamma(t)r(t) + \delta(t)} \tag{3.29}$$

After some algebraic trick, the PDE assumes the following form,

$$A_t(t,T) - \beta(t)B((t,T) + \frac{1}{2}\delta(t)B(t,T)^2 - \{1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2\}r = 0 \tag{3.30}$$

This equation is very general and holds for all t, Tr. To simplify matters further, let us fix our choice of t and T. At this point, given that it must hold for all values of r, his coefficient must be equal to zero.Hence we have the following equation:

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 = -1 \tag{3.31}$$

[13]incomplete in the sense that doesn't take in account all the possible cases of the issue

Moreover, as the PDE must be zero on the whole, also the remaining term must be zero. Hence, also the following equation must hold:

$$A_t(t,T) = \beta(t)B((t,T) - \frac{1}{2}\delta(t)B(t,T)^2 \tag{3.32}$$

If we add the previous boundary conditions for A and B, we have a well defined Cauchy problem: we solve the first equation, which is a Riccati equation and can be solved in closed form and then we plug the solution into the second one and integrate to get A.[14]. This frame contains most of the particular specifications of interest rate models, and we will use it in the next sections to derive closed formula of the classic models of interest rate.

3.4. **Vasicek, 1977.** The main hypothesis of the model is that the instantaneous interest rate is stochastic and his dynamics are given by the following expression:

$$dr(t) = (b - ar)dt + \sigma dw^\star(t) \tag{3.33}$$

The main feature of this specification is the property of mean reversion to the level $\frac{b}{a}$. The drawback of this specification is that with this kind of volatility structure the model can exhibit negative values of the interest rate, which is not a good feature for an interest rate model. The main advantage of this specification, however, is that it can be easily solved using the affine structure property. In fact the two equations of the affine term structure become:

$$B_t(t,T) - a(t,T) = -1 \tag{3.34}$$

with $B(T,T) = 0$ as boundary condition, and

$$A_t(t,T) = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \tag{3.35}$$

with $A(t,T) = 0$ as boundary condition. For every maturity T the first equation is a simply ODE in t and it has the following solution:

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\} \tag{3.36}$$

[14]It is a proved result that if we suppose the drift and the diffusion parameters to be time independent, then affine μ and σ became a necessary condition for the existence of an affine term structure; for a more detailed discussion on the affine structures, see Duffie Kan (1996)

Then integrate the second equation and plug in it the expression for b to get the final result:

$$A(t,T) = \frac{\{B(t,T) - T + t\}(ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a} \tag{3.37}$$

Now that we have specified A and B in terms of the Vasicek model parameters, we can use the affine term structure definition given before to get:

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{3.38}$$

On the basis of this expression there exists a closed form solution of the price of bond options priced consistently with this term structure. This is given by the following:

$$C(t,T,K,S) = B(t,s)N(d) - B(t,T)KN(d - \sigma_p) \tag{3.39}$$

with

$$d = \frac{1}{\sigma_p}\log\{\frac{B(t,S)}{B(t,T)K}\} + \frac{1}{2}\sigma_p \tag{3.40}$$

and

$$\sigma_p = \frac{1}{a}\{1 - e^{-a(s-T)}\}\sqrt{\frac{\sigma^2}{2a}\{1 - e^{-2a(T-t)}\}} \tag{3.41}$$

3.5. **CIR, 1985.** The model proposed by Cox, Ingersoll and Ross is characterized by means of the following SDE for the instantaneous interest rate:

$$dr(t) = (b - ar)dt + \sqrt{r(t)}\sigma dw^\star(t) \tag{3.42}$$

as is easy to notice, the only difference with the Vasicek (1977) model lies in the diffusion parameter which contains a function of *r(t)*. This slight modification indeed embeds several advantages: first of all the short rate cannot have negative values. In fact, the closer the interest rate is to zero, the lesser is the magnitude of the diffusion component, due to the square root specification which allows the mean reverting component to be bigger than the stochastic one. On top of that this characterization of the diffusion component is consistent with the stochastic volatility feature, which is much more satisfactory in accounting for empirical evidence. The next step, then, is to derive the term structure behaviour implied by this law of motion of the short rate. CIR (1985) prove that their model admits a closed form representation of the following form:

$$P(t,T) = e^{A(t,T) - rB(t,T)} \tag{3.43}$$

with

$$A(t,T) = \frac{2ab}{\sigma^2} \ln \left[\frac{2\gamma e^{(\alpha+\gamma)\frac{T-t}{2}}}{(\alpha+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right] \tag{3.44}$$

and

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma} \tag{3.45}$$

On the basis of this term structure it is possible to obtain closed form expressions for call options on zero coupon bonds priced consistently with this model. The price of the European Call Option is thus given by the following:

$$C(r,t,s;T,K) = P(r,t,T)\chi^2(d_1;\iota,\nu) - KP(r,t,T)\chi^2(d_1;\iota,\nu_2) \tag{3.46}$$

with

$$d_1, \iota, \nu, \nu_1 \tag{3.47}$$

parameters consistent with the original model of CIR (1985). For details see the Bibliography. Despite his rather unpleasant shape, this formula resembles the Vasicek (1977) one, except for the fact that we substitute the χ^2 for the Normal distribution. On top of that if we give a closer look at it, it might recall the Black and Scholes (1973) one, and this should not be puzzling.

4. Stochastic Approaches to the Term Structure part 2: A change of Perspective : Hull and White (1992)

The main drawback of the first generation of interest rate models is that they involve several unobservable parameters and do not provide a perfect fit to any arbitrary initial term structure. This represents a very harmful drawback of these models, in so far a slight deviation from the actual rates might cause a big distortion in the resulting price of derivatives because they are much more sensitive; it goes without saying that derivative pricing might be inconsistent with the prices of the zero coupon bond. In order to comply with the real world data, Hull e White (1992), suggested a procedure to change the standard models of the short rate, and provided a closed form extension of the Vasicek (1977) model. In order to understand their intuition, suppose one obtains a reasonably accurate observation of the discount curve which one can assume to be arbitrage free $P(t,u), t < u \leq T$. The following equation,

$$P(t,T) = E^p[e^{-\int_t^T r(s)ds}] \tag{4.1}$$

then implies that the same spot rate process must satisfy the following set of equations:

$$P^\star(t,T_0) = E_t^Q[e^{-\int_t^{T_0} r(s)ds}] \tag{4.2}$$

$$P^\star(t,T_1) = E_t^Q[e^{-\int_t^{T_1} r_(s)ds}] \tag{4.3}$$

$$\ldots = \ldots \tag{4.4}$$

$$P^\star(t,T_n) = E_t^Q[e^{-\int_t^{T_n} r_(s)ds}] \tag{4.5}$$

where $t_0, ...t_n$ are the $n+1$ maturities at which we have reasonably accurate and arbitrage free bond prices. The problem is now to find a short rate process which allows for the solution of such a system of equations. How can we select drift and volatility parameters such that all the equations of the system are solved? We have $n+1$ equations with known left hand sides and only two free parameters to choose. The solution of such a system is not possible unless we admit a so strong interdependence among observed bond prices, that $n-1$ equations are, in fact, redundant. Needless to say, such a model wouldn't capture the information contained in market prices, and would be, therefore, useless for pricing purposes. As a result, the main issue now is to postulate a SDE with more than two parameters in the above system. Hull and White resolved this problem[15] just adding in the mean reverting specification a time dependent parameter θ_t. Doing this allows the model for more flexibility in fitting the process to the real data as we now actually have $n+3$ parameters. In their extended version of the Vasicek (1977) model Hull and White (1992) worked with the following specification of the short rate:

$$dr = (\theta(t) - ar)dt + \sigma dw^\star(t) \tag{4.6}$$

[15] in the simpler version of their model

with a and σ constants and θ a deterministic function of time. The time dependent parameter is chosen in order to fit the theoretical bond prices.This structure is affine, so bond prices are given by the following expression

$$P(t,T) = e^{A(T,t)-B(t,T)r(t)} \tag{4.7}$$

with $A(.,.)$ and $B(.,.)$ satisfying the following conditions:

$$B_t(t,T) = aB(t,T) - 1 \tag{4.8}$$

$$B(T,T) = 0 \tag{4.9}$$

and

$$A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T); A(T,T) = 0 \tag{4.10}$$

The solutions to these equations are given by:

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\} \tag{4.11}$$

$$A(t,T) = \int_t^T \{\frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T)\}ds \tag{4.12}$$

The next step is to fit the theoretical prices of the expression above to the observed one in order to get the parameters of the market. To accomplish this is convenient to use the forward rates. In fact, given the relationship:

$$P(t,T) = e^{-\int_s^T f(t,s)ds} \tag{4.13}$$

we can fit the theoretical forward rate curve, $\{f(0,T); T > 0\}$ w with the market forward rate curve $\{f^\star(0,T); T > 0\}$, with $f^\star$ defined as

$$f^\star(0,T) = -\frac{\partial P^\star(0,T)}{\partial T} \tag{4.14}$$

Keeping into account the following expression, which gives us the forward rates for the class of affine models:

$$f(0,T) = B_T(0,T)r(0) - A_T(0,T) \tag{4.15}$$

and inserting the correspondent expression for A and B given by the solution of the previous differential equations, we get an expression for f(0,T). Fitting this theoretical expression to the empirical one, we finally get:

$$f^\star(0,T) = e^{aT} r\{0\} + \int_0^T e^{-a(T-s)\theta(s)ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2} (4.16)$$

Taking the derivative of both sides w.r.t. T, one can in fact show that

$$\theta(t) = f^\star(0,T) - G^{\cdot}(T) + a\{f^\star(0,T) - G(T)\} \tag{4.17}$$

with

$$G(T) = \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 = \frac{\sigma^2}{2}B^2(0,t) \tag{4.18}$$

At this point is quite straightforward to realize that we have found an expression such that, for a fixed choice of a and the volatility parameter, gives us a term structure with the very desirable property that

$$P(0,T) = P^\star(0,T), \tag{4.19}$$

for all T bigger than zero. The only requirement we have to impose on this model structure is that $P^\star$ is twice differentiable with respect to T. Notice that we have found a martingale measure for the short rate which is consistent with the market data. The next step is to price bonds under this martingale measure; to do this is sufficient to plug the expression for θ in the expression for A and plug it, in his turn in the expression of bond prices. After some boring calculations one can get the following expression for the Hull and White Term Structure:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\{B(t,T)f^\star(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)\} \tag{4.20}$$

where B(.,.) is given by (4.11). To understand the usefulness of this result I report the Hull and White (1992) Bond option price formula.

$$C(t,T,K,S) = B(t,s)N(d) - B(t,T)KN(d-\sigma_p) \tag{4.21}$$

with

$$d = \frac{1}{\sigma_p}\log\{\frac{B(t,S)}{B(t,T)K}\} + \frac{1}{2}\sigma_p \tag{4.22}$$

and

$$\sigma_p = \frac{1}{a}\{1 - e^{-a(s-T)}\}\sqrt{\frac{\sigma^2}{2a}\{1-e^{-2a(T-t)}\}}(4.23)$$

As it is easy to notice it coincides with the Vasicek one; notwithstanding that, this formula contains the term $B(t,s)$ which is now made consistent with arbitrary initial term structure.

4.1. **The Inversion of the yield curve.** As the exposition of the Hull and White (1992) model should have suggested, we want to obtain the parameters of the martingale measure directly from the market; the methodology used to achieve this goal is known as inversion of the yield curve an consists of the following steps:

- Chose a particular model involving the parameter η we want to recover from the market and write the r-dynamics under the martingale measure as dependent on the parameter

$$dr(t) = \mu(t, r(t); \omega)dt + \sigma(t, r(t); \eta)dw(t) \tag{4.24}$$

- Solve, for every conceivable maturity date T, the term structure equation
- Now we have computed the theoretical term structure $\{P(0,T); T \geq 0\}$
- Collect data from the Bond market and get the empirical term structure. Denote this by $\{P^\star(0,T); T \geq 0\}$
- Select the parameter such that the theoretical curve $\{P(0,T); T \geq 0\}$ fits the empirical one $\{P^\star(0,T); T \geq 0\}$ in the best way possible according to a chosen objective function chosen to represent the goodness of fitness. This will give us the parameter we may think chosen by the market $\eta^\star$
- insert this empirical parameter vector into the expression for dr, the dynamics of the short rate. In this way we have chosen the martingale measure we can think of as reasonably chosen by the market
- Use this martingale measure to price interest rate derivatives. This can be done using the Feyman-Kac representation.

5. Stochastic Approaches to the Term Structure part 3: The State of the Art

After the Hull and White (1992) change of perspective, there has been a huge effort from the academic world in order to extend this new market oriented methodology to the widest possible range of characterizations of short rate dynamics in order to comply with the complexities of the real markets. In recent years an innovative approach has been developed to overcame the computational difficulties connected with the inversion of the yield curve necessary to determine the martingale measure chosen by the market. In fact, current works on interest rates share nowadays a common methodology in solving this problem of extending interest rate models to fit any initial term structure: [16] they use forward rates. This can be done thanks to the HJM methodology so called after the two seminal papers by Heath, Jarrow, and Morton, (1990), (1992), who for the first derived the result. HJM is not to be considered a model in itself, but, instead, a general methodology consisting in a thorough exploitation of the informations contained in the forward rate. This is not the whole story. Short rate models use only this one state variable to explain the term structure. From an economic point of view it would be more realistic to consider not only the short rate but also some long rate an possibly some intermediate one. The HJM methodology uses the entire forward rate curve as an infinite dimensional and observable state variable, (Bjork,1998). The next subsection is so devoted to the exposition, in a very heuristic fashion, of this methodology, as I will rely on it for the rest of this survey.

5.1. **HJM Methodology.** The main result of the HJM approach consists in providing the extension of the Black and Scholes (1973) reasoning to the fixed income sector using forward rates. This can be done as there exists a one to one correspondence between instantaneous forward rates and bond prices. Bonds are traded assets, so we can apply the procedure of replacing the drift coefficient with the short rate under a risk neutral probability measure. Passing from spot rates to forward rates, thus, allows us to incorporate directly arbitrage restrictions without specifying in advance the market price of risk. In order to derive heuristically the main result, we will switch to the following relations which exploits the definition

[16] given ad hoc volatility specification

of the instantaneous forward rate.

$$f(t,s) = -\frac{\partial}{\partial s} \ln P(t,s) \tag{5.1}$$

The first thing to notice in this expression is that here doesn't appear any expectation operator, because the forward rate curve is observable at time t. Now let us write the forward rate dynamics as follows:

$$df = \alpha(t,T)dt + \sigma(t,T)dw^\star(t) \tag{5.2}$$

It is possible to demonstrate that the following bond dynamics are implied by the previous expression:

$$dP(t,T) = P(t,T)\{r(t) + D(t,T) + \frac{1}{2}S(t,T)^2\}dt + P(t,T)S(t,T)dw(t) \tag{5.3}$$

with

$$D(t,T) = -\int_t^T \alpha(t,s)ds \tag{5.4}$$

$$S(t,T) = -\int_t^T \sigma(t,s)ds \tag{5.5}$$

The outline of the proof is the following, borrowed from Bjork (1998), and presented in a very euristic fashion. Let me define

$$Y(t,T) = -\int_t^T f(t,s)ds \tag{5.6}$$

Applying Ito's rule to the bond expression in terms of the forward rate we get

$$dP(t,T) = P(t,T)dY(t,T) + \frac{1}{2}P(t,T)\{dY(t,T)\}^2 \tag{5.7}$$

The following step consists in computing $dY(t,T)$. We know that

$$dY(t,T) = -d(\int_t^T f(t,s)ds) \tag{5.8}$$

It can be easily shown by means of standard integral calculus that the following holds:

$$dY(t,T) = r(t)dt - \int_t^T df(t,s)ds \tag{5.9}$$

Recalling the expression for the dynamics of the forward rate we get

$$dY(t,T) = r(t)dt - \left[\int_t^T \alpha(t,s)dt\right]ds - \left[\int_t^T \sigma(t,s)dw(t)\right]ds \tag{5.10}$$

Exchange dt and dw(t) with ds, [17] put r(t) instead of f(t,t)and get

$$dY(t,T) = r(t)dt + D(t,T)dt + S(t,T)dw(t) \tag{5.11}$$

As for the second order terms note that is a result of ordinary and stochastic calculus that $(dt)^2$ vanishes and $(dw)^2$ can be approximated as dt. This implies

$$\{dY(t,T)\}^2 = S(t,T)^2 dt \tag{5.12}$$

Substituting this in the general expression for Bond dynamics gives us that formula. Now I can demonstrate the HJM drift condition. In fact, under a martingale measure, the local rate of return of a bond must be equal to the short rate, i.e.

$$r(t) + D(t,T) + \frac{1}{2}S(t,T)^2 = r(t) \tag{5.13}$$

The last step is now simply to notice that $-D(t,T) = \frac{1}{2}S(t,T)^2$ implies

$$\int_t^T \alpha(t,s)ds = \frac{1}{2}\{\int_t^T \sigma(t,s)ds\}^2 \tag{5.14}$$

Expressing the last expression in terms of α, and deriving both members with respect to T we get the following characterization of the drift of the forward rate dynamics under the martingale measure for every t and T.

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds \tag{5.15}$$

The noticeable fact is that here the drift, determined by arbitrage arguments, depends only on the volatility parameters, and this resembles the Black and Scholes (1973) results. In this sense it can be said that the HJM can be considered the the true extension of their methodology to the fixed income sector. Up to now it might seem that HJM comes in at no cost, but this is not the case. Switching to forward rates and relying only on volatility calibration, has two main drawbacks: firstly, under the risk free measure, the forward rates are biased estimators of the future spot rates; secondly there may be cases in which the spot

[17] It is possible for the linearity property

rate doesn't follow a Markov process. This is a somewhat unpleasant feature of the HJM approach because of the heavy computational difficulties arising in non Markovian contexts. To have an idea of how this can happen consider that we can derive an equation for the spot rate starting from the SDEs for forward rates just relying on the consideration that the spot rate corresponds to the nearest infinitesimal forward loan:

$$r(t) = f(t,t) \tag{5.16}$$

The integral equation for the forward rate at T = t [18] is given by:

$$r(t) = f(0,t) + \int_0^t \sigma(s,t)[\int_s^t \sigma(s,u)du]ds + \int_0^t \sigma(s,t)dw_s \tag{5.17}$$

. Now take the conditional expectation of some future spot rate r_τ at $t < \tau$

$$E_T^Q\{r_\tau\} = E_t^Q\{f(t,\tau\} + E_t^Q\{\int_t^\tau \sigma(s,\tau)\{\int_s^\tau \sigma(s,u)du\}ds\} + E_t^Q\{\int_t^\tau \sigma(s,\tau)dw_s\} \tag{5.18}$$

The first component of the right side is known at time t; hence can be carried out of the expectation operator; the third expectation is zero because it is taken with respect to a Wiener process. The problem is that the second term in general doesn't vanish, and this is indeed the main reason why in general forward rates are biased estimators of the future spot rates under the risk neutral measure. As for the non markovness of the spot rate arising in the HJM framework, consider that r_t depends also on the term

$$\int_0^t \sigma(s,t)[\int_s^t \sigma(s,u)du]ds \tag{5.19}$$

which in general is a complex function of all past forward rate volatilities; it can be demonstrated that in general, the trasition densities of this expression depend on he whole story of the process.

Before closing this subsection, let me give an idea of how this machinery works. Suppose that the process σ is a constant, i.e. $\sigma(t,T) = \sigma$. The drift process according to the HJM result is then given by

$$\alpha(t,T) = \sigma \int_t^T \sigma ds = \sigma^2(T-t) \tag{5.20}$$

[18] to allow for the representation of the spot rate

The integral equation for the forward rates then becomes

$$f(t,T) = f^{\star}(0,T) + \int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dw(s) \tag{5.21}$$

which, for this particular case of diffusion component constant can be written as

$$f(t,T) = f^{\star}(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma dw(t) \tag{5.22}$$

The integral equation for forward rates in T= t, which is also the one for the spot rate is thus given by

$$r_t = f(t,t) = f^{\star}(0,t) + \sigma^2 t\frac{t^2}{2} + \sigma w(t) \tag{5.23}$$

It is straightforward to write the short rate dynamics

$$dr(t) = \{f_T(0,t) + \sigma^2 t\}dt + \sigma dw(t) \tag{5.24}$$

As it is clear from this specification, we have fitted the short rate dynamics to the initial term structure without too many complications for the Ho-Lee model. This is due to the specification of the model; if we want to apply this procedure to other models like CIR(1985), we have to use other analytical devices, as we will see in the following; nevertheless this example shows the effectiveness of this procedure in bringing observable market data to the forefront of arbitrage reasoning.

5.2. **After HJM Change of Perspective.** As mentioned before the non Markovness of the spot rate in the HJM approach can give heavy computational problems. Jeffrey in a 1995 article derived general conditions on the volatility structures in HJM under which markovness is still retained. To be more specific, he provides necessary and sufficient conditions such that one can determine which volatility structures are allowable in a markovian spot interest rate context [19], and the set of allowable initial term structures corresponding to a given volatility structure.[20]

Condition 1 A Volatility structure is allowable in a markovian spot interest rate based paradigm if there exists a pair of functions $\theta(r,t)$ and $h(t,T)$ that satisfy the following

[19] condition 1 in the paper

[20] condition 2 in the paper

equation:

$$\sigma(r,t,T)\int_r^T \sigma(r,t,v)dv = \frac{\sigma(r,t,T)}{\sigma(r,t,t)}\theta(r,t) + \frac{\partial}{\partial t}\left[\int_0^r \frac{\sigma(m,t,T)}{\sigma(m,t,t)}dm\right] + \tag{5.25}$$
$$+h(t,T) + \frac{1}{2}\sigma(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\sigma(r,t,T)}{\sigma(r,t,t)}\right]$$

The second condition states that given a volatility structure that satisfies condition 1, the correspondent initial forward rate curve must be of the following form

$$f(r,0,T) = \int_0^r \frac{\sigma(m,0,T)}{\sigma(m,0,0)}dm + J(T) \tag{5.26}$$

where

$$J(T) = -\int_0^T h((s,T)ds \tag{5.27}$$

for any admitted h(t,T) in condition 1.

In the following Jeffrey proves a Theorem which allows us to distinguish two main cases. In fact theorem 1 states that:

(i)if $\sigma(r,t,T)$ is not of the form $\xi(t,T\sigma(r,t,t)$, then there is only a valid pair of functions $\theta(r,T)$ and h(t,T) which satisfy condition 1, hence the volatility structure uniquely determines the initial term structure because of condition 1.

(ii) if $\sigma(r,t,T)$ is of the form $\xi(t,T\sigma(r,t,t)$, then the set of valid pairs of func tions $\theta(r,t)$ and $h(t,T)$ in condition 1 is given by:

$$\theta(r,T) = -\{\frac{\partial}{\partial t}\int_0^r \frac{\sigma(m,t,T)}{\sigma(m,t,t)}dm\}_{T=t} - C(t) \tag{5.28}$$

and

$$h(t,T) = \xi(t,T)(C(t) - h_p(t,t)) + h_p(t,T) \tag{5.29}$$

where h_p indicates any particular valid function $h(.,.)$ in condition 1. In this second case J(T) can be chosen to fit any observable initial term structure satisfying the condition of being differentiable at least once in maturity. To put it in a straightforward way this is the only case in which the model has enough elasticity to fit any observable term structure. Starting from the insights of this work, Mari(2003) has proposed a perturbative extension of a model for Bond Prices within the the affine class. This model is set up with the property of consistency with arbitrary initial term structures; as a result, only the case (ii) of Jeffrey is considered. Besides, as the author wants the model to account for the stochastic volatility feature, only the case of $K(t) \neq 0$ is considered. [21] Affine structures posses very interesting properties: first of all they are mathematically very tractable; as a consequence they allow for risk analysis and estimation via closed form solutions of PDE or via solutions of ODE of the first order; moreover they can be estimated using maximum likelihood techniques. In the model it is assumed that the discount factor is a smooth function of the spot rate and of maturity T. Under the risk neutral measure, the stochastic dynamics of the term structure is given by

$$\frac{dP(r(T),t;T)}{P} = r(t)dt + \sigma_P(r(t),t,T)dw^{\star}(t) \tag{5.30}$$

with

$$P(r(0),0,T) = P^{\star}(0,T) \tag{5.31}$$

Mari (2003) proved that the model can be fitted consistently with arbitrary initial term structures and the implied spot rate follows a Markov process iff the following condition, which is a particular instance of the Jeffrey (1995) results, holds:

$$\sigma_P(r(t),t,T) = \sqrt{h(t)+k(t)r(t)}B(t,T) \tag{5.32}$$

with

$$B(t,T) = 2\frac{C'(t)-A(t)}{k_1(t)}[\frac{1}{C(t)} - \frac{1}{\int_t^T A(u)du + C(t)}] \tag{5.33}$$

and

$$A(t) = \frac{1}{2}(C'(t) + -\sqrt{C'^2(t) - 2k(t)C^2(t)(5.34)}$$

where $h(t), k(t)$, and $C(t)$ are functions of time that can be arbitrarily chosen which must satisfy the condition $C'(t)^2 \geq 2k(t)C^2(t)$. Under this condition Mari (2003) shows that the solution of the Cauchy problem (5.30, 5.31) for the determination of the term structure is

[21] case 2, p.631 Jeffrey,1995

given by:

$$P(r(t),t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\left[f^*(0,t)B(t,T) - \int_0^t H(u)B(u,T)du + \right. \tag{5.35}$$
$$\left. + \frac{1}{2}\int_0^t \sigma^2(f^*(0,u),u)B^2(u,T)du\right] e^{-r(t)B(t,T)}$$

where $H(t)$ is the solution of the Volterra integral equation of the first kind:

$$\int_0^t H(u)B(u,t)du = G(t) \tag{5.36}$$

with

$$G(t) = -\frac{1}{2}\int_0^t \sigma^2(f^*(0,u),u)B^2(u,t)du \tag{5.37}$$

and f^* as usual is the initial forward rate curve. As a Corollary, Mari (2003) proves that the dynamics of the spot rate is described by

$$dr(t) = \big[a(t) - b(t)r(t)\big]dt + \sqrt{h(t) + k(t)r(t)}dw(t) \tag{5.38}$$

where

$$a(t) = \frac{\partial f^*(0,t)}{\partial t} + b(t)f^*(0,t) - H(t), \tag{5.39}$$

$$b(t) = -\frac{\partial^2 B(t,T)}{\partial T^2}|_{T=t} \tag{5.40}$$

The innovation of the paper consists in the explicit determination of the function $a(t)$ which is the term accounting for the initial term structure, and in bringing to the forefront the Volterra equation, as a device to overcame the obstacle met by Hull and White (1992) and Jeffrey (1995). In the following, Mari (2003), goes on considering some applications: Gaussian Models, the CIR volatility structure and the generalized CIR volatility structure, in which $h \neq 0$. The problem is that, in general, Volterra equation does not admit closed form solution, except for simple case, as the Vasicek one. In fact for the extended Vasicek model, this method give the same solution of the Hull and White (1992). As for the generalized CIR model, a perturbative solution of the Volterra equation is proposed.

To sum up, a relevant contribution of this work consists in providing a solution to the very general bond price equation, characterized as a Cauchy problem; moreover, within the class of models characterized by a generalized CIR term structure, a solution of the Volterra equation is provided via perturbation methods. This represent a very general frame by means of which most of affine models can be made consistent with arbitrary initial conditions, and can account for the stochastic volatility feature. This represents a big achievement in the theory of interest rate dynamics and can be considered the state of the art of this specialized, but still continually evolving branch of Finance.

6. Concluding Comments and a Proposal

The modern approach to interest rate dynamics and the pricing of contingent claims is certainly one of the youngest branch of economic theory; nevertheless research in this field has made considerable advances in the last twenty five years. The first generation of models has been replaced by more advanced frame works which have become more effective in taking into account what the real data say in a more consistent way; the Hull and White (1992) insight and the HJM switch to forward rates are considered success stories, but a lot of work remains to be done in the attempt to overcame heavy computational difficulties arising from the simultaneous attempt to use stochastic volatility structures, which are closer to reality, and the need to allow the models to be consistent with arbitrary initial term structures. The Extended CIR has recently been characterized from an analytical point of view and there is ample room for further research in this direction: first of all it would be very interesting to price the most common contingent claims according to that paradigm; secondly one could confront the price obtained with the actual prices in the market, and see if there are big discrepancies. On the basis of this comparison it would then be possible to asses the practical relevance of this new characterization of the term structure. The model is quite general so, in principle, it is possible to obtain a lot of different characterizations of the term structure dynamics as a basis for pricing contingent claim.

References

[1] Aoki, M.(2002), A new Model of Industry Dynamics *Working Paper*

[2] Bjork, T. *Arbitrage Theory in Continuous Time.* Oxford University Press,1999

[3] Black, F., (1976), The pricing of Commodity Contracts *Journal of Financial Economics, 167-179*

[4] Black,F.and Scholes, (1973) M.The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*,81, 673-654, 1973

[5] Brennan, M.J. & Schwartz, E.S. (1979). A continuous Time Approach to Pricing Bonds *Journal of Banjing and Finance*, July, 133-155

[6] Cox,J., Ingersoll.J., Ross, S. (1981). A re-examination of traditional Expectation Hypotheses about the term structure of Interest Rates. *The Journal of Finance*, 36 769-799

[7] Cox,J., Ingersoll, J., Ross, S. (1985b), A Theory of the Term Structure of Interest Rates *Econometrica*,53,385-408

[8] DeFelice, F., Moriconi, F., (1991). *La Teoria dell'immunizzazione Finanziaria*, Il Mulino Editore

[9] Duffie,D., *Dynamic Asset Pricing Theory* Princeton University Press, 1996

[10] Duffie, D.,Kan, R., (1996). A Yield Factor Model of Interest Rates,*Mathematical Finance, 6, 379-406*

[11] Harrison,J. & Kreps,J., (1981) Martingales and Arbitrage in Multiperiod Security Markets. *Journal of Economic Theory 11, 418-443*

[12] Harrison, J. & Pliska, J.(1981) Martingales and Stochastic Integrals in The Theory of Continuous Trading. *Stochastic Processes Applications 11, 215-260*

[13] Hicks, J., *Value and Capital*, Oxford University Press, 1939

[14] Heath,D., Jarrow,R. & Morton, A.(1992) Bond Pricing and the Term Structure of Interest rates *Econometrica, 60:1,77-106*

[15] Ho,T. Lee,S.(1986) Term Structure Movements and Pricing Interest Rate Contingent Claims. *Journal of Finance 41,1011-1029*

[16] Hull, J., (1997) “emphOptions, Futures and Other Derivatives (3rd Edition) Prentice Hall Englewwod and Cliffs, N.J.

[17] Hull, J.,White,A., (1990). Pricing Interest Rate Derivative Securities *The Review of Financial Studies, Vol.3, Issue 4, 573-592*

[18] Jamshidian, F. (1989). An Exact Bond Pricing Formula. *The Journal of Finance*,44, 205-209

[19] Jeffrey, A.(1995). Single Factor Heath Jarrow Morton Term Structure Models Dased on a Markov Spot Interest Rate Dynamics *Journal of Financial and Quantitative Analysis, 30,*

619-642

[20] Karatzas I. Shreve, S. (1988). *Brownian Motion and Stochastic Calculus.*Springer Verlag, New York Heidelberg Berlin

[21] Longstaff, F.A. 6 Swartz, E.S.(1992)Interest Rate Volatility and the Term Structure *Journal of Finance 40, 1259-1282*

[22] Mari, C.(2003). Single Factor Models with Markovian Spot Interest Rate: an Analytical Treatment*Decisions in Economics and Finance, forthcoming*

[23] Mari, C.,Reno',R., (2002). Extending the CIR Model in the Affine Class, *Working Paper*

[24] Modigliani, F., Sutch, R.,(1966). Innovations in Interest Rate Policy *American Economic Review, (May 1966), pp.33-58*

[25] Salih, Neftci. *An Introduction to the Mathematics of Financial Derivatives.* Academic Press second edition, 2000

[26] Vasicek,O. (1977) An equilibrium Characterization of the Term Structure. *Journal of Financial Economics, 5, 177-188*