# Interest Rate Options

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## Contents

# 1. Caps and Floors

## 1.1. Defintions

A cap is a call option on the future realisation of a given underlying LIBOR rate. More specifically, it is a collection (or strip) of caplets, each of which is a call option on the LIBOR level at a specified date in the future. Similarly, a floor is a strip of floorlets, each of which is a put option on the LIBOR level at a given future date. Caps and floors are widely traded OTC instruments. As explained below, they provide protection against widely fluctuating interest rates – a cap, for instance, is insurance against rising interest rates. Caps and floors also reflect market views on the future volatility of LIBOR rates.

## 1.2. Plain Vanilla Caps

In this section, we discuss plain vanilla caps in further detail.

For concreteness, suppose the underlying interest rate is the $\tau-$ maturity LIBOR. Let $L(t,T,\tau)$ denote[1] the forward LIBOR at time $t$ for the accrual period $[T,T+\tau]$. The spot LIBOR at time $T$ is then, by definition, $L(T,T,\tau)$. This rate fixes at time $T$, and a dollar invested at this rate pays $1+\tau L(T,T,\tau)$ at time $T+\tau$. The maturity $\tau$ is expressed in terms of fractions of a year – for example, $\tau = 0.25$ for the 3-month LIBOR.

[1]**Notation**: In $L(t,T,\tau)$, the first argument is current time, the second argument is the start date for the accrual period, and the third argument is the length of the accrual period.

### 1.2.1. Caplets

As mentioned earlier, a cap is a strip of caplets. The price of a cap is the sum of its constituent caplet prices, and so we focus on these first. A caplet is a call option on $L$. Specifically, a caplet with maturity date $T$ and strike rate $K$ has the following payoff: at time $T+\tau$, the holder of the caplet receives[2]

$$\zeta_T = \tau \left( L\left(T,T,\tau\right) - K \right)^+$$

Note that the caplet expires at time $T$, but the payoff is received at the end of the accrual period, i.e. at time $T+\tau$. The payoff is day-count adjusted. The liabilites of the holder of this caplet are always bounded above by the strike rate $K$, and clearly if interest rates increase, the value of the caplet increases, so that the holder benefits from rising interest rates.

By the usual arguments, the price of this caplet is given by the discounted risk-adjusted expected payoff. If $\{P(t,T) : T \geq t\}$ represents the observed term structure of zero-coupon bond prices[3] at time $t$, then the price of the caplet is given by

$$\zeta_t = \tau P\left(t, T+\tau\right) \tilde{\mathbb{E}}_t \left[ \left( L\left(T,T,\tau\right) - K \right)^+ \right]$$

In this equation[4], the only random term is the future spot LIBOR, $L\left(T,T,\tau\right)$. The price of the caplet therefore depends on the distributional assumptions made on $L\left(T,T,\tau\right)$. One of the standard models for this is the Black model, described in the appendix. According to this model, for each maturity $T$, the risk-adjusted relative changes in the forward LIBOR $L\left(t,T,\tau\right)$ are normally distributed with specified constant volatility $\sigma_T$, i.e.

$$\frac{dL\left(t,T,\tau\right)}{L\left(t,T,\tau\right)} = \sigma_T d\tilde{W}\left(t\right)$$

As shown in the appendix, this implies a lognormal distribution for $L\left(T,T,\tau\right)$, and under this modelling assumption the price of the $T-$ maturity caplet is given by

$$\zeta_t = \tau P\left(t, T+\tau\right) \left\{ L\left(t,T,\tau\right) \mathcal{N}\left(d_1^T\right) - K\mathcal{N}\left(d_2^T\right) \right\}$$

where

$$d_{1,2}^T = \frac{\log\left(L\left(t,T,\tau\right)/K\right) \pm \frac{1}{2}\sigma_T^2\left(T-t\right)}{\sigma_T\sqrt{T-t}}$$

and

$$\mathcal{N}\left(z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du$$

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[2]**Notation**: $x^+ = \max\{x,0\} = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$

[3]**Notation**: In $P\left(t,T\right)$, $t$ represents current time and $T$ is maturity date. A dollar at time $T$ is worth $P\left(t,T\right)$ dollars at time $t$.

[4]**Notation**: $\tilde{\mathbb{E}}_t\left[.\right]$ is the expectation operator, conditioned on all information at time $t$, hence the subscript, and computed using the risk-adjusted probabilities $\tilde{\mathbb{P}}$, hence the superscript.

### 1.2.2. Caps

To construct a standard cap on the $\tau$ – maturity LIBOR with strike $K$ and maturity $T$, we proceed as follows. Suppose the current time is $t$. Starting with $T$, we proceed backwards in steps of length $\tau$, and let $n$ be the number of complete periods of length $\tau$ between $t$ and $T$. Thus, we get a set of times

$$\begin{aligned}
T_0 &= t + \delta; \delta < \tau \\
T_1 &= T_0 + \tau \\
T_2 &= T_1 + \tau = T_0 + 2\tau \\
&\vdots \\
T_n &= T = T_0 + n\tau
\end{aligned}$$

We then construct a portfolio of $n$ caplets, struck at $K$, with maturities $\{T_0, T_1, ..., T_{n-1}\}$. These are called the fixing dates or caplet maturity dates. The dates $\{T_1, T_2, ..., T_n\}$ are called payment dates. The cap is then just equal to this strip of caplets.

The price of a cap is equal to the price of its constituent caplets. If $\zeta_i(t)$ denotes the price at time $t$ of a caplet with maturity date $T_i$ (and payment date $T_i + \tau$), then the price of the cap is

$$\zeta(t, T) = \sum_{i=0}^{n-1} \zeta_i(t) = \sum_{i=0}^{n-1} \tau P(t, T_{i+1}) \left[ L(t, T_i, \tau) \mathcal{N}\left(d_1^{T_i}\right) - K\mathcal{N}\left(d_2^{T_i}\right) \right]$$

The only quantity that cannot be directly observed in this pricing formula is the set of forward rate volatilities, $\sigma_{T_i}$. Thus

$$\zeta(t, T) = \zeta\left(t, T; \sigma_{T_0}, \sigma_{T_1}, ..., \sigma_{T_{n-1}}\right)$$

and a given set of forward rate volatilities produces a unique price. If we can find a single number $\sigma$ such that

$$\zeta(t, T) = \zeta\left(t, T; \sigma_{T_0}, \sigma_{T_1}, ..., \sigma_{T_{n-1}}\right) = \zeta(t, T; \sigma, \sigma, ..., \sigma)$$

then this $\sigma$ is called the implied or Black volatility for the $T$ – maturity cap. The observed prices of caps of various maturities are inverted numerically to obtain a term structure of Black volatilities, and these implied volatilites are quoted on the market.

### 1.2.3. Bootstrapping the Forward Volatility Curve

The forward volatility curve describes information about individual caplet volatilities contained in the term structure of implied cap volatilities. It therefore unravels the information contained in the caps, and is useful in pricing other vanilla or exotic options on LIBOR rates.

The technique used to extract caplet volatilities is called bootstrapping. The idea is essentially that a 10-year cap contains all the caplets in a 5-year cap. We work recursively, using

the caplet volatilities in short maturity caps to infer the volatility for the next maturity caplet. Suppose we have a cap volatility term structure as follows.

| Maturity (yrs) | Implied Cap Volatility |
|---|---|
| 1 | $\kappa_1$ |
| 2 | $\kappa_2$ |
| $\vdots$ | $\vdots$ |
| $n$ | $\kappa_n$ |

The volatility of the 1-year caplet is just $\sigma_1 = \kappa_1$. To get the volatility of the 2-year caplet, we use the fact that the 2-year cap consists of a 1-year cap and a 2-year caplet added to it.

$$\text{2-yr Caplet Price} = \text{2-yr Cap Price} - \text{1-yr Cap Price}$$

Since both the quantities on the right hand side are known, we can use this equation to infer an implied volatility for the 2-year caplet. Proceeding in exactly the same manner, we can infer the individual caplet volatilities for all maturities. Interpolation between quoted maturities is either piecewise linear or smoothed.

### 1.2.4. Caplet as a Put Option on a Zero-Coupon Bond

A caplet is a call option on an interest rate, and since bond prices are inversely related to interest rates, it is natural to be able to view a caplet as a put option on a zero coupon bond. Specifically, the payoff of a caplet is

$$\tau \left( L\left(T, T, \tau\right) - K \right)^{+}$$

This payoff is received at time $T + \tau$. The LIBOR rate prevalent over the accrual period $[T, T + \tau]$ is $L\left(T, T, \tau\right)$. It follows that, at time $T$, the price of the caplet is

$$\text{Caplet} = \frac{1}{1 + \tau L\left(T, T, \tau\right)} \tau \left( L\left(T, T, \tau\right) - K \right)^{+}$$

The price of a zero-coupon bond, on the other hand, is expressed as

$$P\left(T, T + \tau\right) = \frac{1}{1 + \tau L\left(T, T, \tau\right)}$$

from which it follows that

$$\begin{aligned} \text{Caplet} &= P\left(T, T + \tau\right) \tau \left( \frac{1}{\tau} \left[ \frac{1}{P\left(T, T + \tau\right)} - 1 \right] - K \right)^{+} \\ &= \left(1 + \tau K\right) \left( \frac{1}{1 + \tau K} - P\left(T, T + \tau\right) \right)^{+} \end{aligned}$$

This is just $1 + \tau K$ units of a put option on the $T + \tau$ – maturity zero-coupon bond with strike $\left(1 + \tau K\right)^{-1}$. Thus a caplet is a put option on a zero-coupon bond. A cap, therefore, is a basket of put options on zero-coupon bonds of various maturities.

### 1.2.5. Hedging Caps

A cap is a basket of options on a strip of forward LIBORs, and so is sensitive to changes in these. By the nature of the payoff of each caplet, this sensitivity is similar to that of call options on their underlying stock. A long cap position benefits from rising interest rates, and so a hedging instrument must lose value if interest rates rise. Appropriate hedging instruments include FRA's (receive fixed pay floating), futures strips (long) and swaps (receive fixed). The amount of hedge depends on the delta or hedge ratio. The broad strategy is to allocate more money to the hedge if interest rates rise and unwind the hedge if interest rates fall.

To illustrate the basic concept of delta-hedging, consider hedging with a futures strip. We shall focus on hedging an individual caplet. The payoff from the caplet is determined by

$$V(T, L_T) = (L(T, T, \tau) - K)^+ = (L_T - K)^+$$

By no-arbitrage, the present value of the caplet is $V(t, L_t)$, where we have adopted the shorthand notation $L_t = L(t, T, \tau)$. By the Black model assumption,

$$dL_t = \sigma L_t d\tilde{W}$$

The terminal value of a futures contract is $F_T = 1 - L_T$. One of the consequences of the Black modelling assumptions is that we approximate the present value of the futures contract with

$$F_t = 1 - L_t$$

Consider a portfolio consisting of one caplet and long $\Delta$ units of the $T$ – maturity futures contract. The value of this portfolio is

$$\Pi_t = V(t, L_t) + \Delta(1 - L_t)$$

By Ito's lemma,

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 L_t^2 \frac{\partial^2 V}{\partial L^2}\right) dt + \sigma L_t \left(\frac{\partial V}{\partial L} - \Delta\right) d\tilde{W}$$

By choosing

$$\Delta = \frac{\partial V}{\partial L}$$

we can knock out the randomness in the value of the portfolio. This is the delta-hedge. Additionally, since we are hedging two sets of cashflows which are initialised to be equal, we must have $d\Pi_t = 0$, and solving the resultant PDE (which is just a Kolmogorov backward equation) gives another equivalent way of calculating the price of a caplet.

The owner of a cap is always short the market, i.e. as bond prices rally (increase), the price of a cap decreases, as does its delta, and caplets become more out-of-the-market (OTM). A cap is, however, long vega – its value increases with volatility. The value of a cap also increases with increasing maturity, as the holder now owns a basket containing more caplets, and hence has more options.

## 1.3. Floors

### 1.3.1. Pricing and Hedging

A floor is a strip of floorlets, each of which is a put option on the LIBOR at a given date in the future. Pricing and hedging of floors is exactly complementary to the treatment of caps. The price of a floor with similar structure to the plain vanilla cap discussed before is given by:

$$\phi(t,T) = \sum_{i=0}^{n-1} \tau P(t,T_{i+1}) \left[ K\mathcal{N}\left(-d_2^{T_i}\right) - L(t,T_i,\tau)\mathcal{N}\left(d_1^{T_i}\right) \right]$$

Just as a caplet is a put option on a pure discount bond, similarly a floorlet is a call option on such a bond. The hedging instruments for floors are the same as for caps, except that positions are reversed since the holder of a floor benefits from falling interest rates. The owner of a floor is always long the market and long vega, i.e. benefits from rising volatility. The value of a floor also increases with maturity, as the number of put options increases.

### 1.3.2. Put-Call Parity

Consider a caplet $\zeta$ and a floorlet $\phi$, each maturing at time $T$, with the same strike rate $K$. Construct a portfolio

$$\pi = \zeta - \phi$$

The payoff from this portfolio, received at the end of the accrual period, is

$$\pi_T = \tau \left[ (L(T,T,\tau) - K)^+ - (K - L(T,T,\tau))^+ \right] = \tau \{ L(T,T,\tau) - K \}$$

This is just a cashflow from a payers swap. Thus we have the following version of put-call parity appropriate to caps and floors:

$$\text{Cap} - \text{Floor} = \text{Payers Swap}$$

### 1.3.3. At-the-money (ATM) Caps and Floors

From the above result on put-call parity, we can write

$$\Pi(t) = \zeta(t,T;K) - \phi(t,T;K) = \sum_{i=0}^{n-1} \tau P(t,T_{i+1}) \left[ L(T,T,\tau) - K \right]$$

which is just the value of a forward starting payers swap. The strike rate that makes the value of this swap zero is given by

$$K = \frac{\sum_{i=0}^{n-1} P(t,T_{i+1}) L(T,T,\tau)}{\sum_{i=0}^{n-1} P(t,T_{i+1})}$$

and this is defined to be the ATM strike for the $t$ – maturity cap/floor. In other words, the strike of an ATM cap or floor is the break-even $T - T_0$ year swap rate $T_0 - t$ years forward.

## 1.4. Digital Caps

### 1.4.1. Pricing

A digital cap is a strip of digital caplets, each of which is a digital call on the underlying LIBOR rate. Consider a digial caplet maturing at time $T$. The payoff from this caplet, received at the end of the accrual period $T+\tau$, is

$$\delta_T = \tau \mathcal{H}\left(L\left(T,T,\tau\right) - K\right)$$

where $\mathcal{H}$ is the Heaviside function[5]. The price of the digital cap is therefore given by

$$\delta\left(t,T\right) = \sum_{i=0}^{n-1} P\left(t,T_{i+1}\right)\mathcal{N}\left(d_2^{T_i}\right)$$

where $d_2^{T_i}$ and $\mathcal{N}\left(.\right)$ are as earlier.

### 1.4.2. Hedging

One way to hedge digital caps is by mirroring the dynamic strategy for vanilla caps, i.e. delta hedging with a strip of futures, or FRAs, or a swap.

Another way is to use a static hedge, which approximates the digital call with a portfolio of vanilla calls (this is called a call spread). The static hedge for a digital caplet with strike $K$ consists of buying $n$ vanilla caplets struck at $K-\varepsilon_1$ and selling $n$ vanilla caplets struck at $K+\varepsilon_2$. The approximation is summarised in the table below:

$$n = \frac{1}{\varepsilon_1+\varepsilon_2}$$

| LIBOR | Digital Caplet | Call Spread | Error |
|---|---|---|---|
| $L < K-\varepsilon_1$ | 0 | 0 | 0 |
| $K-\varepsilon_1 \leq L < K$ | 0 | $n\left(L-\left(K-\varepsilon_1\right)\right)$ | $\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}$ |
| $K \leq L < K+\varepsilon_2$ | 1 | $n\left(L-\left(K-\varepsilon_1\right)\right)$ | $\frac{-\varepsilon_2}{\varepsilon_1+\varepsilon_2}$ |
| $L \geq K+\varepsilon_2$ | 1 | 1 | 0 |

The hedge improves as $\varepsilon_1, \varepsilon_2 \to 0$, but this makes $n$ very large, i.e the size of the transaction increases, making the hedge expensive.

The vega profile of a digital caplet is not as simple as a vanilla cap. If the digital cap is out of the money (OTM), then increasing volatility means that there is a greater probability that the LIBOR will end up higher than the strike, which means that OTM digital caps are long volatility. The situation is reversed for in the money (ITM) digital caps, since increased volatility means that there is a greater probability of the option expiring OTM. Thus ITM digital caps are short volatility. We need a vega profile for each caplet.

[5]Notation: The Heaviside function is defined as follows:

$$H\left(x\right) = \begin{cases} 1 \text{ if } x \geqslant 0 \\ 0 \text{ if } x < 0 \end{cases}$$

## 1.5. Other Exotic Caps and Floors

### 1.5.1. Knock-In Caps

A knock-in cap consists of a strip of knock-in caplets. The in barrier can be up or down. For example, an up-and-in (UI) caplet is one that pays off only if the spot LIBOR is above a predetermined level $\hat{K}$, greater than the strike of the cap, $K$. In other words, the payoff from each caplet is (ignoring accrual factor)

$$(L(T,T,\tau)-K)^{+}\mathcal{H}\left(L(T,T,\tau)-\hat{K}\right)$$

It is easy to see that this caplet can be hedged using a static portfolio consisting of a digital caplet and a vanilla caplet, each struck at $\hat{K}$.

### 1.5.2. LIBOR Reset Caps

Each caplet in a LIBOR reset cap is struck at the previous LIBOR fixing. This means that the payoff from the $i$–th caplet is

$$\zeta_i = \left(L(T_i,T_i,\tau)-L(T_{i-1},T_{i-1},\tau)\right)^{+}$$

Thus the strike for the $i$–th caplet is unknown until the $(i-1)$–th caplet expires. LIBOR reset caps are mostly sensitive to the short end of the caplet volatility curve.

### 1.5.3. Auto Caps

An auto cap has the same structure as a vanilla cap, i.e. it consists of $n$ appropriately constructed caplets, except that the holder only receives payoff from $m<n$ of these. The cap disappears once the specified number of exercises is reached. Exercises are mandatory. These caps are cheaper than regular caps with $n$ caplets and more expensive than regular caps with $m$ caplets. Markov functional models, which we do not discuss here, are used to price these heavily path-dependent instruments.

### 1.5.4. Chooser Caps

Chooser caps have the same structure as auto caps, except that the holder has the right to choose which caplets to exercise. The added optionality makes them more expensive than auto caps.

### 1.5.5. CMS Caps and Floors

A CMS cap is a strip of call options on a given CMS rate (see the section on Constant Maturity Swap for more on this). The structure of a CMS cap is very similar to that of a plain vanilla cap. It consists of a set of caplets, whose payoff is given by

$$\zeta_T^{CMS} = \tau\left(R^{CMS}(T,T,m)-K\right)^{+}$$

where $R^{CMS}$ is the CMS rate that fixes at time $T$, for a tenor of $m$ periods. As before, we use the Black model to price these caplets. This time, however, the evolution of the convexity-adjusted forward starting swap rate is specified as

$$\frac{dR^{CMS}(t,T,m)}{R^{CMS}(t,T,m)} = \sigma_{T\times m} d\tilde{W}(t)$$

Using exactly the same calculations as before, it is straightforward to check that the price of the CMS cap is given by

$$\zeta^{CMS}(t,T) = \sum_{i=0}^{n-1} \tau P(t,T_{i+1}) \left[R^{CMS}(t,T_i,m)\mathcal{N}\left(d_1^{T_i}\right) - K\mathcal{N}\left(d_2^{T_i}\right)\right]$$

where

$$d_{1,2}^{T_i} = \frac{\log\left(R^{CMS}(t,T_i,m)/K\right) \pm \frac{1}{2}\sigma_{T_i\times m}^2 (T_i - t)}{\sigma_{T_i\times m}\sqrt{T_i - t}}$$

# 2. Swap Options

## 2.1. Swaps: A Brief Review of Essentials

A swap is an agreement between two counterparties to exchange a series of payments at regular intervals for a specified period of time. The payments are based on a notional underlying principal. One counterparty pays a fixed coupon rate $K$ and the other pays a variable or floating rate, which is usually LIBOR-based.

The lifetime of the swap is called its tenor. An investor is said to hold a payer swap if s/he payes fixed and received floating, and a receiver swap if the reverse is true. The time-table for payments can be represented schematically as follows (assuming unit notional principal)

| Time | Fixed Coupon | Floating Coupon | Cashflow |
|---|---|---|---|
| $T_0$ | 0 | 0 | 0 |
| $T_1$ | $\tau K$ | $\tau L(T_0,T_0,\tau)$ | $\tau\lvert K - L(T_0,T_0,\tau)\rvert$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $T_n$ | $\tau K$ | $\tau L(T_{n-1},T_{n-1},\tau)$ | $\tau\lvert K - L(T_{n-1},T_{n-1},\tau)\rvert$ |

Consider the position of the holder of a payer swap. The value of the payer swap at time $t < T_0$ (the first LIBOR fixing date) is given by

$$V^P(t,T_n) = \sum_{i=0}^{n-1} \tau P(t,T_{i+1}) \{L(t,T_i,\tau) - K\}$$

where $K$ is the fixed coupon rate. To give this swap zero initial value, we can set

$$K = R(t,T_0,T_n) = \frac{\sum_{i=0}^{n-1} P(t,T_i) L(t,T_i,\tau)}{\sum_{i=0}^{n-1} P(t,T_i)}$$

This rate $R(t, T_0, T_n)$ which gives zero value to a swap starting at time $T_0$ and expiring after $n$ payments at $T_n$, is called the forward par swap rate[6] for a swap with tenor $T_n - T_0$. The spot starting par swap rate, or just the swap rate, is the fixed coupon rate that gives zero intial value to a swap starting at the present time. This is denoted by $R(t, t, T)$ for a swap with tenor $T - t$.

## 2.2. Swaptions

### 2.2.1. Definitions

A swaption is an option on a swap. Swaptions are of two types: a payer swaption allows its holder to exercise into a payer swap at maturity, i.e. an agreement to pay fixed and receive floating cashflows for a specified period of time, called the swap tenor. Similarly, a receiver swaption is an option to exercise into a receiver swap, paying floating and receiving fixed rate payments for a specified time. At maturity, the holder of a swaption could exercise into swaps having several possible tenors. For this reason, swaptions must specify not just the expiry time $T$ of the option, but also the tenor $\hat{T}$ of the swap resulting from exercise. Thus swaption prices, volatilities etc are quoted on a matrix.

### 2.2.2. Payoff Structure

Consider a $T \times \hat{T}$ payer swaption. Let $T = T_0, T_1, ..., T_{n-1}$ be fixing dates and $T_1, T_2, ..., T_n = \hat{T}$ be cashflow dates for the swap. If $K$ is the fixed swap coupon and $\tau$ the constant accrual factor between fixing and payment dates, then the value at time $t$ of the payer swap is

$$V\left(t, \hat{T}\right) = \sum_{i=0}^{n-1} \tau P(t, T_i) \left[L(t, T_i, \tau) - K\right]$$

The payoff from the payer swaption is therefore given by

$$\Pi^{PAY}\left(T, T \times \hat{T}\right) = V\left(T, \hat{T}\right)^{+}$$

If we consider a forward-starting par swap, then the coupon rate $K$ is given by $R\left(t, T, \hat{T}\right)$. Substituting from the definition of this rate into the payer swap underlying the swaption, we get

$$V\left(t, \hat{T}\right) = \left(R\left(t, T, \hat{T}\right) - K\right) \sum_{i=0}^{n-1} \tau P(t, T_i)$$

[6]**Notation**: In $R\left(t, T, \hat{T}\right)$, $t$ is current time, $T$ is the swap start date, i.e. the date of the first LIBOR fixing, and $\hat{T}$ is the swap expiry, i.e. the date on which the final payments are exchanged.

The payoff from the swaption becomes

$$\begin{aligned} PAY &= \left\{ \left( R\left(T,T,\hat{T}\right) - K \right) \sum_{i=0}^{n-1} \tau P\left(T,T_i\right) \right\}^{+} \\ &= \left( R\left(T,T,\hat{T}\right) - K \right)^{+} \times \sum_{i=0}^{n-1} \tau P\left(T,T_i\right) \\ &= \left( R\left(T,T,\hat{T}\right) - K \right)^{+} \times \mathrm{PV01} \end{aligned}$$

The summation factor $\sum_{i=0}^{n-1} \tau P\left(T,T_i\right)$ is called the PV01, and represents the present value of a basis point paid on the swap cash flow dates. Thus a payer swaption is just a call option on the forward swap rate, with strike $K$. Similarly, a receiver swaption is a put option on the forward swap rate.

### 2.2.3. Pricing

The Black model is the market standard for pricing swaptions. The two crucial assumptions in the model are first, that the forward swap rate $R\left(t,T,\hat{T}\right)$ is driven by a zero drift geometric Brownian motion as before,

$$\frac{dR\left(t,T,\hat{T}\right)}{R\left(t,T,\hat{T}\right)} = \sigma_{T\times\hat{T}} d\tilde{W}\left(t\right)$$

and second, that discounting is constant. This implies that PV01 does not change through time.

Now, by using exactly the same calculations as for vanilla caplets, it is easy to see that the price of a payer swaption is given by

$$\begin{aligned} \Pi^{PAY}\left(t,T\times\hat{T}\right) &= P\left(t,T\right) \times \mathrm{PV01} \times \tilde{\mathbb{E}}_t \left[ \left( R\left(T,T,\hat{T}\right) - K \right)^{+} \right] \\ &= P\left(t,T\right) \times \mathrm{PV01} \times \left[ R\left(t,T,\hat{T}\right) \mathcal{N}\left(d_1^T\right) - K\mathcal{N}\left(d_2^T\right) \right] \end{aligned}$$

where

$$d_{1,2}^T = \frac{\log\left(R\left(t,T,\hat{T}\right)/K\right) \pm \frac{1}{2}\sigma_{T\times\hat{T}}^2 \left(T-t\right)}{\sigma_{T\times\hat{T}}\sqrt{T-t}}$$

The only unobservable is the forward rate volatility $\sigma_{T\times\hat{T}}$. However, there is a 1-1 correspondence between this volatility and the resultant price of a swaption, and this fact is used to invert observed swaption prices and obtain a matrix of flat implied volatilities, also known as Black or lognormal volatilities.

A receiver swaption is similar to a payer swaption, except that it can be expressed as a put option on the forward starting swap rate. The price of an otherwise identical receiver swaption is given by

$$\Pi^{RCV}\left(t,T\times\hat{T}\right) = P\left(t,T\right) \times \mathrm{PV01} \times \left[ K\mathcal{N}\left(-d_2^T\right) - R\left(t,T,\hat{T}\right)\mathcal{N}\left(-d_1^T\right) \right]$$

Details are exactly as for vanilla floorlets, and may be found in the appendix.

### 2.2.4. Put-Call Parity and Moneyness for Swaptions

Let $\Pi^{RCV}\left(t, T \times \hat{T}\right)$ be a receiver swaption with strike $K$ and $\Pi^{PAY}\left(t, T \times \hat{T}\right)$ be an identical payer swaption. Consider the portfolio

$$\Pi\left(t, T \times \hat{T}\right) = \Pi^{PAY}\left(t, T \times \hat{T}\right) - \Pi^{RCV}\left(t, T \times \hat{T}\right)$$

It is a simple exercise to verify that

$$\Pi\left(T, T \times \hat{T}\right) = \left(R\left(T, T, \hat{T}\right) - K\right) \times \text{PV01}$$

Or in words,

Payer Swaption – Receiver Swaption = Payer Swap

By no-arbitrage, the value of the portfolio $\Pi$ must just be equal to that of a payer swap. This relation can be used as an alternative to direct integration to find the price of a receiver swaption.

ATM strike is defined to be the value of $K$ which makes the values of the payer and receiver swaptions equal. Put-call parity now implies that this must be the same rate that gives a forward-starting swap zero value. In other words, the ATM strike is just equal to the forward starting par swap rate $R\left(t, T, \hat{T}\right)$.

### 2.2.5. Hedging

Swaptions are options on forward swap rates – a payer swaption, for instance, is a call option on the forward swap rate. The relationship between option price and the level of forward rates is therefore similar to that of call and put (for receiver swaptions) options.

The owner of a payer swaption is always short the market, i.e. as the market rallies (interest rates decrease), the payer swaption becomes more OTM, the delta increases and the value of the swaption decreases. Conversely, the holder of a receiver swaption is long the market. Both payer and receiver swaptions are long vega, and gain value from increased volatility.

Hedging instruments are usually combinations of forward starting swaps. A payer option, for instance, gains value as interest rates increase, and so its hedging instrument must lose value as this happens. A $5y \times 10y$ payer swaption can be hedged by receiving fixed on a 15y par swap and paying fixed on a 5y par swap.

## 2.3. Constant Maturity Swaps

### 2.3.1. Definition

A CMS swap differs from a regular swap in that the fixed leg is always indexed to a particular maturity swap rate. For example, the floating leg may be indexed to the 6-m LIBOR, and the fixed leg to the 5-y swap rate throughout the life of the swap.

### 2.3.2. Pricing

For a regular swap, the par swap rate is such that the initial value of the contract is zero. For a $T$–year CMS swap, we need to determine the spread over or below the $T$–year swap rate that is needed to set the initial value of the swap to zero[7]. We first use the usual bootstrapping methods to extract the zero-coupon rates from an observed term structure of swap rates. From these zero rates, we calculate the forward starting $T$–year swap rates at all the coupon dates.

We now need to add a convexity correction to these rates to rates. The intutive reason for this is that a forward starting swap reacts differently to changing interest rates than a CMS swap. Consider a CMS coupon received in a swap. The hedge should be a forward starting swap receiving fixed and paying floating rate. While the CMS coupon is a linear function of prevailing rates, the forward swap is a convex function of these rates. This means that a CMS swap will be worth more than a regular swap if rates move in either direction. As a result, the counterparty receiving CMS payments should have to compensate the CMS payer owing to this convexity advantage.

### 2.3.3. Approximate CMS Convexity Correction

Let $R\left(t, T, \hat{T}\right)$ be the forward swap rate. We assume a flat yield curve, parallel displacements and constant discounting. Though these and some other assumptions made here appear simplistic, they provide us with insight into the relevant ideas and give an approximate rule of thumb for the convexity correction. For this reason, these assumptions are often implicit in market practice. The change in value of CMS coupons is given by

$$dV^{CMS} = P(t, T)\left[dR + d\kappa\right]$$

Here, $\kappa$ is the convexity adjustment. With PV01 defined as before, write PV01 $= B(R)$. The change in value of the forward swap is then given by

$$dV^{FWD} = P(t, T)\, dB = P(t, T)\left[B'(R)\, dR + \frac{1}{2}B''(R)\, dR^2 + ...\right]$$

Under the Black assumption, $dR = \sigma R dW$. Consider a portfolio

$$\Pi = V^{CMS} + \Delta V^{FWD}$$

Then

$$d\Pi = P(t, T)\left[\left(1 + \Delta B'(R)\right)\sigma R dW + d\kappa + \frac{1}{2}\Delta\sigma^2 R^2 B''(R)\, dt\right]$$

Choosing $\Delta = -1/B'(R)$ gives the hedge ratio, needed to hedge the CMS swap with a forward start. With this choice of $\Delta$, the two portfolios will be compensated for convexity if, in addition, we have

$$d\kappa = \frac{1}{2}\sigma^2 R^2 \frac{B''(R)}{B'(R)} dt$$

[7]Saurav Sen would like to acknowledge Dr Jamil Baz for introducing him to this methodology.

Ignoring the time-dependence of forward rates, we integrate this to get the convexity correction

$$\kappa = \frac{1}{2}\frac{B''(R)}{B'(R)}\sigma^2 R^2 (T-t) = \frac{1}{2}\frac{\Gamma_{\text{PV01}}}{\Delta_{\text{PV01}}}\sigma^2 R^2 (T-t)$$

This is the amount to add to the forward rates to get CMS rates. As the reader may have noted, the calculation above makes a number of assumptions: $P(t,T)$ is left constant; $B''(R)\sigma^2 R^2/B'(R)$ is treated as independent of time when we integrate $d\kappa$; the duration and convexity measures assume parallel displacements of the yield curve and we assume that $R$ varies while $P(0,T)$ does not. These assumptions turn out to be relatively benign, as comparisons with more sophisticated models have shown.

### 2.3.4. Pricing (continued)

To fix ideas, we recap with some simplified notation. Suppose we want to price a 5-year CMS swap. We start with an initial term structure of swap rates $\{R_1^0, R_2^0, ....., R_{30}^0\}$. From this, we bootstrap an initial zero-coupon curve, $\{Z_1^0, ..., Z_{30}^0\}$. Then we use this zero curve to calculate the forward 5-year swap rates, $\{R_5^1, R_5^2, ..., R_5^5\}$. Applying the convexity correction, we get the CMS rates $\{C_5^1, C_5^2, ..., C_5^5\}$. Now we add a spread $\varepsilon$ to the CMS rates to ensure that the fixed leg of the swap is at par, i.e. find $\varepsilon$ such that

$$1 = \frac{C_5^1 - \varepsilon}{1 + Z_1^0} + ... + \frac{C_5^5 - \varepsilon}{\left(1 + Z_5^0\right)^5}$$

This completes the determination of convexity correction ans spread needed to give the CMS swap zero initial value.

### 2.3.5. CMS Summary

A CMS is a non-generic interest rate swap. Although two counterparties still exchange a series of payments based on a notional principal for a specified amount of time called the swap tenor, one counterparty has liabilities indexed to some LIBOR rate while the other pays a constant maturity swap rate minus some spread. The spread is chosen to give the CMS swap zero initial value.

The payer CMS swap is sensitive to the slope of the yield curve, and will gain from a flattening of the curve. As this happens, forward swap rates decline at the long end of the curve. This lowers the value of the CMS paying leg and increases therefore the swap value. The spread is a function of the slope and the volatility of the yield curve.

A receiver CMS is long convexity, volatility and gamma.

## 2.4. Other Swap Options

### 2.4.1. LIBOR in Arrears Swaps

In a regular swap, the floating leg LIBOR fixes at the start and pays at the end of each accrual period. LIBOR-in-arrears fixes and pays at the start of the accrual period. This instrument is attractive if the forward curve is upward sloping. The floating cashflow is of the form

$$v_T = \tau L(T, T, \tau)$$

and is received at time $T$. Investing this at the prevalent LIBOR rate for one period is a riskless investment, whose value at the end of the accrual period is

$$\pi_{T+\tau} = \tau L(T, T, \tau)(1 + \tau L(T, T, \tau))$$

The present value of this cashflow is therefore

$$\begin{aligned}\pi_t &= P(t, T+\tau)\tilde{\mathbb{E}}_t[\tau L(T, T, \tau)(1 + \tau L(T, T, \tau))] \\ &= P(t, T+\tau)\left\{\tau\tilde{\mathbb{E}}_t[L(T, T, \tau)] + \tau^2\tilde{\mathbb{E}}_t\left[L(T, T, \tau)^2\right]\right\}\end{aligned}$$

Under the standard Black modelling assumption of lognormal forward rates

$$dL_t = \sigma L_t d\tilde{W}_t$$

where $L_t = L(t, T, \tau)$, we can show that (see calculations in the appendix)

$$\pi_t = P(t, T+\tau)\left\{\tau L(t, T, \tau) + \tau^2 e^{\sigma_T^2(T-t)} L(t, T, \tau)^2\right\}$$

The first term is just the present value of a LIBOR cashflow. The second term is a convexity term. If the volatility were zero, we would have

$$\begin{aligned}\pi_t^0 &= \tau P(t, T+\tau) L(t, T, \tau)\{1 + \tau L(t, T, \tau)\} \\ &= \tau P(t, T+\tau) L(t, T, \tau)\frac{P(t, T)}{P(t, T+\tau)} \\ &= \tau P(t, T) L(t, T, \tau)\end{aligned}$$

which is just the present value of a regular LIBOR cashflow, adjusted for day count.

### 2.4.2. Bermudan Swaptions

These are mainly embedded in callable bond issues. The holder has the right to call a bond at predefined periodic dates. The coupon of the embedded swap is higher than the par swap rate due to the embedded long option position. Bermudan swaptions are hedged using European swaptions.

### 2.4.3. Hybrid Structures

It is possible to incorporate many different optionalities into caps, floors and swaptions. The more common structures include barrier or trigger options, which come into existence only if the LIBOR reaches a particular rate, and callable caps, where the holder is long the right to cancel.

# Appendix: The Black Model

## A.1. Model Specification

The Black model specifies the evolution of forward rates (these could be LIBORs, swap rates, CMS rates, etc) under the risk-adjusted probability measure $\tilde{\mathbb{P}}$ as follows

$$\frac{dF(t,T,\tau)}{F(t,T,\tau)} = \sigma_T d\tilde{W}(t)$$

Thus for each maturity $T$, the relative changes in forward rates have independent normally distributed increments with constant variance $\sigma_T^2$ per unit time. By Ito's lemma, it can be seen that

$$\begin{aligned} \log F(T,T,\tau) &= \log F(t,T,\tau) - \frac{1}{2}\sigma_T^2 (T-t) + \sigma_T \tilde{W}(T-t) \\ \text{i.e. } F(T,T,\tau) &= F(t,T,\tau) \exp\left\{ -\frac{1}{2}\sigma_T^2 (T-t) + \sigma_T \tilde{W}(T-t) \right\} \end{aligned}$$

This means that conditional on $F(t,T,\tau)$, the distribution of $F(T,T,\tau)$ is lognormal with parameters

$$\begin{aligned} \mu &= \log F(t,T,\tau) - \frac{1}{2}\sigma_T^2 (T-t) \\ \sigma^2 &= \sigma_T^2 (T-t) \end{aligned}$$

or equivalently that $\log F(T,T,\tau)$ is normally distributed with mean $\mu$ and variance $\sigma^2$, conditional on $\log F(t,T,\tau)$.

## A.2. Pricing Vanilla Calls

Caplets, payer swaptions etc are all call options on some underlying forward rate. To this end, we derive a formula for pricing vanilla calls on forward rates. The payoff is given by

$$I = \tilde{\mathbb{E}}_t \left[ (L(T,T,\tau) - K)^+ \right]$$

In this section, we derive a closed form expression for this price that is consistent with the Black model. By definition

$$I = \int_K^\infty (F(T,T,\tau) - K)\, w_F(F)\, dF$$

where $w_F$ is the conditional probability density function for $F(T,T,\tau)$, which is lognormal with parameters $\mu$ and $\sigma^2$. Writing $x_t = \log F(t,T,\tau)$ and $x_T = \log F(T,T,\tau)$, this integral becomes

$$I = \int_{\log K}^{\infty} \left(e^{x_T} - K\right) w_X(x_T)\, dx_T$$

Here, $w_X$ is the conditional density function of $x_T$ given $x_t$. By the previous discussion, this is normal with mean $\mu$ and variance $\sigma^2$. Thus

$$I = \int_{\log K}^{\infty} e^{x_T} w_X(x_T)\, dx_T - \int_{\log K}^{\infty} K w_X(x_T)\, dx_T = \text{Term 1} - \text{Term 2}$$

Term 2 is simply $K$ times the probability that $x_T$ is greater than $\log K$. Thus

$$\begin{aligned}
\text{Term 2} &= K \Pr\left[x_T \geq \log K\right] \\
&= K \Pr\left[\log F(t,T,\tau) - \frac{1}{2}\sigma_T^2 (T-t) + \sigma_T \sqrt{T-t} Z \geq \log K\right] \\
&= K \Pr\left[Z \geq \frac{\log\left(K/F(t,T,\tau)\right) + \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}}\right]
\end{aligned}$$

where $Z \sim N(0,1)$ is a standard normal random variable. If

$$\mathcal{N}(z) = \Pr\left[Z \leq z\right]$$

represents the cumulative distribution function for a standard normal random variable, then by symmetry we know that

$$\Pr\left[Z \geq z\right] = \mathcal{N}(-z)$$

Thus it follows that

$$\text{Term 2} = K\mathcal{N}\left(\frac{\log\left(F(t,T,\tau)/K\right) - \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}}\right) = K\mathcal{N}\left(d_2^T\right)$$

Now for Term 1:

$$\text{Term 1} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log K}^{\infty} e^{x_T - \frac{1}{2\sigma^2}(x_T-\mu)^2} dx_T$$

The term in the exponent is

$$x_T - \frac{(x_T-\mu)^2}{2\sigma^2} = \left(\mu + \frac{1}{2}\sigma^2\right) - \frac{\left(x_T - (\mu+\sigma^2)\right)^2}{2\sigma^2}$$

Now $\mu + \frac{1}{2}\sigma^2 = \log F(t,T,\tau)$, and so substituting this into the integral we get

$$\begin{aligned}
\text{Term 1} &= F(t,T,\tau) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log K}^{\infty} e^{-\frac{\left(x_T - \left(\mu+\sigma^2\right)\right)^2}{2\sigma^2}} dx_T \\
&= F(t,T,\tau) \frac{1}{\sqrt{2\pi}} \int_{\frac{\log K - (\mu+\sigma^2)}{\sigma}}^{\infty} e^{-\frac{1}{2}z^2} dz
\end{aligned}$$

where we have used the substitution $z = \frac{x_T - (\mu + \sigma^2)}{\sigma}$. The lower limit of the integral simplifies to

$$\frac{\log K - \log F(t,T,\tau) - \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}}$$

Thus

$$\begin{aligned}
\text{Term 1} &= F(t,T,\tau) \Pr\left[Z \geq \frac{\log(K/F(t,T,\tau)) - \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}}\right] \\
&= F(t,T,\tau)\, \mathcal{N}\left(\frac{\log(F(t,T,\tau)/K) + \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}}\right) \\
&= F(t,T,\tau)\, \mathcal{N}\left(d_1^T\right)
\end{aligned}$$

Putting everything together,

$$\begin{aligned}
I &= F(t,T,\tau)\, \mathcal{N}\left(d_1^T\right) - K\mathcal{N}\left(d_2^T\right) \\
\text{where } d_{1,2}^T &= \frac{\log(F(t,T,\tau)/K) \pm \frac{1}{2}\sigma_T^2 (T-t)}{\sigma_T \sqrt{T-t}} \\
\text{and } \mathcal{N}(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} du
\end{aligned}$$

## A.3. Pricing Vanilla Puts

The price of a vanilla put with strike $K$ is determined by

$$\begin{aligned}
J &= \int_0^K (K - F(T,T,\tau))\, w_F(F)\, dF \\
&= \int_0^{\log K} (K - e^{x_T})\, w_X(x_T)\, dx_T
\end{aligned}$$

where the notation is the same as before. Using the earlier results, we get the following:

$$\begin{aligned}
\text{Term 1} &= K \int_0^{\log K} w_X(x_T)\, dx_T \\
&= K \left[\int_0^{\infty} w_X(x_T)\, dx_T - \int_{\log K}^{\infty} w_X(x_T)\, dx_T\right] \\
&= K\left[1 - \mathcal{N}\left(d_2^T\right)\right] \\
&= K\mathcal{N}\left(-d_2^T\right)
\end{aligned}$$

Also,

$$\begin{aligned}
\text{Term 2} &= \int_0^{\log K} e^{x_T} w_X(x_T)\, dx_T \\
&= \int_0^{\infty} e^{x_T} w_X(x_T)\, dx_T - \int_{\log K}^{\infty} e^{x_T} w_X(x_T)\, dx_T \\
&= \text{Term 2a} + \text{Term 2b}
\end{aligned}$$

Term 2a is the moment generating function of a normal distribution with mean $\mu$ and variance $\sigma^2$, and so is equal to

$$e^{\mu+\frac{1}{2}\sigma^2} = e^{\log F_t} = F_t := F(t,T,\tau)$$

From the derivation of call prices, Term 2b is

$$F_t \mathcal{N}\left(d_1^T\right)$$

It follows that

$$\text{Term 2} = F_t\left(1 - \mathcal{N}\left(d_1^T\right)\right) = F_t \mathcal{N}\left(-d_1^T\right)$$

The price of a vanilla put is therefore given by

$$J = K\mathcal{N}\left(-d_2^T\right) - F(t,T,\tau)\,\mathcal{N}\left(-d_1^T\right)$$

## A.4. Digital Call Pricing

$$\delta_t = \tau P(t,T+\tau)\,\tilde{\mathbb{E}}_t\left[\mathcal{H}\left(F(T,T,\tau) - K\right)\right]$$

With the same notation as before,

$$\begin{aligned}
&\tilde{\mathbb{E}}_t\left[\mathcal{H}\left(F(T,T,\tau) - K\right)\right] \\
&= \int_K^\infty 1.w_F(F)\,dF \\
&= \int_{\log K}^\infty w_X(x_T)\,dx_T \text{ where } x_T = \log F(T,T,\tau) \\
&= \Pr\left[x_T \geq \log K\right] \\
&= \Pr\left[\log F(t,T,\tau) - \frac{1}{2}\sigma_T^2(T-t) + \sigma_T\sqrt{T-t}Z \geq \log K\right] \\
&= \Pr\left[Z \geq \frac{\log\left(K/F(t,T,\tau)\right) + \frac{1}{2}\sigma_T^2(T-t)}{\sigma_T\sqrt{T-t}}\right] \\
&= \mathcal{N}\left(d_2^T\right)
\end{aligned}$$

## A.5. LIBOR-in-Arrears Calculations

From the Black model,

$$F_T = F_t e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\tilde{W}_{T-t}}$$

It follows that

$$\tilde{\mathbb{E}}_t\left[F_T^2\right] = F_t^2 e^{-\sigma^2(T-t)}\tilde{\mathbb{E}}_t\left[e^{2\sigma\sqrt{T-t}Z}\right] = F_t^2 e^{\sigma^2(T-t)}$$

Here, we have used the fact that the moment generating function of a standard normal distribution is

$$\mathbb{E}\left[e^{\theta Z}\right] = e^{\frac{1}{2}\theta^2}$$