Some Extra Notes on Duration & Modified Duration.

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The price of a bond (assuming semi-annual coupons) is given by:

$$P = C(1+i)^{-1} + C(1+i)^{-2} + (P+C)(1+i)^{-3}$$

where i = the **half-yearly** interest rate (for example if semi-annual compounding and a yield of 12%, then $i = 6\%$).

We can differentiate the bond price with respect to the interest rate to get:

$$\frac{\partial P}{\partial i} = -1C(1+i)^{-2} - 2C(1+i)^{-3} - 3(P+C)(1+i)^{-4}$$

$$\frac{\partial P}{\partial i} = -(1+i)^{-1}(1C(1+i)^{-1} + 2C(1+i)^{-2} + 3(P+C)(1+i)^{-3})$$

$$\frac{\partial P}{\partial i} = -(1+i)^{-1} D_{Macauley}.P$$

Modified duration is defined by convention as $MD = -\frac{\partial P}{\partial i} / P$, that is:

$$MD = (1+i)^{-1} D_{Macauley}$$

Note that we discount the duration by a single coupon period's discount rate to get the modified duration. <u>The one period interest rate we use depends on the frequency of the coupon payments</u>. So if it is a quarterly coupon, the rate i would be the quarterly interest rate, if annual then use the annual interest rate etc. If the bond pays a continuous coupon, you can write out the PV of the cash flows and differentiate the expression containing exponential functions to find out what happens. Why does the period change? Do the differentiation with a different frequency coupon eg quarterly, and you will see.

To get an expression for convexity we can differentiate again to get an expression for $\frac{\partial^2 P}{\partial i^2}$.

Convexity is defined as a function of the second derivative.

Why is (was) modified duration and duration used? Using the Taylor expansion we can write the price of a bond after a small change in interest rate (di) as:

$$P(i+di) = P(i) + \frac{\partial P}{\partial i} di + \frac{1}{2}\frac{\partial^2 P}{\partial i^2} di^2 + \ldots$$

So to get the percentage change in bond price for a given percentage change in interest rates, we can rearrage the above to get:

$$\frac{P(i+di) - P(i)}{P(i)} = \frac{\partial P}{\partial i}\frac{1}{P(i)} di + \frac{1}{2}\frac{\partial^2 P}{\partial i^2}\frac{1}{P(i)} di^2 + \ldots$$

which can be written as:

$$\%\Delta P = -MD(\%\Delta i) + \frac{1}{2} Convexity(\%\Delta i)^2$$

The change in interest rate should really be the same interest rate method that you used to calculate duration, ie if you calculated duration using effective rates, express the change in interest rates as an effective rate.

Note however that the Taylor series approximation assumes a PARALLEL shift in yield curves, which is a strong assumption. Hence I would **never** recommend using modified duration or convexity as a real-life risk management method, and most sophisticated bond dealers do not use it. See Suleyman Basak's fixed interest & financial engineering courses for a good guide on what you **should** do in real life if you get a fixed interest job (also I think that financial engineering is the single best course at LBS).

What if we used an annual effective rate rather than a half-yearly interest rate to calculate the price and the modified duration?

The annual effective rate y that is equivalent to the half yearly interest rate i is defined as $1+y=(1+i)^2$. Note that $\frac{\partial y}{\partial i}=2(1+i)=2(1+y)^{-0.5}$.

Now the price in this case will be given as:

$$P=C(1+y)^{-0.5}+C(1+y)^{-1}+(P+C)(1+y)^{-1.5}$$

Differentiating gives:

$$\frac{\partial P}{\partial i}=\frac{\partial P}{\partial y}\frac{\partial y}{\partial i}$$

$$\frac{\partial P}{\partial i}=\left[-0.5C(1+y)^{-1.5}-1C(1+y)^{-2}-1.5(P+C)(1+y)^{-2.5}\right]2(1+y)^{-0.5}$$

Substituting $1+y=(1+i)^2$ gives the following:

$$\frac{\partial P}{\partial i}=\left[-1C(1+i)^{-2}-2C(1+i)^{-3}-3(P+C)(1+i)^{-4}\right]$$

$$\frac{\partial P}{\partial i}=-(1+i)^{-1}(1C(1+i)^{-1}+2C(1+i)^{-2}+3(P+C)(1+i)^{-3})$$

$$MD=(1+i)^{-1}D_{Macauley}$$

which is the same as before (as we hope it should be since the economics of what we are doing is identical, we are just using a different measure of interest rates).