# Pricing and Hedging Convertible Bonds Under Non-probabilistic Interest Rates

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Abstract

Two of the authors (DE and PW) recently introduced a non-probabilistic spot interest rate model. The key concepts in this model are the non-diffusive nature of the spot rate process and the uncertainty in the parameters. The model assumes the worst possible outcome for the spot rate path when pricing a fixed-income product. The model differs in many important ways from the traditional approaches of fully deterministic rates (as assumed when calculating yields, durations and convexities) or stochastic rates governed by a Brownian motion. In this new model, delta hedging and unique pricing play no role, nor does any market price of risk term appear. In this paper we apply the model to the pricing of convertible bonds. Later we show how to optimally hedge the interest rate risk; this hedging is not dynamic but static. We show how to solve the governing equation numerically and present results.

## 1. Introduction

The current state of interest rate modelling is unsatisfactory for a number of reasons. The two of particular concern are the estimation of parameters and the dynamic hedging of contracts. It is well known that parameter estimation in the fixed-income world is prone to all kinds of errors. Not only are parameters difficult to estimate but they are also notoriously unstable. One of the key parameters is the market price of interest rate risk. This can be interpreted, depending on your viewpoint, as either an elegant quantity that links the random behaviour of diverse products, or as a giant carpet under which to brush all your estimation problems. The role of correlation is also problematic. Almost all one-factor interest rate theories assume that a bond of one maturity can be successfully hedged with a bond of another maturity. A two- or more factor model will just require a few more bonds, but the result of perfect hedging is maintained. In practice the markets are not so helpful. Correlation is even more unstable than volatility and perfect dynamic hedging is not possible. This is known by all practitioners, but is not seen in the theory.

These difficulties with classical models are what led us in Epstein & Wilmott (1997, 1998) to introduce a non-stochastic, non-probabilistic, yet non-deterministic model for the spot rate that has no part for delta hedging or correlation. The model and simple extensions are described in the next section. The inspiration behind the model was the work of Avellaneda, Levy & Paras (1995) and Lyons (1995) on uncertain volatility in the equity derivatives world, and the work of Avellaneda & Paras (1996) on optimal static hedging, the Lagrangian uncertain volatility model. Although these works provided the inspiration, the ideas cannot be transferred immediately to the fixed-income world because the quantity we are modelling, the spot interest rate, is not a traded quantity.

The layout of this paper is as follows. In Section 2 we describe the model and its extensions as applied to simple zero-coupon bonds. In Section 3 we apply the model to pricing a convertible bond. We will show how to delta hedge with the asset, but leave the interest rate risk unhedged. In Section 4 we briefly explain how to solve the equation numerically, giving results in Section 5. In Section 6 we explain the concept of optimal static hedging and how it is used to improve the value of the convertible bond. In Section 7 we give results for the convertible bond after statically hedging the interest rate risk. In Section 8 we draw conclusions.

## 2. A non-probabilistic interest rate model

In place of a probabilistic description of the spot interest rate and a unique price for any fixed income product, we will simply disallow certain rate paths, make no statement about those paths that are permitted, and determine the path that gives a particular contract the worst value. The simplest form of the model for the spot rate $r$ is the following:

$$\begin{aligned} & r^- \le r \le r^+ \\ & c^- \le \frac{dr}{dt} \le c^+. \end{aligned} \tag{1}$$

In words, these state that the spot interest rate cannot leave the range ($r^-$,$r^+$) and cannot increase or decrease at a speed outside the range ($c^-$,$c^+$). This model does not capture the Brownian nature of the spot rate over short time scales. To accommodate this we make the following simple modification to the model. We introduce the new variable $r'$ which is to be constrained by (1) and let the real spot interest rate, still denoted by $r$, move within a prescribed distance of $r'$. That is

$$|r - r'| \le e.$$

A trajectory for $r$ is shown in Figure 1. It is possible to further generalise the model by allowing $r'$ to jump discretely from time to time. We omit the details of this ultimate model, referring the reader to Wilmott (1998) for details. We note, however, that it will be virtually impossible to dismiss this model by a statistical examination of data, something that cannot be said of any other interest rate model currently being used.

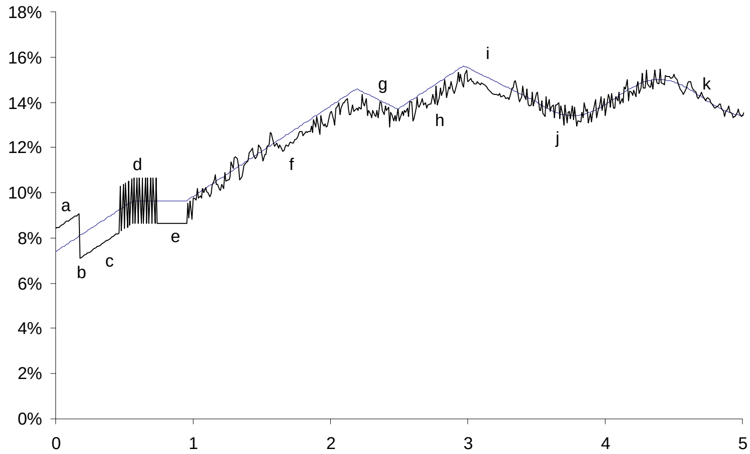


Figure 1: A possible evolution of the short-term interest rate.

Note that the model does not have any stochastic terms. The 'randomness' in the value of fixed-income products appears via the uncertainty in the path of the spot rate. The pricing equation for any non-path-dependent contract is

$$\frac{\partial V}{\partial t} + c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r} - \left(r + e(V)\right)V = 0 \tag{2}$$

where

$$c(x) = \begin{cases} c^- & x \geq 0 \\ c^+ & x < 0 \end{cases}$$

$$e(x) = \begin{cases} e & x \geq 0 \\ -e & x < 0 \end{cases} \qquad (3)$$

Here $V(r,t)$ is the value of the interest rate product when the spot rate is $r$ and time is $t$. This equation is derived in Epstein & Wilmott (1997, 1998) and Wilmott (1998) by assuming that the *worst-case* path for the spot rate is realised and that cashflows are discounted at the rate given by this path. The *best-case* price for the present value of the cashflows is given by the solution of the same equation but with the inequalities in (3) all reversed. That we can derive a worst and a best price is the explanation behind our earlier statement that we will not find a unique price for a product. In practice we would only be interested in the worst-case value, since this corresponds to a conservative pricing. Note that the best-case value for a product corresponds to the worst-case value for a product having cashflows of the opposite signs. That is, the best price for a long position is the same as the worst price for a short position. This is easily confirmed by changing the sign of $V$ in Equation (2).

The important points to note about this equation are that it is hyperbolic and nonlinear, unlike most partial differential equations in finance. The absence of diffusion makes the numerical solution slightly harder than the Black-Scholes equation (Black-Scholes, 1973). The nonlinearity is of crucial importance in making the model of practical use, playing the same role as in the uncertain volatility model explained in Avellaneda & Paras (1996). Because we are considering the worst-case value for a product we will necessarily find a value that is far below the market price and we would therefore never entertain the idea of buying it. However, because of the nonlinearity in the equation the value of a portfolio of contracts can be different from the sum of the values of the individual components. Thus by adding other traded contracts to the portfolio containing our original product we will alter the marginal value of the original. Moreover, this marginal value can be maximised and, in many cases, the original product becomes appealing. We will see this 'optimal static hedging' in action later.

## 3. Delta hedging and pricing a convertible bond

The convertible bond is like a coupon-bearing bond in that the holder receives coupon payments at specified periods. It also has equity characteristics since the holder can at specified times exchange the bond for a quantity of some asset, the underlying asset. This exchange is called conversion. When the asset value is low there is little reason to convert the bond and so it behaves like a simple, non-convertible, coupon-bearing bond. When the stock price is high the option to convert gives the bond a value closer to the value of the relevant quantity of the underlying asset. In some cases it is optimal to convert the bond before maturity. Mathematically the conversion of the bond is like the early exercise of an American option, and can be thought of as a free boundary problem, a variational problem, a stochastic control problem or an optimal stopping time problem depending on your preferred branch of mathematics. We will omit all the details since the conclusion is clear.

If the underlying asset $S$ satisfies

$$dS = mS\ dt + sS\ dX_1$$

then the pricing equation for the bond is

$$\frac{\partial V}{\partial t} + \frac{s^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r} + \left(r + e\left(V - S\frac{\partial V}{\partial S}\right)\right)\left(S\frac{\partial V}{\partial S} - V\right) - DS\frac{\partial V}{\partial S} = 0 \qquad (4)$$

where $V(S,r,t)$ is the bond price,

$$c(x) = \begin{cases} c^- & x \geq 0 \\ c^+ & x < 0 \end{cases} \qquad (5)$$

$$e(x) = \begin{cases} e & x \geq 0 \\ -e & x < 0 \end{cases}$$

and $D$ is the dividend yield on the asset, here assumed constant and continuously paid. Again, this is the worst possible present value for the contract. The best value, which won't concern us much, would be given by the solution of Equation (4) with reversal of all the inequalities in (5).

The derivation of the equation assumes that the random moves in the underlying asset are delta hedged away by holding $-\Delta$ of the underlying asset where

$$\Delta = \frac{\partial V}{\partial S}.$$

In contrast, the interest rate risk is not hedged, we are assuming the worst outcome are far as the spot rate path is concerned.

Optimal conversion into $n$ of the stock is assured by insisting that

$$V(S,r,t) \geq nS$$

at all times that conversion is permitted and that the delta of the bond with respect to the asset is everywhere continuous.

The final condition at maturity of the bond $T$ is that the bond value is equal to the principal, assumed to be 1, plus the last coupon

$$V(S,r,T) = 1 + c.$$

Across each coupon date $t_i$ the bond falls by the amount of the coupon. Thus

$$V(S,r,t_i^-) = V(S,r,t_i^+) + c.$$

This completes the specification of the convertible bond model under the risk-neutral measure for the asset and the worst-case scenario for the interest rate.

## 4. Numerical treatment of the pricing equation

Equation (4) can be solved using either a trinomial scheme (see Epstein & Wilmott (1998) for details) or by the more versatile explicit finite-difference method. Such methods are also well explained in the academic literature (see, for example, Wilmott 1998). The only points that we will make concern the discretizations used and the stability of the method.

All derivatives with respect to $S$ are approximated by central differences. Because the partial differential equation exhibits diffusion in the $S$ direction there will be no instability due to these discretizations provided the timestep is sufficiently small.

However, there is no diffusion in the $r$ direction. This is unusual for a financial partial differential equation with non-deterministic interest rates. Because there is only a single $r$ derivative in the equation, in the finite difference scheme this is approximated by a one-sided difference. This difference is chosen to be either a forward or a backward difference in order to satisfy the CFL condition (Morton & Mayers, 1994) to ensure stability. Moreover, the parameters $c^+$ and $c^-$ are chosen to be constant and such that $c^- = -c^+$, so that the grid spacing in the $r$ and $t$ directions corresponded to characteristic directions *exactly*. Therefore the only errors due to the $r$ discretization are caused by rounding errors and not by inaccuracy of the numerical scheme.

The early exercise condition of continuity of price and gradient with respect to $S$ are ensured by the simple cut-off condition of replacing the option value at each timestep with the maximum of the updated value and the value on conversion. This method is valid even when the equation is nonlinear of the form of Equation (4).

## 5. Preliminary results

We are now going to consider the pricing of a specific convertible bond using several popular models for interest rates. We will examine the robustness of the prices to variations in parameters. We then price the convertible bond using the above-explained non-probabilistic model.

The underlying asset has value 100, the volatility is 15% and the dividend yield is 4%. Note that in this paper we are not questioning the accuracy of the asset price parameters.

We value a convertible bond with a maturity of 25th November 2001 (where today is 14th May 1998). The bond has principal 1, can be converted into 0.01 of the asset and pays a coupon of 3% every six months until expiry. The spot rate is 7%.

The following numerical results have an asset step of 10 and an interest rate step (where appropriate) of 0.1%.

a) **Constant interest rate of 7%** In the first example the interest rate is a constant, there are no dynamics either deterministic or stochastic.

Convertible bond value: **1.131**

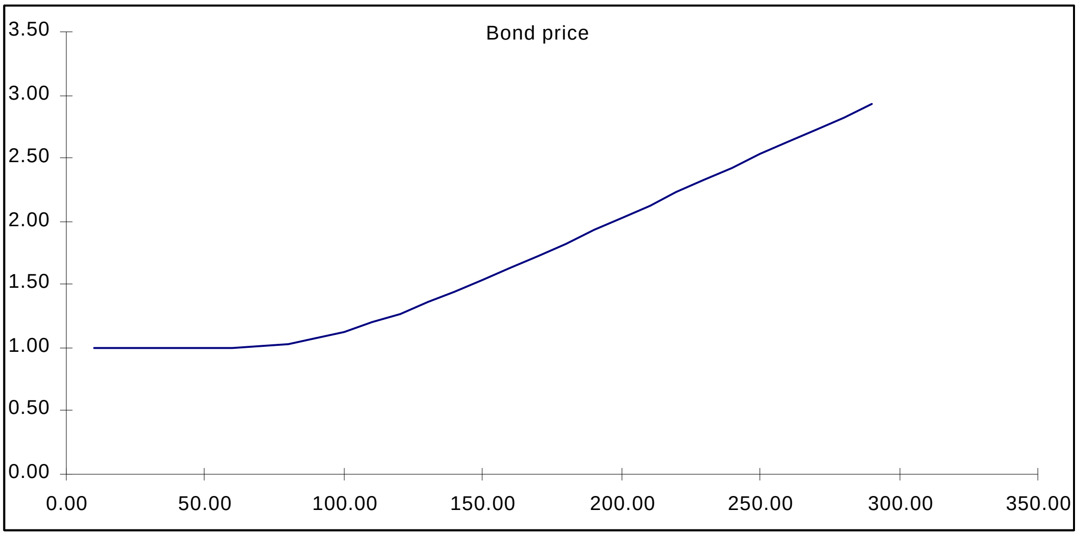


Figure 2: Convertible bond value with a constant interest rate

b) **Deterministic forward rate** given by a linear interpolation of rates determined from the following zero-coupon bond yield curve:

| Maturity | Yield |
|---|---|
| 1 m | 0.07000 |
| 6 m | 0.07447 |
| 1 yr | 0.07016 |
| 2 yr | 0.06631 |
| 5 yr | 0.06224 |
| 7 yr | 0.06121 |
| 10 yr | 0.06037 |
| 30 yr | 0.05990 |

Table 1

Convertible bond value **1.147**

Sensitivity to parallel shifts in the yield curve:

| Yield curve shift | Convertible bond value |
|---|---|
| +2% | 1.118 |
| +1% | 1.132 |
| 0% | 1.147 |
| -1% | 1.165 |
| -2% | 1.185 |

Table 2

c) **The Vasicek model** of the form

$$dr = (a - br)dt + u\ dX_2$$

with a=0.007, b=0.1, ν=0.02 and a correlation of 0.1.

Convertible bond value **1.137**

Sensitivity to *a*, *b* and ν:

| a | Convertible bond value |
|---|---|
| 0.009 | 1.132 |
| 0.008 | 1.134 |
| 0.007 | 1.137 |
| 0.006 | 1.139 |
| 0.005 | 1.141 |

Table 3

| b | Convertible bond value |
|---|---|
| 0.12 | 1.139 |
| 0.11 | 1.138 |
| 0.10 | 1.137 |
| 0.09 | 1.135 |
| 0.08 | 1.134 |

Table 4

| ν | Convertible bond value |
|---|---|
| 0.030 | 1.142 |
| 0.025 | 1.139 |
| 0.020 | 1.137 |
| 0.015 | 1.135 |
| 0.010 | 1.133 |

Table 5

d) **Vasicek model fitted to the yield curve** of Table 1, with b=0.1, ν=0.02 and a correlation of 0.1.

Convertible bond value **1.161**

Sensitivity to *b* and ν:

| b | Convertible bond value |
|---|---|
| 0.12 | 1.164 |
| 0.11 | 1.163 |
| 0.10 | 1.161 |
| 0.09 | 1.160 |
| 0.08 | 1.158 |

Table 6

| ν | Convertible bond value |
|---|---|
| 0.030 | 1.166 |
| 0.025 | 1.163 |
| 0.020 | 1.161 |
| 0.015 | 1.159 |
| 0.010 | 1.157 |

Table 7

e) **ACKW model**

The Apabhai, Choe, Khennach, Wilmott (1995) one-factor model is based on a statistical analysis of the US spot interest rate and the yield curve slope at the short end over 20 years. The model fits the average dynamics of the yield curve.

Convertible bond value **1.113**

f) **Epstein-Wilmott model** We choose

$$0.03 \le r \le 0.2$$

$$-0.04 \le \frac{dr}{dt} \le 0.04.$$

Thus the spot interest rate is not allowed outside the range 3 to 20%, nor to grow or decay faster than 4% pa. These numbers are chosen to be realistic for US short-term interest rates. For this example we take ε to be zero.

We find that the worst price attainable is **1.072** and the best price **1.191**.

A sensitivity analysis is not relevant for this model.

## 6. Optimal static hedging

The results of the previous section are interesting in showing the magnitude of uncertainty in the value of the convertible bond when there is some doubt about the value of parameters. There will always be such doubt in financial modelling and this must be faced when pricing. The Epstein-Wilmott interest rate model builds uncertainty into the pricing from the beginning. The worst-best price range for the convertible bond is valid provided interest rates do not violate the various bands that we have imposed. Again, we stress that the values for the bands have been chosen to be realistic. Nevertheless, the range predicted by the model is too great to be of practical use; faced with such a range no-one would buy or sell such a bond.

The Epstein-Wilmott model in its basic form is not of much use. The model only becomes practical when combined with a static hedge. Recall that we are delta hedging the stock price risk but have so far left the interest rate risk unhedged. This will now be remedied, but not by any form of dynamic hedging.

The idea that we shall now incorporate was first implemented by Avellaneda & Paras (1996) in their uncertain volatility model but is applicable in any situation where the governing equation is nonlinear, such as is the case here. In words, the nonlinearity in the pricing equation means that the value of a portfolio of contracts is not necessarily the same as the sum of the values of the individual components. To the authors' knowledge the Epstein-Wilmott model is the only simple nonlinear pricing model for interest rate products.

The static hedging of the convertible bond works as follows. We shall construct a portfolio consisting of the original convertible bond and some zero-coupon bonds. We use zero-coupon bonds just for illustration but the same principle can be used to hedge with coupon-bearing bonds or even swaps, caps and floors, with little extra effort. The important point about the static hedging is that the hedging contracts are traded, meaning that they can be bought (or sold) for an amount that is known from the market.

By adding a quantity $q$, say, of a zero-coupon bond with maturity $T_Z$ to the portfolio we affect the pricing problem by adding the jump condition

$$V(S,r,T_Z-) = V(S,r,T_Z+) + q,$$

assuming that the principal of the bond is 1. If $q$ is positive the combined value of the convertible bond and the zero-coupon bond will be greater than the value of the convertible bond alone. The *marginal* value of the convertible bond is therefore

$$V(S_0, r_0, t_0) - qZ_0$$

where '0' denotes the values of parameters today, the time of pricing, and $Z_0$ is the market price of the zero-coupon bond. Having found the marginal value of the convertible bond when hedged with a quantity $q$ of a market-traded zero-coupon bond it is natural to ask how to choose $q$. The obvious choice is that which gives the convertible bond the highest marginal value, hence the phrase *optimal* static hedging. The idea is easily extended to hedging with bonds of any maturities and this is illustrated in the example of the next section.

## 7. Results after hedging

Apart from the Epstein-Wilmott model, all of the models used in Section 5 are linear. The prices derived for the convertible bond are therefore not affected by the market prices of zero-coupon bonds except insofar as these prices are used to calibrate some of the models. The fitting of the current yield curve by this calibration contrasts greatly with the 'fitting' of the Epstein-Wilmott model. In the classical models we are in effect saying that we believe the market prices to be correct and that our model can only be correct if the outputs match the market. In the nonlinear Epstein-Wilmott model we are more realistically saying nothing about the validity of the market prices, only that the traded bonds may be useful in our portfolio to eliminate some risk due to uncertainty in the spot rate 'process.' In the extreme case that we price a traded contract by our model we would inevitably hedge it one-for-one, eliminating all model dependence and finding a theoretical price exactly the same as the market price. No fitting in the classical sense is required. This important property of nonlinear pricing equations is discussed fully in Avellaneda & Paras (1996), Epstein & Wilmott (1997, 1998) and Wilmott (1998).

The universe of hedging bonds is the same as that used in Section 5 when we fitted classical models. The yield curve for these bonds is shown in Table 1.

We hedge with the 6 month and the 1, 2 and 5 year zero-coupon bonds. The optimal worst-case hedges for these bonds are:

| Bond Maturity | Bond Yield | Worst-case hedge |
|---|---|---|
| 6m | 0.07447 | 0.283 |
| 1yr | 0.07016 | 0.053 |
| 2yr | 0.06631 | -0.421 |
| 5yr | 0.06224 | -0.290 |

Table 8

With these hedges in place, we find that the new worst-case price is **1.112** (unhedged was 1.072) and the best-case price is **1.193** (was 1.191).

## 8. Conclusions

The present value of all the coupons and the principal, valued off an interpolated yield curve, is approximately 1.026. The added value due to the convertability in the bond is thus the difference between the bond value and the present value of all the coupons plus principal. The bond values and the added value according to each of the models are summarized in the following table.

| Model | CB Value | Added Value due to Conversion |
|---|---|---|
| Constant interest rate | 1.131 | 0.105 |
| Deterministic yield curve | 1.147 | 0.121 |
| Vasicek, unfitted | 1.137 | 0.111 |
| Vasicek, fitted | 1.161 | 0.135 |
| ACKW | 1.113 | 0.087 |
| EW, worst case, unhedged | 1.072 | 0.046 |
| EW, best case, unhedged | 1.191 | 0.165 |
| **EW, worst case, optimally hedged** | **1.112** | **0.086** |
| **EW, best case, optimally hedged** | **1.193** | **0.167** |

Table 9

We have applied a nonlinear, non-probabilistic interest rate model to the pricing of a convertible bond. The resulting worst-case scenario valuation produced results which were far lower in price than those found using typical interest rate models. Moreover, unlike the worst-case scenario approach, these models were found to be quite sensitive to their parameters, the values of which can often be uncertain.

The theoretical prices and particularly the added value are significantly different across different models. The optimally hedged worst-case added value is 0.086 which should be seen as a benchmark against which to compare the other values. The historically accurate ACKW gives a value close to this worst case (0.087) and the other models are considerably higher. It is particularly interesting to note that in this example the more ‘fitting’ that is done the higher the price, and the more empirically justified the model (ACKW) the lower the price. It would seem that the less ‘fudging’ the more conservative the price. Of course, this would be reversed for a yield curve sloping the other direction.

If we believe that the Epstein-Wilmott model gives a conservative price range then we would hope that other models give prices within this range. The ability to find definitive bounds for the value of the convertible bond (through the Epstein-Wilmott model) may prove invaluable in the task of validating some of the more complex stochastic models.

Through the process of static hedging, we were able to significantly increase the worst-case scenario price for the convertible bond. In addition, the process found an optimal static hedge. This hedge could be applied to a convertible bond portfolio to reduce the inherent interest rate risk, regardless of which model were chosen to price the resulting portfolio.

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