Fundamental Properties of Bond Prices in Models of the Short-Term Rate

Antonio Mele
London School of Economics

This article develops restrictions that arbitrage-constrained bond prices impose on the short-term rate process in order to be consistent with given dynamic properties of the term structure of interest rates. The central focus is the relationship between bond prices and the short-term rate volatility. In both scalar and multidimensional diffusion settings, typical relationships between bond prices and volatility are generated by joint restrictions on the risk-neutralized drift functions of the state variables and convexity of bond prices with respect to the short-term rate. The theory is illustrated by several examples and is partially extended to accommodate the occurrence of jumps and default.

A standard approach to modeling the term structure of interest rates is to derive sets of arbitrage-free bond prices using as an input an exogenously given short-term rate process [see, e.g., Vasicek (1977) and Cox, Ingersoll, and Ross (1985)]. While this approach is widely used, there is no theoretically sound work establishing systematic answers to such fundamental questions as: When are bond prices a decreasing function of the short-term rate? When are bond prices a convex function of the short-term rate? Are bond prices a decreasing function of the short-term rate volatility? This article demonstrates that it is possible to develop results addressing these questions in relation to all sets of economically admissible (i.e., no-arbitrage) bond prices.

The key characteristic of the theory developed in this article is its generality. As in the seminal contributions of Merton (1973), Cox and Ross (1976) and Jagannathan (1984) in the option literature, I derive general properties of bond prices while imposing as few assumptions as possible on the state processes driving the economy. Related articles that

I dedicate this work to the memory of my beloved father Ettore, who passed away when I started to produce its latest revisions. The first draft of the article was written when I was a visiting fellow at the Economics Department of Princeton University (Spring 2000). I thank Yacine Aït-Sahalia and José Scheinkman for stimulating conversations and helpful suggestions. I am also grateful to Kyriakos Chourdakis, Alan Lewis, and Jaouad Sahbani for valuable remarks made on early versions of the manuscript, and seminar participants at the University of Exeter and at the 2002 American Finance Association meeting, especially Francis Longstaff, for useful comments. I benefited considerably from the detailed comments of an anonymous referee, Kenneth Singleton (the editor), and Maureen O'Hara (the executive editor), particularly from their suggestions on how to improve the presentation and how to consolidate the analysis of multifactor models and jumps phenomena. This work was partly supported by a Fulbright research fellowship that I gratefully acknowledge. The usual disclaimer applies. Address correspondence to Antonio Mele, London School of Economics, Houghton Street, London WC2A 2AE, UK, or e-mail: a.mele@lse.ac.uk.

The Review of Financial Studies Fall 2003 Vol. 16, No. 3, pp. 679–716, DOI: 10.1093/rfs/hhg011

also derive general properties of bond prices while imposing minimal assumptions on the primitive state process dynamics are Dybvig, Ingersoll and Ross (1996) and Dunn and Spatt (1999), but the results of the present article in the term structure domain are totally new.

The class of stochastic processes typically used to analyze general properties of option prices is the one of diffusion processes [see, e.g., Bajeux-Besnainou and Rochet (1996, sect. 5), Bergman, Grandy, and Wiener (1996), Romano and Touzi (1997), and El Karoui, Jeanblanc-Picqué and Shreve (1998)]. As is well-known, diffusion processes are also the workhorse in the term-structure field. Especially in the last decade, many new diffusion models of the short-term rate have been proposed. These models aim to achieve two related objectives. First, they aim to provide increasingly accurate empirical explanations of the short-term rate dynamics [see, e.g., Chan et al. (1992), Aït-Sahalia (1996b), Andersen and Lund (1997), Conley et al. (1997)]. Second, they aim to explore the term structure implications of empirically interesting short-term rate dynamics [see, e.g., Longstaff and Schwartz (1992), Aït-Sahalia (1996a), Andersen and Lund (1997), Stanton (1997), Boudoukh et al. (1998), Jiang (1998), Dai and Singleton (2000)]. Because the data-generating process is inherently difficult to detect empirically, an additional useful approach is to classify theoretical properties of bond prices according to the general qualitative features of the primitive state process dynamics. This is exactly the approach I develop here.

In the present article, the only assumption made is that the primitive state processes are diffusion processes satisfying some basic regularity conditions. Precisely, in the framework analyzed here, the short-term rate and its instantaneous *stochastic* volatility form a joint Markov process.[1] As the title suggests, the main concern is to develop "fundamental" results relating arbitrage-free bond price movements to changes in the short-term rate and its instantaneous volatility. This article does not consider non-Markovian settings [e.g., Heath, Jarrow, and Morton (1992)]. However, extending the approach followed here to more general situations is a promising area for future research. A step in that direction is made in Section 4, where it is shown how easily this article's approach may be extended to treat multidimensional models including unobserved factors with no immediate economic interpretation.

A major insight derived in this article is that the bond price reaction to random changes of the state variables can be represented through a series of joint restrictions on both the risk-neutralized drift function of the state

[1] Surveys on continuous-time stochastic volatility option pricing models can be found in Lewis (2000) or Fouque, Papanicolaou, and Sircar (2000). Recent work on specification, estimation, and filtering methods applied to stochastic volatility models for the short-term rate include Dai and Singleton (2000) and Mele and Fornari (2000).

variables and convexity of bond prices with respect to the short-term rate. The simplest illustration of this phenomenon is provided by Proposition 1, which shows that bond prices are convex in the short-term rate under a bound on convexity of the short-term rate risk-neutralized drift [see Section 3 and Equations (8) and (9)). Another important illustration regards the relationship between stochastic volatility and bond prices:[2]

(A) If the drift function of the short-term rate process under the risk-neutral measure is strictly decreasing in volatility, and bond prices are decreasing and convex in the short-term rate, then bond prices are positively related to random volatility changes [see Section 3, Equation (12)];

(B) At short maturity dates, bond prices are negatively (positively) related to random volatility changes if the drift function of the short-term rate process under the risk-neutral measure is strictly increasing (decreasing) in volatility (see Section 3, Proposition 3).

While property (B) is intuitive, properties (A) and (B) taken together are rather different from properties known in the option pricing field. In the stochastic volatility option pricing domain, for instance, convexity of option prices with respect to the underlying asset price is a sufficient condition for option prices to be increasing in volatility [see Romano and Touzi (1997)]. Section 2 then provides a heuristic explanation of property (B), and Sections 3 and 4 contain conditions and examples for bond prices and volatility to be negatively related at any arbitrary maturity date, thus strengthening property (B).

The relationship between bond prices and volatility also helps to explain the origins of given relationships between bond prices and the short-term rate. Even in a two-factor setting, bond prices may not be negatively related to short-term rate movements. Propositions 1 and 2 show that such a classical inverse relationship holds when the risk-neutralized drift and the diffusion functions of volatility are independent of the short-term rate (and/or in the very intuitive case of short-term maturity dates). If this is not the case, the relationship between medium-long-term bond prices and the short-term rate is further qualified according to the relationship between bond prices and volatility (see Section 3).

In a three-factor framework, the bond price behavior is essentially of the same nature as the one described above (see Section 4, Propositions 4 and 5). Within such a more general setup, however, given relationships between factors and bond prices are described by means of more complex

[2] For previous numerical exercises aiming at unveiling the relationship between bond prices and volatility, see Litterman, Scheinkman and Weiss (1991), Chen (1996), Andersen and Lund (1997), and Mele and Fornari (2000, chap. 5).

conditions on the joint factor dynamics. Finally, the previous analysis is robust to the introduction of jumps phenomena and the possibility of default as long as the parameters of the jump size distribution and the various hazard rates are "sufficiently independent" of the level of the state variables (see Section 5, Proposition 6).

Throughout the article, a number of models are examined to illustrate specific aspects of the theory. For instance, the affine Longstaff and Schwartz (1992) (two-factor) model and two of the affine (three-factor) models of Dai and Singleton (2000) display in an exemplary way the main properties of bond prices deduced by the theory. In the setting of these models, the main features of the theory can be described by means of algebraic formulae and analytical theoretical test conditions. In turn, such a description sheds new light on these affine models. Affine models such as the familiar model examples of this article, are certainly not entirely exhaustive. Yet similar to the models that Lewis (2000, p. 4) suggested to be relevant in the option pricing area, affine models appear to be economically important in the term-structure domain. Not only are these models easy to solve, but as shown in this article, their solution is also "typical", that is it displays many interesting qualitative properties that we expect in more general (nonlinear) settings.

The article is organized as follows. The next section introduces the model's primitives. Section 2 provides a selected heuristic overview of the article's results. Section 3 contains the central analysis of the article. Section 4 contains extensions pertaining to three-factor models. Section 5 deals with cases in which the state variables exhibit discontinuities. Section 6 concludes. Four appendices gather proofs, examples, and results omitted in the main text.

1. The Model

I consider a model in which the short-term rate and its instantaneous volatility form together a sufficient statistic for the state variables generating uncertainty in the economy. Such a model is important to study because it displays in a transparent way many interesting qualitative properties of more complex three-factor models. The reader may refer to Section 4 to learn additional properties that are specific to three-factor models.

I begin by describing the risk-neutral measure space under which the discounted bond prices are Q-martingales. This is $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F} = \{\mathcal{F}(\tau)\}_{\tau \in [t,T]}$ is taken to be the Q-augmentation of the natural filtration $\sigma((W, B)(u), u \leq \tau)$ generated by two independent Q-Brownian motions W, B (with $\mathcal{F} = \mathcal{F}(T)$ and $T < \infty$). I suppose that the short-term rate r and volatility y are diffusion processes, that is, Markov processes with continuous sample paths [see, e.g., Karlin and Taylor (1981, p. 157)]. They form a strong solution to the following stochastic differential system:

$$\begin{pmatrix} dr(\tau) \\ dy(\tau) \end{pmatrix} = \begin{pmatrix} b(r(\tau), y(\tau)) \\ \varphi(r(\tau), y(\tau)) \end{pmatrix} d\tau + \begin{pmatrix} \sigma^{(1)}(r(\tau), y(\tau)) & \sigma^{(2)}(r(\tau), y(\tau)) \\ \psi^{(1)}(r(\tau), y(\tau)) & \psi^{(2)}(r(\tau), y(\tau)) \end{pmatrix} \begin{pmatrix} dW(\tau) \\ dB(\tau) \end{pmatrix}, \quad (1)$$

for $\tau \in (t, T]$, where (r, y) take values in $\mathbb{R}_{++}$ and $\mathbb{R}$, $(r(t), y(t)) \equiv (x, s)$, and b, $\sigma^{(j)}$, φ and $\psi^{(j)}$ are functions satisfying regularity conditions ensuring the existence of a strong solution [e.g., definition 2.1, Karatzas and Shreve (1991, p. 285)] to the preceding system. Let $\sigma(r, y) \equiv \sum_{j=1}^{2} \sigma^{(j)} (r, y)^2/2$ and $\psi(r, y) \equiv \sum_{j=1}^{2} \psi^{(j)}(r, y)^2/2$. I suppose that $\partial\sigma/\partial y \geq 0$ and $\partial\sigma/\partial r \geq 0$.

To relate the drift functions in Equation (1) to the corresponding drift functions defined under the physical measure space, recall the well-known result stating that in the absence of arbitrage opportunities, there exist functions Λ^1 and Λ^2 such that the drift functions in Equation (1) can be written as

$$\begin{cases} b(r, y) = \hat{b}(r, y) + \sum_{j=1}^{2} \sigma^{(j)}(r, y) \cdot \Lambda^j(r, y) \\ \varphi(r, y) = \hat{\varphi}(r, y) + \sum_{j=1}^{2} \psi^{(j)}(r, y) \cdot \Lambda^j(r, y), \end{cases} \quad (2)$$

where $\hat{b}(\cdot, \cdot)$ and $\hat{\varphi}(\cdot, \cdot)$ denote the drift functions under the physical measure space, and $\Lambda = (\Lambda(\tau) \equiv (\Lambda^1(r(\tau), y(\tau)), \Lambda^2(r(\tau), y(\tau)))_{\tau \in [t,T]}$ is a $\mathcal{F}(\tau)$-adapted process that satisfies standard regularity conditions.[3] Under a boundedness condition given in note 3 on Λ, and the condition that for all $(r, y) \in \mathbb{R}_{++} \times \mathbb{R}$: $\sigma^{(1)}(r, y) = \sigma^{(2)}(r, y) = \psi^{(1)}(r, y) = \psi^{(2)}(r, y) = 0, \sum_{j=1}^{2} \sigma^{(j)}(r, y)\Lambda^{(j)}(r, y) = \sum_{j=1}^{2} \psi^{(j)}(r, y)\Lambda^{(j)}(r, y) = 0$, any otherwise arbitrary functional form of Λ^i will prevent arbitrage opportunities similar to those first discussed by Cox, Ingeroll and Ross (1985, sect. 5; henceforth CIR).

Naturally the fact that the technical starting point in Equation (1) is the risk-neutral measure does not imply that only a risk-neutral world is being considered here. As is well known, the economic interpretation of Λ^1 and Λ^2 is the one of risk premia demanded by agents to be compensated for the stochastic fluctuations of the two Brownian motions under the physical

[3] Formally, physical measure P (say) and measure Q are equivalent measures with Radon–Nikodym derivative of Q with respect to P on $\mathcal{F}(T)$ given by $dQ/dP = \exp(\int_t^T \Lambda(r(\tau), y(\tau)) dU(\tau) - \int_t^T \|\Lambda(r(\tau), y(\tau))\|^2 d\tau/2)$, where $U = (W, B)^{\mathrm{T}}$ and $\Lambda\ (r(\tau), y(\tau))_{\tau \in [t,T]}$ is $\mathcal{F}(\tau)$ adapted and satisfies the Novikov's condition [Karatzas and Shreve (1991, corollary 5.13, p. 199)]: $E\{\exp(\int_t^T \|\Lambda(r(\tau), y(\tau))\|^2 d\tau/2)\} < \infty$, where $E\cdot$ denotes expectation under measure P.

measure, and Λ^i are both nil when agents are risk neutral. Equations (2) thus summarize the mapping between the fundamentals (law of motion of the state variables and models of risk aversion) and the risk-neutral drifts in Equation (1). Consequently all results in this article impose joint restrictions on both the law of motion of the state variables under the physical measure and models of risk aversion [see, e.g., Equations (13), (18), and (19)].[4]

Let $u(x, s, t, T)$ denote the rational price of a pure discount bond expiring at $T \geq t$ when the short-term rate and its instantaneous volatility are (x, s) at time t. (Only pure discount bonds are considered in this article.) In a frictionless economy without arbitrage opportunities, u is the solution to the following partial differential equation:

$$\begin{cases} 0 = \left(\frac{\partial}{\partial \tau} + L - r\right) u(r, y, \tau, T), & \forall (r, y, \tau) \in \mathbb{R}_{++} \times \mathbb{R} \times [t, T) \\ u(r, y, T, T) = 1, & \forall (r, y) \in \mathbb{R}_{++} \times \mathbb{R}, \end{cases} \tag{3}$$

where $\partial \cdot / \partial \tau + L \cdot$ is the infinitesimal generator of Equation (1), with

$$Lu = bu_1 + \varphi u_2 + \sigma u_{11} + \psi u_{22} + (\sigma^{(1)} \psi^{(1)} + \sigma^{(2)} \psi^{(2)}) u_{12},$$

where $u_1 \equiv \partial u / \partial r$, $u_{11} \equiv \partial^2 u / \partial r^2$, $u_2 \equiv \partial u / \partial y$, and so on. No transversality conditions are imposed here.[5] Throughout this article, it will also be assumed that the coefficients of the infinitesimal generator of Equation (1) are such that the bond price and its partial derivatives can be computed via the celebrated Feynman–Kac representation theorem [e.g., Karatzas and Shreve (1991, p. 366)]. Regularity conditions ensuring the feasibility of such a representation as well as related regularity conditions are spelled out in Mele (2002, appendices A, B, and C).

Let $\vartheta(\tau) \equiv \vartheta(\tau, z; \omega)$, $\tau \in [t, T]$ denote the solution flow of the first stochastic differential in Equation (1) at τ starting at $z \equiv (x, s)$ at the point $\omega \in \Omega$. Under the regularity conditions mentioned before, there is a unique $C^{2,1}(\mathbb{R}_{++} \times \mathbb{R}, [0, T])$ solution to Equation (3) that admits the following Feynman–Kac stochastic representation:

$$u(x, s, t, T) = \mathbb{E}\left\{ \exp\left(- \int_t^T \vartheta(\tau) d\tau \right) \right\}, \tag{4}$$

where $\mathbb{E}$ is the expectation operator taken under measure Q.

By differentiating the bond price function in Equation (4),

$$u_1(x, s, t, T) = -\mathbb{E}\left\{ \left(\int_t^T \frac{\partial \vartheta}{\partial x}(\tau) d\tau \right) \exp\left(- \int_t^T \vartheta(\tau) d\tau \right) \right\}, \tag{5}$$

---

[4] Recent empirical studies focusing on the estimation of objects defined under the physical measure as well as the risk-neutral measure include Dai and Singleton (2000) and Mele and Fornari (2000, chap. 5) in the term-structure domain; and Chernov and Ghysles (2000) and Mele and Fornari (2001) in the option pricing field.

[5] If 0 and ∞ (respectively $-\infty$ and ∞) are inaccessible in finite expected time for r (y), the behavior of u as r and/or y approach their inaccessible boundaries is implicitly determined by Equation (3).

and

$$u_{11}(x,s,t,T) = \mathbb{E}\left\{\left[\left(\int_t^T \frac{\partial\vartheta}{\partial x}(\tau)d\tau\right)^2 - \int_t^T \frac{\partial^2\vartheta}{\partial x^2}(\tau)d\tau\right] \times \exp\left(-\int_t^T \vartheta(\tau)d\tau\right)\right\}, \tag{6}$$

where the "sensitivity processes" $\partial\vartheta/\partial x$, $\partial\vartheta/\partial s$, etc., are taken to share the same "diffusion" properties as Equation (1) and in particular are such that $\exists T_+ > t : \mathbb{E}\{\sup_T |\int_t^T (\partial\vartheta/\partial x)(\tau)d\tau|\} < \infty$ for all $T < T_+$ (the same condition being satisfied by the other sensitivity processes). These last conditions are very mild and will be used to study the local properties of Equation (4).[6]

Equations (5) and (6) are convenient ways to represent partials of the bond price with respect to the short-term rate. Together with Equations (3) and (4), these basic equations form the starting point of the analysis in this article.

2. How do Bond Prices React to Random Volatility Changes?

A binomial example may illustrate very simply some aspects of the relationship between bond prices and volatility. Consider a risk-neutral tree in which the next period interest rate is either $i^+ = i + d$ or $i^+ = i - d$ with equal probability, where i is the current interest rate level and $d > 0$. The price of a two-period bond is $u(i, d) = m(i, d)/(1+i)$, where $m(i, d) = E\{1/(1+i^+)\}$ is the expected discount factor of the next period. By Jensen's inequality, $m(i, d) > 1/(1+E\{i^+\}) = 1/(1+i) = m(i, 0)$. Therefore two-period bond prices increase upon activation of randomness. More generally, two-period bond prices are always increasing in the "volatility" parameter d in this example (see Figure 1).

This phenomenon can be connected with an important result derived by Jagannathan (1984, pp. 429–430) in the option pricing area. The Jagannathan's insight was that in a two-period economy with identical initial underlying asset prices, a terminal underlying asset price $\tilde{y}$ is a mean-preserving spread of another terminal underlying asset price $\tilde{x}$ [in the Rothschild and Stiglitz (1970) sense] if and only if the price of a call

[6] Additional technical conditions that do not have an immediate economic interpretation are that $\forall (x,s) \in \mathbb{R}_{++} \times \mathbb{R}$, $\exists T_+ > t : \mathbb{E}\{\sup_{T\in[\tau,T_+]} |(\partial\vartheta(T)/\partial x) - \vartheta(T)\int_\tau^T (\partial\vartheta(u)/\partial x)du|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |b_{(3),11}(\tau)(\partial\vartheta(T)/\partial x)|$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |b_{(3),11}(\tau)\int_\tau^T (\partial\vartheta(u)/\partial x)du|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |\varphi_{(2),1}(\tau)((\int_\tau^T (\partial\vartheta(u)/\partial s)du|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |\psi_{(2),1}(\tau)((\int_\tau^T (\partial\vartheta(u)/\partial s)du)^2 - \int_\tau^T (\partial^2\vartheta(u)/\partial s^2)du)|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |\sigma_{(1),2}(\tau)((\int_\tau^T (\partial\vartheta(u)/\partial x) du)^2 - \int_\tau^T (\partial^2\vartheta(u)/\partial x^2)du)|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |\sigma_{(1),2}(\tau)(\partial\vartheta(T)/\partial x)\int_\tau^T (\partial\vartheta(u)/\partial x)du|\}$, $\mathbb{E}\{\sup_{T\in[\tau,T_+]} |\sigma_{(1),2}(\tau) (\partial^2 \vartheta(T)/\partial x^2)|\}$ exist for all $\tau \in [t, T_+]$ (with $b_{(1),2}$ and $\sigma_{(1),2}$ also satisfying the same conditions satisfied by $b_{(3),11}$) (see Appendix A for the notation). These conditions are also very mild and are required to use both the Fubini–Tonelli and the Lebesgue's dominated convergence theorems to study the local behavior of the pricing function (4) [see, for instance, Equation (B4) in Appendix B].

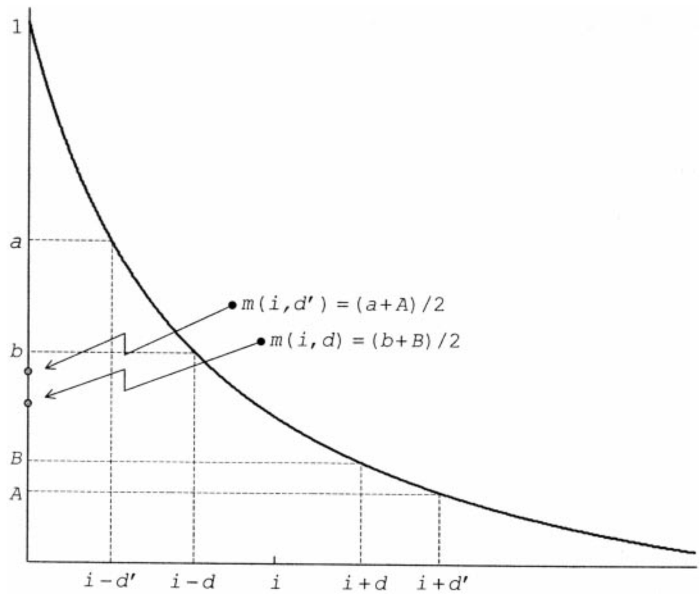

Figure 1
A connection with the Rothschild–Stiglitz–Jagannathan theory: the simple case in which convexity of the discount factor induces bond prices to be increasing in volatility
If the risk-neutralized interest rate of the next period is either $i^+ = i + d$ or $i^+ = i - d$ with equal probability, the random discount factor $1/(1+i^+)$ is either B or b with equal probability. Hence $m(i, d) = E\{1/(1+i^+)\}$ is the midpoint of $\overline{bB}$. Similarly if volatility is $d' > d$, $m(i, d')$ is the midpoint of $\overline{aA}$. Since $\overline{ab} > \overline{BA}$, it follows that $m(i, dt') > m(i, d)$. Therefore the two-period bond price $u(i, v) = m(i, v)/(1+i)$ satisfies $u(i, d') > u(i, d)$ for $d' > d$.

option on $\tilde{y}$ is higher than the price of a call option on $\tilde{x}$. This is so because if $\tilde{y}$ is a mean-preserving spread of $\tilde{x}$, then $E(f(\tilde{y})) > E(f(\tilde{x}))$ for f increasing and convex.[7]

An argument such as the one illustrated in Figure 1 is thus theoretically appealing, yet it depends too much on the assumption that the expected short-term rate is independent of d. Consider indeed a multiplicative setting in which either $i^+ = i(1+d)$ or $i^+ = i/(1+d)$ with equal probability. Litterman, Scheinkman, and Weiss (1991) showed that in such a setting, bond prices are decreasing in volatility at short maturity dates and increasing in volatility at long maturity dates. This is so because expected future interest rates increase over time at a strength positively related to d. At short maturity dates, such an effect dominates the convexity effect

[7] To make such a connection more transparent in terms of the Rothschild and Stiglitz (1970) theory, let $\tilde{m}_d(i^+) = 1/(1+i^+)$ denote the random discount factor when $i^+ = i \mp d$. Clearly $x \to -\tilde{m}_d(x)$ is increasing and concave, and so we must have $E(-\tilde{m}_{d''}(x)) < E(-\tilde{m}_{d'}(x)) \Leftrightarrow d' < d''$, which is what demonstrated in Figure 1. In Jagannathan (1984), f is increasing and convex, and so we must have $E(\tilde{f}(y)) > E(f(\tilde{x})) \Rightarrow \tilde{y}$ is riskier than (or a mean-preserving spread of) $\tilde{x}$.

illustrated in Figure 1. At longer maturity dates, the convexity effect dominates.

This article provides a theoretically sound explanation of the previous and related phenomena. Consider for example the two-factor *stochastic volatility* model of the previous section, and differentiate Equation (3) with respect to volatility (y). The result is that the first partial of the bond price with respect to volatility, $u_2(r, y, \tau, T)$, is the solution to the following partial differential equation:

$$\begin{cases} 0=\left(\frac{\partial}{\partial\tau}+L^1-k^1\right)u_2(r,y,\tau,T)+\{b_2(r,y)u_1(r,y,\tau,T) \\ \quad +\sigma_2(r,y)u_{11}(r,y,\tau,T)\}, \quad \forall(r,y,\tau)\in\mathbb{R}_{++}\times\mathbb{R}\times[t,T) \\ u_2(r,y,T,T)=0, \qquad \forall(r,y)\in\mathbb{R}_{++}\times\mathbb{R} \end{cases} \tag{7}$$

where L^1 and k^1 play the same role as L and r in Equation (3) (see Appendix A for the precise definitions of L^1 and k^1). By an application of the *maximum principle* [e.g., Friedman (1975)],[8] u_2 is then *always* positive under the assumption that $b_2u_1+\sigma_2u_{11}$ is positive for each $(r, y, \tau)\in\mathbb{R}_{++}\times\mathbb{R}\times[t, T)$, thus confirming property (A) stated in the introduction.

Arguably, assuming that $b_2u_1+\sigma_2u_{11}>0$ is restrictive. For example, Proposition 2 shows that $u_1<0$ at short maturity dates. When $u_{11}>0$, u_2 is then positive only with $b_2<0$ at short-maturity dates. This conclusion is correct even if $u_{11}<0$. Indeed, the sign of $b_2u_1+\sigma_2u_{11}$ is the result of a conflict between slope (u_1) and convexity (u_{11}) effects, but as maturity shrinks to zero, u_{11} tends to zero more rapidly than u_1 (see Lemma A3 in Appendix A). Proposition 3 then establishes that bond prices are decreasing (increasing) in volatility at short maturity dates when $b_2>0$ ($b_2<0$). This clarifies property (B) stated in the introduction and illustrates the previous multiplicative tree model where (1) expected future rates increase with d, and (2) the expected discount factor is decreasing in these rates. These two conditions correspond here to the conditions that (1) $b_2>0$, and (2) $u_1<0$.

As is clear, volatility changes do not generally represent a mean-preserving spread for the risk-neutral distribution in the term structure framework considered here. The seminal contribution of Jagannathan (1984) suggests that this is generally the case in the option pricing domain and in a diffusion environment [as in the celebrated Black and Scholes (1973) model]. In a stochastic volatility diffusion setting (with no dividends), for example, Romano and Touzi (1997, Theorem 3.1, p. 406)

---

8 As regards the type of problems studied here, the maximum principle can be stated informally as follows: given a function h with a constant sign for all $(r, y, \tau)\in\mathbb{R}_{++}\times\mathbb{R}\times[t, T]$, if another function f satisfies $(\partial/\partial\tau+L-k)f+h=0$ for all $\tau\in[t,T)$ and $f=0$ at T [as in Equation (7) above], then f has the same sign as h (see, also, Lemma A1 in Appendix A).

found that if an option price is strictly convex in the underlying stock price, then it is strictly increasing with respect to stock price volatility. In the framework this article analyzes, the short-term rate is not a traded asset. Therefore the risk-neutral drift function b is generally constrained to depend on volatility, and as Equation (7) reveals, such a phenomenon generates slope effects. As shown in Sections 3 and 4, in the presence of a sufficiently high level of the market risk premium, slope effects may even dominate at any finite maturity date, thus making bond prices decrease with volatility at any arbitrary maturity date.

3. Theory

This section develops the central results of the article. Its objective is to examine how random volatility changes affect bond prices dynamics. Section 3.1 provides results relating bond prices to short-term rate movements. Section 3.2 analyzes the relationship between bond prices and volatility.

3.1 Bond prices and short-term rate movements

The main objective of this subsection is to develop conditions under which bond prices are strictly decreasing and convex in the short-term rate. The following result reveals that these properties hold when (stochastic) volatility is sufficiently independent of the short-term rate and/or when b does not exhibit too many nonlinearities:

Proposition 1. *If $\varphi_1(r,y)=\psi_1(r,y)=0$ for every $r,\ y\in\mathbb{R}_{++}\times\mathbb{R}$, then bond prices are strictly decreasing in the short-term rate. Suppose further that $\partial^2\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}(r,y)/\partial r^2=0$; then bond prices are strictly convex (concave) in the short-term rate if $\max_{r,y\in\mathbb{R}_{++}\times\mathbb{R}}b_{11}(r,y)<2$ $(\min_{r,\ y\in\mathbb{R}_{++}\times\mathbb{R}}b_{11}(r,\ y)>2)$. Finally, relax all the previous assumptions; then $\forall\ (x,\ s)\in\mathbb{R}_{++}\times\mathbb{R}$ such that $-\infty<b_{11}(x,\ s)<2$ $(2<b_{11}(x,s)<\infty)$, there exists a maturity date $\hat{T}$ depending on x, s such that bond prices are strictly convex (concave) in the short-term rate at maturity dates less than $\hat{T}$.*

The previous proposition can be used to examine a number of well-known models. Consider, for example, the scalar diffusion case. This case is obtained by setting $b(r,\ y)\equiv b(r)$ and $\sigma^{(1)}(r,y)\equiv\sqrt{2a(r)}$, $\sigma^{(2)}(r,\ y)\equiv 0$ in Equation (1), $\hat{b}(r,y)\equiv\hat{b}(r)$ and $\Lambda^1(r)\equiv\Lambda^1(r,\ y)$ in Equation (2), and letting the price function in Equation (3) be simply of the form $u(x,\ t,\ T)$. Proposition 1 then predicts that in the scalar diffusion case, bond prices are always strictly decreasing in the short-term rate. Furthermore, bond prices are strictly convex (concave) in the short-term rate if

$$\max_{r\in\mathbb{R}_{++}} b''(r)<2 \quad (\min_{r\in\mathbb{R}_{++}} b''(r)>2); \tag{8}$$

and for each current short-term rate level $r(t)=x$ such that

$$-\infty < b''(x) < 2 \quad (2 < b''(x) < \infty), \tag{9}$$

there exists a maturity date $\hat{T}$ depending on x such that bond prices are strictly convex (concave) in the short-term rate at maturity dates less than $\hat{T}$.

As is well known, bond prices are by construction a convex function of the short-term rate in affine models (i.e., models in which b and a are affine functions). Equations (8) and (9) clearly confirm this fact in the scalar diffusion setting. Equations (8) and (9) can be used to check bond price convexity in nonlinear models. Consider, for example, the model proposed by Ahn and Gao (1999). The authors take $\hat{b}(r)=\beta_0(\beta_1-r)r$ and $\sqrt{2a(r)}=\eta r^{3/2}$, and specify the risk premium function as $\Lambda^1(r)= -(\lambda_1 r^{-1/2}+\lambda_2 r^{1/2})/\eta$. This implies that for all $r\in\mathbb{R}_{++}$, $b''(r)=-2(\beta_0+\lambda_2)$. It can be shown that a stationarity condition for this model imposes that $\beta_0+\lambda_2\geq 0$. Such a condition is satisfied by the estimates reported by the authors (see their Tables 3 and 5). Equation (8) then predicts that in this model, bond prices are strictly convex in the short-term rate. More generally we should expect that global concavity of bond prices does not occur in many other models, since stationarity of r under the risk-neutral measure typically rules out global convexity of b.

In contrast, a nonlinear model not displaying the (global) convexity property is the one considered by Chapman, Long, and Pearson (1999, pp. 779–780). The authors take $\hat{b}(r)=\beta_0+\beta_1 r+\beta_2 r^2+\beta_3 r^{-1}$, $\sqrt{2a(r)}=\eta r^{3/2}$, and $\sqrt{2a(r)}\Lambda^1(r)=-(\lambda_0 r^{3/2}+\lambda_1 r^{5/2}+\lambda_2 r^{7/2})$. The drift specification is the one used by Aït-Sahalia (1996b) and Conley et al. (1997). The diffusion specification is the one considered by Chan et al. (1992) and Conley et al. (1997), and is a special case of the diffusion function examined by Aït-Sahalia (1996b). For this model I use the coefficient values reported by Chapman, Long, and Pearson (1999), and apply the theoretical test condition of Equation (9) to find that for all $r(t)=x<5.04\%$ and for all $r(t)=x>27.23\%$, there exist maturity dates for which bond prices are strictly concave in the short-term rate.

The last scalar diffusion example considered here is the "double square-root" model of Longstaff (1989). For technical reasons developed in Appendix B, I consider a variant of this model that sets $\hat{b}(r)=\mu-\beta\sqrt{r}$, $\sqrt{2a(r)}=\eta\sqrt{r}$, and $\Lambda=0$, with $\mu>\eta^2/2$. Again applying Equation (9), I find that for all $r(t)=x<\beta^{2/3}/4$, there exist maturity dates for which bond prices are strictly concave in the short-term rate.

As regards stochastic volatility models, the condition of Proposition 1 which states that $\varphi_1(\cdot,\cdot)=\psi_1(\cdot,\cdot)=0$ is important. Consider, for example, the affine Longstaff and Schwartz (henceforth LS) (1992) model. This model imposes that the process $(y(\tau)/r(\tau))_{\tau\geq 0}$ be in (α, β), and that

$$\begin{cases} dr(\tau) = (b_0 - b_1 r(\tau) + b_2 y(\tau))d\tau + \alpha\sqrt{\dfrac{\beta r(\tau) - y(\tau)}{\alpha(\beta-\alpha)}}\cdot dW(\tau) \\ \qquad + \beta\sqrt{\dfrac{y(\tau) - \alpha r(\tau)}{\beta(\beta-\alpha)}}\cdot dB(\tau) \\ dy(\tau) = (\varphi_0 + \varphi_1 r(\tau) - \varphi_2 y(\tau))d\tau + \alpha^2\sqrt{\dfrac{\beta r(\tau) - y(\tau)}{\alpha(\beta-\alpha)}}\cdot dW(\tau) \\ \qquad + \beta^2\sqrt{\dfrac{y(\tau) - \alpha r(\tau)}{\beta(\beta-\alpha)}}\cdot dB(\tau) \end{cases} \tag{10}$$

where $\alpha > 0$, $\beta > 0$, and b_j and φ_j are constants (see Appendix B). To fix ideas, assume that $\varphi_1 < 0$ (see Table 1 for the case $\varphi_1 > 0$). When $\varphi_1 < 0$, future expected volatility drops after an increase in r. Now suppose that bond prices are decreasing in volatility. Can it be the case that the effect on the term structure induced by an increase in r is offset by the negative feedback effect that r has on the drift function of volatility? May bond prices increase following an increase in the short-term rate?

The next result [stated in the general case of Equation (1)] shows the intuitive property that even in less-favorable cases, bond prices still behave as in the scalar diffusion setting at short maturity dates:

Proposition 2. *There always exists a maturity date T_* depending on the current short-term rate and volatility level $(r(t), y(t)) = (x, s)$ such that bond prices are strictly decreasing in the short-term rate for each $T \leq T_*$.*

The impossibility to exclude that bond prices are *never* increasing in the short-term rate in Equation (1) arises exactly because of the short-term rate feedbacks on the drift and volatility of the short-term rate volatility. As regards volatility drift feedbacks, for instance, the proof reveals that the negative feedbacks in the LS model are negligible at short maturity

Table 1
Cases (b) and (d) identify necessary conditions for bond prices to be positively related to the short-term rate at medium-long maturity dates when the volatility of volatility is decreasing in r and bond prices are convex in volatility

	Risk-neutral drift of volatility	
	Increasing in the short-term rate	Decreasing in the short-term rate
$\partial u/\partial y > 0$	(b)	(a)
$\partial u/\partial y < 0$	(c)	(d)

dates because volatility has a negligible impact on bond prices at short maturity dates. At longer maturity dates, such feedbacks may be important. To illustrate this phenomenon in the context of the LS model, let $\varpi \equiv \partial\phi/\partial x$ denote the first partial of the stochastic volatility flow $\phi(\tau;\, x,\, s) = y(\tau)$ with respect to the initial condition of the short-term rate $r(t) = x$. It can be shown that

$$\mathbb{E}(\varpi(\tau)) = \frac{\varphi_1}{\overline{\theta}}\{\exp(-\theta_-(\tau - t)) - \exp(-\theta_+(\tau - t))\}, \quad \tau \in (t, T], \qquad (11)$$

where $\overline{\theta} \equiv \sqrt{(b_1 - \varphi_2)^2 + 4\varphi_1 b_2}$, $\theta_\mp \equiv (b_1 + \varphi_2 \mp \overline{\theta})/2$. Next, suppose that $\theta_- > 0$ (a stability condition ensured by the condition that $b_1\varphi_2 > b_2\varphi_1$). Equation (11) then reveals that an increase in the short-term rate has no effect at $\tau = t$, yet as time unfolds, it has on average a progressively higher (negative) impact on volatility until time $\tau = t + \log(\theta_+/\theta_-)^{1/\overline{\theta}}$, where function $\mathbb{E}(\varpi(\tau))$ attains its minimum. Now suppose that $\partial u/\partial y < 0$ (see Appendix B for numerical examples ensuring this). Because discounted bond prices are Q-martingales, they might then be *positively* related to short-term rate movements at medium-long maturity dates. This possibility is illustrated by entry (d) of Table 1.[9] LS (1992) noticed that their model predicts that bond prices may react positively to short-term rate movements at medium-long maturity dates. In Appendix B, I provide further technical details on how to use the theory of this section to clarify the origins of this property.

3.2 Bond prices and volatility

The objective of this subsection is to examine the mechanism generating given relationships between bond prices and volatility. As indicated in Section 2, a sufficient condition under which bond prices are increasing in volatility at any finite maturity date is that

$$\text{For all } (r, y, \tau) \in \mathbb{R}_{++} \times \mathbb{R} \times [t, T),$$
$$b_2(r, y)u_1(r, y, \tau, T) + \sigma_2(r, y)u_{11}(r, y, \tau, T) > 0. \qquad (12)$$

As an example, LS (1992, Table II, p. 1278) reported parameter estimates of their model [Equation (10)] guaranteeing that $u_1 < 0$. The authors also reported a negative estimate of b_2. Similar findings were reported by Chapman, Long, and Pearson (1999, pp. 800–801). Since Equation (10) is an affine model, $u_{11} > 0$. Because $\sigma(r, y) = y/2$ in Equation (10), Equation (12) then implies that given this kind of parameter estimates, bond prices can never be decreasing in volatility in the LS model.

---

[9] Since the LS model [Equation (10)] is affine, the bond pricing function is always convex in the state variables. Furthermore, in Equation (10) the volatility of volatility is $\psi_1 r + \psi_2 y$, where $\psi_1 \equiv -\alpha\beta(\alpha + \beta)/2$ (see Appendix B). Therefore case (d) in Table 1 is the relevant case to refer to.

Table 2
Examples of stochastic volatility models in which the risk-neutralized drift function of the short-term rate is increasing in volatility

	$b(r, y)$	$\sigma^{(1)}(r, y)$	$\sigma^{(2)}(r, y)$	$\varphi(r, y)$	$\psi^{(1)}(r, y)$	$\psi^{(2)}(r, y)$
FV	$\iota - \theta r + \lambda_1 y$	$\sqrt{y}$	0	$w - (\varphi_2 + \lambda_2 \psi_0) y$	$\rho \psi_0 \sqrt{y}$	$\psi_0 \sqrt{y} \sqrt{1 - \rho^2}$
AL	$\iota - \theta r + \lambda_1 e^{y/2} r$	$e^{y/2} r^{\gamma}$	0	$w - \varphi_2 y$	ψ_0	0
MF	$\iota - \theta r + \lambda_1 y^{1/\delta} r$	$y^{1/\delta} \sqrt{r}$	0	$w - \varphi_2 y - \lambda_2 \psi_0 y \sqrt{r}$	$\psi_0 y$	0

When $b_2 > 0$, the situation is radically different. Such a situation may arise within the LS model [Equation (10)] under a set of alternative parameter values. In general, it arises whenever the interest rate risk premium $\sum_{j=1}^{2} \sigma^{(j)}(r, y) \cdot \Lambda^j(r, y)$ is positively valued *and* increases sufficiently rapidly with volatility [see Equation (2)]. If $\hat{b}_2 \equiv 0$, the condition that $b_2 > 0$ is automatically satisfied whenever the interest rate risk premium is increasing in volatility. Table 2 describes three models that have $\hat{b}_2 \equiv 0$ [FV: Fong and Vasicek (1991); AL: Andersen and Lund (1997); MF: Mele and Fornari (2000)]. In these models, λ_1 and λ_2 are risk-premia coefficients; ι, θ, γ, δ, w, φ_2, ψ_0 are constants; and $\delta \in [1, \infty)$, $\theta \in (0, \infty)$, $\rho \in (-1, +1)$. The constant λ_1 is typically found to be positive to accommodate main stylized features of the entire term structure of interest rates, at least in the context of models with zero correlation. In the context of three-factor models with nonzero correlations, Dai and Singleton (2000) provided mixed empirical evidence on the sign of such coefficients. Such pieces of evidence are discussed and used to illustrate the multifactor theory of the next section.

The following proposition provides a theory on how bond prices react to random volatility changes in models that may make $b_2 > 0$, independently of the sign of σ_2.

Proposition 3. (Weak term structure augmenting (decreasing) volatility property). *For each current short-term rate and volatility level* $(r(t), y(t)) = (x, s) \in \mathbb{R}_{++} \times \mathbb{R}$ *such that* $b_2(x, s) \in (0, \infty)$ $(b_2(x, s) \in (-\infty, 0))$ *and* $0 < |\sigma_2(x, s)| < \infty$, *there exists a maturity date* $\overline{T}$ $(\underline{T})$ *depending on* (x, s) *such that bond prices are strictly decreasing* (*increasing*) *in volatility for all maturity dates less than* $\overline{T}$ $(\underline{T})$.

When do bond prices satisfy a sort of "strong" form of the term structure augmenting (decreasing) volatility property? The answer clearly depends on whether Equation (12) is satisfied, and is model specific. Under fairly regular conditions, however, all the model examples of this section predict that bond prices may be decreasing in volatility at any arbitrary maturity date corresponding to sufficiently high levels of the interest rate risk premium (the case of bond prices increasing in volatility has a similar interpretation).[10] This is the case, for instance, in the LS Equation (10) (see Appendix B for details). Also, in the AL model of Table 2, $u_1 < 0$ and $u_{11} > 0$ for $\tau \in [t, T)$, and

$$u_2(x, s, t, T) = \frac{1}{2}\mathbb{E}\left\{ \int_{t}^{T} \kappa^1(t, \tau)(-u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T)) \right. \\ \times \exp(\phi_{(1)}(\tau))\vartheta_{(1)}(\tau)^{2\gamma}\left[\frac{u_{11}}{-u_1}(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T) \right. \\ \left.\left. -\lambda_1 \exp\left(-\tfrac{1}{2}\phi_{(1)}(\tau))\vartheta_{(1)}(\tau)^{1-2\gamma}\right)\right] d\tau \right\}, \tag{12}$$

where κ^1, $\vartheta_{(1)}$, $\phi_{(1)}$ are as in Appendix A (see Lemma A1). Under the conditions introduced in Appendix B for a related problem [see Equation (B10)], a fixed point argument can then be employed [as in Equation (B11)] to ensure that there exist sufficiently high values of λ_1 depending on T, x, s that make $u_2 < 0$ for any finite T. In numerical work, AL found that $u_2 < 0$ for all maturity dates up to thirty years.[11] The authors attribute this finding to (1) the presence of a positively priced risk premium, and (2) high mean reversion in volatility. As demonstrated here, the first conjecture of the authors is correct.

Finally, consider the Fong and Vasicek (1991) model. In this model it is possible to analytically find uniform bounds for the critical maturity date $\overline{T}$ of Proposition 3. Let $\lambda_1 > 0$. In Appendix B, I show that

$$\overline{T} = \begin{cases} t - \log(1 - 2\theta\lambda_1)^{1/\theta}, & \text{if } \lambda_1 \in \left(0, \frac{1}{2\theta}\right) \\ \text{any strictly positive real number, otherwise.} \end{cases} \tag{13}$$

Equation (13) says that the persistence of the term-structure augmenting volatility property increases with the risk-premium coefficient λ_1. They also reveal that with a sufficiently high interest rate risk premium (viz $\lambda_1 \geq 1/2\theta$), bond prices are always negatively related to random volatility changes. Finally, Equation (13) also shows that in this model, it is the short-term rate persistence that plays an important role in explaining the term-structure augmenting volatility property, not the volatility persistence.

4. Stochastic central tendency models in a three-factor setting

The original purpose of three-factor models including a stochastic central tendency was to make the short-term rate revert toward a stochastically

[10] Conditions and methods of proofs are similar to the ones developed for the scalar diffusion case [see Mele (2002, appendix B)].

[11] As regards the MF model, the authors reported results that can be interpreted similarly. Also, note that the mentioned numerical exercises of AL concerned a three-factor model. However, the main qualitative properties of that model can be analyzed with the tools of this section, since AL assumed that the third factor evolves independently of r and y (see the next section).

moving long-term value [see, e.g., Balduzzi et al. (1996), Chen (1996), and Andersen and Lund (1997)]. From an empirical standpoint, the inclusion of a third factor is also important since most of the U.S. yield curve variation seems to be driven by three principal components [e.g., Litterman and Scheinkman (1991)]. More recently, Dai and Singleton (2000) produced empirical evidence that within the class of three-factor affine models, U.S. historical interest rate behavior can only be adequately represented by models with a rich feedback structure between the state variables and with correlated Brownian motions. An example of such a model is

$$\begin{cases} dr(\tau) = \{\kappa(\ell(\tau) - r(\tau)) - (\lambda_A + \lambda_B \sigma_{rv}) y(\tau) - \lambda_C \sigma_{r\theta}\} d\tau \\ \qquad\qquad + \sqrt{y(\tau)} dW(\tau) + \sigma_{rv} \eta \sqrt{y(\tau)} dB(\tau) + \sigma_{r\theta} \zeta dZ(\tau) \\ dy(\tau) = \{\mu \bar{v} - (\mu + \lambda_B) y(\tau)\} d\tau + \eta \sqrt{y(\tau)} dB(\tau) \\ d\ell(\tau) = \{\nu(\bar{\theta} - \ell(\tau)) - \lambda_C - \lambda_A \sigma_{\theta r} y(\tau)\} d\tau + \sigma_{\theta r} \sqrt{y(\tau)} dW(\tau) + \zeta dZ(\tau), \end{cases} \tag{14}$$

where W, B, Z are independent Brownian motions under the risk-neutral measure, and the notation for the various constants is the one used by the authors (with the exception of the risk-premia coefficients λ_A, λ_B, λ_C).

In this section I generalize both Equation (1) and Equation (14) and take as primitive:

$$\begin{pmatrix} dr(\tau) \\ dy(\tau) \\ d\ell(\tau) \end{pmatrix} = \begin{pmatrix} b(r(\tau), y(\tau), \ell(\tau)) \\ \varphi(r(\tau), y(\tau), \ell(\tau)) \\ \varepsilon(r(\tau), y(\tau), \ell(\tau)) \end{pmatrix} d\tau + V(r(\tau), y(\tau), \ell(\tau)) \begin{pmatrix} dW(\tau) \\ dB(\tau) \\ dZ(\tau) \end{pmatrix}, \quad \text{for } \tau \in (t, T], \tag{15}$$

where V is a 3×3 matrix with $[V]_{1j}(r, y, \ell) \equiv \sigma^{(j)}(r, y, \ell)$, $[V]_{2j}(r, y, \ell) \equiv \psi^{(j)}(r, y, \ell)$, and $[V]_{3j}(r, y, \ell) \equiv \pi^{(j)}(r, y, \ell), j = 1, 2, 3$; $(r(t), y(t), \ell(t)) = (x, s, c)$, and the various drift and diffusion coefficients satisfy the same conditions as those of Equation (1) [the risk premia are defined similarly as in Equation (2)]. Let the price function be $u(x, s, c, t, T)$ finally, I set $\sigma(r, y, \ell) \equiv \sum_{j=1}^{3} \sigma^{(j)}(r, y, \ell)^2/2$, $\psi(r, y, \ell) \equiv \sum_{j=1}^{3} \psi^{(j)}(r, y, \ell)^2/2$, and $\pi(r, y, \ell) = \sum_{j=1}^{3} \pi^{(j)}(r, y, \ell)^2/2$.

Naturally it is generally impossible to interpret one of the unobserved factors as "stochastic volatility" in Equation (15). This would be possible when, say, $\partial\sigma/\partial\ell = 0$ and $\partial\sigma/\partial y > 0$, in which case *only* factor y could be interpreted as a stochastic volatility factor, as in Equation (14) and in Equation (17) below. Naturally such interpretative (and arbitrary) constraints will not be imposed to derive Propositions 4 and 5 below.

Proposition 4. (*Weak term structure augmenting* (*decreasing*) *unobservable factor property*). *For each current factor level* $(r(t), y(t), \ell(t)) = (x, s, c) \in$

$\mathbb{R}_{++} \times \mathbb{R} \times \mathbb{R}$ *such that* $b_i(x, s, c) \in (0, \infty)$ ($b_i(x,s,c) \in (-\infty, 0)$) *and* $0 < |\sigma_i(x, s, c)| < \infty$, $i = 2, 3$, *there exists a maturity date* $\overline{T}$ ($\underline{T}$) *depending on* (x, s, c) *such that bond prices are strictly decreasing* (*increasing*) *in factor* j, $j = y, \ell$, *for all maturity dates less than* $\overline{T}$ ($\underline{T}$).

The previous result generalizes Proposition 3, and is due to phenomena very similar to those mentioned in Section 2: precisely, the bond price reaction behavior at short maturity dates is still led by slope effects [see Equation (C2) in Appendix C]. However, the conditions guaranteeing the existence of a strong version of Proposition 4 are more complex than Equation (12).

Proposition 5. (*Strong term structure augmenting* (*decreasing*) *unobservable factor property*). *For any* $T > t$, *bond prices are strictly decreasing* (*increasing*) *in factor* j, $j = y, \ell$, *for all maturity dates up to* T *if for all* $(r, y, \ell, \tau) \in \mathbb{R}_{++} \times \mathbb{R} \times \mathbb{R} \times [t, T)$,

$$\begin{aligned}
&\frac{\partial b}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \sigma}{\partial y}\frac{\partial^2 u}{\partial r^2} + \frac{\partial \varepsilon}{\partial y}\frac{\partial u}{\partial \ell} + \frac{\partial \pi}{\partial y}\frac{\partial^2 u}{\partial \ell^2} \\
&\quad + \frac{\partial^2 u}{\partial r \partial \ell} \cdot \sum_{j=1}^{3} \frac{\partial}{\partial y} \sigma^{(j)} \pi^{(j)} < 0 \ (>0) \quad (\textit{factor } y) \\
&\frac{\partial b}{\partial \ell}\frac{\partial u}{\partial r} + \frac{\partial \sigma}{\partial \ell}\frac{\partial^2 u}{\partial r^2} + \frac{\partial \varphi}{\partial \ell}\frac{\partial u}{\partial y} \\
&\quad + \frac{\partial \psi}{\partial \ell}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial r \partial y} \cdot \sum_{j=1}^{3} \frac{\partial}{\partial \ell} \sigma^{(j)} \psi^{(j)} < 0 \ (>0) \quad (\textit{factor } \ell)
\end{aligned} \tag{16}$$

Comparing the first condition in Equation (16) with Equation (12) reveals that the new terms arising from the presence of a third factor are (1) slope and convexity of bond prices with respect to factor ℓ; and (2) correlation terms. As shown in the examples below, these new terms may also explain the origins of given relationships between bond prices and factor y (the analysis for factor ℓ is identical).

An example of models that can be analyzed quite easily with the help of Equation (16) is one proposed in Dai and Singleton (2000):

$$\begin{cases}
dr(\tau) = \{\kappa_{rv}\overline{v} + \kappa(\ell(\tau) - r(\tau)) - (\kappa_{rv} + \lambda_A + \lambda_B \sigma_{rv}) y(\tau)\} d\tau \\
\qquad\qquad + \sqrt{y(\tau)} dW(\tau) + \sigma_{rv} \eta \sqrt{y(\tau)} dB(\tau) \\
dy(\tau) = \{\mu \overline{v} - (\mu + \lambda_B) y(\tau)\} d\tau + \eta \sqrt{y(\tau)} dB(\tau) \\
d\ell(\tau) = \{\nu \overline{\theta} + \kappa_{\theta v}(\overline{v} - y(\tau)) - (\nu + \lambda_C)\ell(\tau)\} d\tau + \zeta \sqrt{\ell(\tau)} dZ(\tau).
\end{cases} \tag{17}$$

The authors reported a negative estimate of both $-\kappa$ and $-b_2 \equiv \kappa_{rv} + \lambda_A + \lambda_B\, \sigma_{rv}$ (see their Table III, p. 1965). Therefore, Proposition 4 predicts that in Equation (17), bond prices are negatively related to changes in both factors y and ℓ at short maturity dates. In addition, in Appendix C, I show that if $\kappa > 0$ and $\kappa_{\theta v} < 0$ (as in the estimates reported by the authors), Proposition 5 then predicts that a sufficient condition for bond prices to be *decreasing* in volatility at any finite maturity date is that

$$\frac{-2(\kappa_{rv} + \lambda_A + \lambda_B \sigma_{rv})}{1 + \eta^2 \sigma_{rv}^2} > \frac{1}{\kappa}, \tag{18}$$

which is satisfied by the estimates reported by the authors. Equation (18) generalizes Equation (13) exactly because it keeps track of the rich feedback and correlation structure of Equation (17): only when $\kappa_{rv} = \sigma_{rv} = 0$ does Equation (18) reduce to Equation (13). Also, in this model bond prices are negatively related to changes in factor ℓ at any finite maturity date (see Appendix C).

An issue that deserves a special mention here is that the factor ℓ dynamics may be important in explaining the sign of the left-hand side of Equation (16). This is not so evident in Equations (17) and (18) [however, see Equation (C7) in Appendix C for more details on this], but can be clearly seen at work within Equation (14). Precisely, Dai and Singleton (2000, Table II, p. 1964) reported a positive estimate of $-b_2 \equiv \lambda_A + \lambda_B \sigma_{rv}$ in Equation (14), and according to Proposition 4, $u_2 > 0$ at short maturity dates. Furthermore, in Appendix C, I show that if $\lambda_A + \lambda_B \sigma_{rv} > 0$, $\lambda_A \sigma_{\theta \mathrm{r}} < 0$, and $\kappa > \nu$ (as in the estimates reported by the authors), then Proposition 5 predicts that $u_2 > 0$ at any maturity date whenever

$$\frac{\lambda_A + \lambda_B \sigma_{rv}}{-\lambda_A \sigma_{\theta r}} > \frac{\kappa}{\nu} \cdot \frac{\kappa}{\kappa - \nu}. \tag{19}$$

Because of the high value of the ratio κ/ν estimated by the authors, however, Equation (19) is not satisfied. Intuitively, a high κ means that the third term in Equation (16) for factor y is also high in absolute value. This is so because $(\partial \varepsilon/\partial y) \cdot (\partial u/\partial \ell) = -(\partial \varepsilon/\partial y) \cdot (\int m_{\vartheta,c}(v) dv) \cdot u$, where $m_{\vartheta,c}(v)$ is the partial of the short-term rate ϑ with respect to the initial condition c of factor ℓ; and in the model estimated by Dai and Singleton, $\int m_{\vartheta,c}$ is positive and increasing in κ [see Appendix C, Equation (C4)]. In fact, in Appendix C, I use Equation (16) to show that bond prices are positively related to volatility up to nine months. However, I also show that Equation (19) can be further elaborated to provide another condition for bond prices to be decreasing in volatility at longer maturity dates. I then find that the latter condition is satisfied at least for maturity dates of two years.

Back to the general theory, the final remark of this section is that the bond price reaction to short-term rate movements is governed by the same logic presented in Section 3. Particularly, results similar to Propositions 1 and 2 also apply here (see the proofs of Propositions 4 and 5; as regards generalizing Proposition 1, the appropriate condition is that the coefficients of y and ℓ do not depend on r). As regards the Dai and Singleton (2000) Equations (14) and (17), for instance, bond prices are always decreasing in r.

5. On Jumps and Default

This section examines the robustness of the theory developed in the previous sections to the introduction of jump phenomena. I shall consider two settings: jump-diffusion models of the short-term rate, and models of defaultable bonds. Jump-diffusion models of the short-term rate have traditionally attracted the interest of modelers because they may capture sudden changes in market liquidity conditions and/or discontinuous information releases. A first general equilibrium treatment of this kind of model appears in Ahn and Thompson (1988). A detailed list of references on recent empirical work in this area can be found in Das (2000). As regards defaultable bonds, I shall make reference to "reduced form" models [see, for instance, Duffie and Singleton (1999) and the references therein], in which default is considered as an exogenously given rare event.

5.1 Jump-diffusion models

Let the short-term rate be the solution to

$$dr(\tau) = b^J(r(\tau))d\tau + \sqrt{2a_1(r(\tau))}dW(\tau) + a_2(r(\tau)) \cdot \mathcal{S} \cdot dN(\tau), \tau \in (t, T],$$
$$r(t) = x, \qquad (20)$$

where b^J is the jump-adjusted risk-neutral drift, a_1 is a strictly positive "diffusion" function, a_2 is a bounded "jump" function with bounded derivatives, and N is a Cox process with intensity function (or "hazard rate") of the form $v(r)$, where v is bounded with bounded derivatives. Finally, $\mathcal{S}$ is a random variable with a fixed probability measure on $\mathbb{R}$ with density p and expectation operator $\mathbb{E}_{\mathcal{S}}$. [See, e.g., Jacod and Shiryaev (1987, pp. 142–146) for a succinct discussion of diffusion processes with jumps.]

We have

Proposition 6. *Let the short-term rate be the solution to Equation* (20)*. The following statements are true:*

(a) *There exists a maturity date T_* depending on the current short-term rate level $r(t) = x$ such that bond prices are strictly decreasing in x for each maturity date $T \leq T_*$.*

(b) *Assume that for all* $r \in \mathbb{R}_{++}$, $a_2'(r) = 0$. *Then bond prices are strictly decreasing in the short-term rate if for all* $r \in \mathbb{R}_{++}$, $-1 \leq v'(r) \leq 0$; *when for all* $r \in \mathbb{R}_{++}$, $v'(r) > 0$, *bond prices are strictly decreasing in the short-term rate if for each* $(r, \tau) \in \mathbb{R}_{++} \times [t, T)$, $v'(r)/(1 + v'(r)) < u(r,\tau,T)/\mathbb{E}_{\mathcal{S}}\{u(r + a_2\mathcal{S}, \tau, T)\}$, *where* a_2 *is a constant.*

(c) *Assume that for all* $r \in \mathbb{R}_{++}$, $a_2'(r) = 0$ *and* $v'(r) = 0$. *Then the conclusions about bond price (global) convexity of Proposition* 1 (*applied to* b^J) *are also valid here.*

(d) *Consider two economies A and B which only differ because for each* $r \in \mathbb{R}_{++}$, $v^A(r) > v^B(r)$ *and let* $u^j, j = A, B$, *be the corresponding bond price functions. Finally, assume that* b^J *does not depend on v. Then, for each* $(r,\tau) \in \mathbb{R}_{++} \times [t,T)$, $u^A(r,\tau,T) < (>)\ u^B(r, \tau, T)$ *whenever* $\mathbb{E}_{\mathcal{S}}\{u^B(r + a_2(r)\mathcal{S},\tau,T)\} < (>)\ u^B(r, \tau, T)$.

Under the same conditions of Proposition 6(c), the *local* convexity properties of Proposition 1 [see Equations (8) and (9)] still hold here (with respect to b^J) for sets of bond prices satisfying the standard regularity condition that $\partial u_1(x,t,T)/\partial T|_{T=t} = -1$. As regards multifactor models, the analysis of Sections 3 and 4 is unaffected as long as intensity and jump functions are independent of the state variables [by a simple extension of Equation (D1) in Appendix D to the multidimensional case; Proposition 6(a) will then hold even when intensity and jump functions depend on factor levels, and local analysis is unchanged for all bond prices satisfying standard regularity conditions as $\partial u_{11}(x, s, t, T)/\partial T|_{T=t} = 0$].

5.2 Defaultable bonds

Assume that under a risk-neutral measure, the short-term rate $(r(\tau))_{\tau \in [t,T]}$ is a diffusion process and that the event of default at each instant in time is exactly the same as process N considered in the previous subsection (with intensity process v). In case of default at point τ, the holder of the bond receives a recovery payment $\overline{u}$ that I assume to be of the form $\overline{u}(\tau) \equiv \overline{u}(\tau, r(\tau))$. Let the pre default bond price function be $u^{\mathrm{pre}}(r,\tau,T)$, and set $\overline{u} = (1 - l) \cdot u^{\mathrm{pre}}$ for some process l in [0,1]. Let $\mathbb{E}^* \cdot$ be the expectation operator taken with reference to the information sets $\sigma(r(u) : t \leq u \leq \tau)$ only. As shown by Duffie and Singleton [1999, Equation (10), p. 696],

$$u^{\mathrm{pre}}(x, t, T) = \mathbb{E}^*\left\{\exp\left(-\int_t^T (r(\tau) + l(\tau) \cdot v(r(\tau)))d\tau\right)\right\} \qquad (21)$$

and in Appendix D, I indicate a new method of proof of Equation (21) that is related to a remark by Lando (1998, p. 107). Such a proof reveals that u^{pre} follows the same kind of dynamics followed by the bond price function of the previous subsection [for technical details, compare Equation (D1) with Equation (D3) in Appendix D]. Therefore the conclusions of Proposition 6 also apply to $u^{\mathrm{pre}}(r, \tau, T)$ once the expectation

$\mathbb{E}_{\mathcal{S}}\{u^{\cdot}(r+a_2(r)\mathcal{S},\ \tau,\ T)\}$ of the previous subsection is replaced with $\overline{u}^{\cdot}(\tau)$, and the condition $a_2{}'=0$ stated in Proposition 6 [parts (b) and (c)] is ignored.

6. Conclusion

This article analyzed theoretical properties of standard, parsimonious term structure models in which bond prices are modeled starting from the knowledge of the stochastic evolution of the short-term rate. Consistent with an approach followed by many authors over nearly 30 years, the short-term rate was assumed to be a diffusion process. The objective here was to study arbitrage restrictions, together with additional properties, such as convexity of bond prices with respect to the short-term rate, implied restrictions on the dynamics of the short-term rate, and other (possibly unobservable) factor processes. In addition to providing a theoretical construction of how rationally formed bond prices move in reaction to observable and unobservable factor changes, the theory developed in this article imposes new testable restrictions on the joint dynamics of bond prices, fundamentals, and models of risk aversion. Typical examples of such restrictions are summarized in property (B) stated in the introduction or in Table 1. According to property (B), for instance, it cannot be the case that the risk-neutral drift of the short-term rate is increasing in volatility *and* that bond prices are increasing in volatility at short maturity dates: this is a testable implication of any stochastic volatility model of the short-term rate in a diffusion setting. Recently there has been increasing interest in asking data to give detailed information on the validity of *generic* properties of asset pricing models. Bakshi, Cao and Chen (2000), for instance, recently submitted the fundamental, "general properties of option prices" of Bergman, Grundy, and Wiener (1996) to a thorough empirical examination. Similar empirical studies can be conducted within the context analyzed in this article, especially when the scope is to deepen our understanding of the relationship between bond prices and volatility.

Appendix A

This appendix makes an iterated use of the Feynman–Kac representation theorem for the bond price and its partial derivatives, when they come in the form of a solution to partial differential equations. Naturally the Feynman–Kac representation theorem does not imply the existence of a solution to partial differential equations per se. As pointed out in Section 1, Mele (2002, Appendices A, B, and C) contains precise conditions justifying all operations taken in the present appendix.

Lemma A1. (Feynman–Kac representation of the partial derivatives of the bond price with respect to the short-term rate and volatility). *Let* $w^1(x, s, t, T) \equiv u_2(x,s,t,T)$, $w^2(x,s,t,T) \equiv u_1(x,s,t,T)$, $w^3(x,s,t,T) \equiv u_{11}(x,s,t,T)$. We have.

$$w^i(x,s,t,T)=\mathbb{E}\left\{\int_t^T \kappa^i(t,\tau)h^i(\vartheta_{(i)}(\tau),\phi_{(i)}(\tau),\tau,T)d\tau\right\},\quad i=1,\ldots,3,$$

where

$$\begin{cases} \kappa^1(t,\tau) = \exp\left\{-\int_t^\tau (\vartheta_{(1)}(u) - \varphi_2(\vartheta_{(1)}(u), \phi_{(1)}(u)))du\right\} \\ \kappa^2(t,\tau) = \exp\left\{-\int_t^\tau (\vartheta_{(2)}(u) - b_1(\vartheta_{(2)}(u), \phi_{(2)}(u)))du\right\} \\ \kappa^3(t,\tau) = \exp\left\{-\int_t^\tau (\vartheta_{(3)}(u) - 2b_1(\vartheta_{(3)}(u), \phi_{(3)}(u)) - \sigma_{11}(\vartheta_{(3)}(u), \phi_{(3)}(u)))du\right\} \end{cases} \tag{A1}$$

$$\begin{cases} h^1 = b_{(1),2}w^2 + \sigma_{(1),2}w^3 \\ h^2 = -u + \varphi_{(2),1}w^1 + \psi_{(2),1}u_{22} \\ h^3 = -(2 - b_{(3),11})w^2 + \varphi_{(3),11}w^1 + \psi_{(3),11}u_{22} \\ \qquad + 2\psi_{(3),1}u_{122} + \left(2\varphi_{(3),1} + \frac{\partial^2}{\partial\vartheta^2}\left\{\sigma_{(3)}^{(1)}\psi_{(3)}^{(1)} + \sigma_{(3)}^{(2)}\psi_{(3)}^{(2)}\right\}\right)u_{12} \end{cases} \tag{A2}$$

$(\vartheta_{(i)}, \phi_{(i)})_{i=1}{}^3$ *are solutions to the following stochastic differential systems*:

$$\begin{cases} d\vartheta_{(1)} = \left(b_{(1)} + \frac{\partial}{\partial\phi}\sum_{j=1}^{2}\sigma_{(1)}^{(j)}\psi_{(1)}^{(j)}\right)d\tau + \sigma_{(1)}^{(1)}dW + \sigma_{(1)}^{(2)}dB \\ d\phi_{(1)} = \left(\varphi_{(1)} + \psi_{(1),2}\right)d\tau + \psi_{(1)}^{(1)}dW + \psi_{(1)}^{(2)}dB \end{cases} \tag{A3i}$$

$$\begin{cases} d\vartheta_{(2)} = (b_{(2)} + \sigma_{(2),1})d\tau + \sigma_{(2)}^{(1)}dW + \sigma_{(2)}^{(2)}dB \\ d\phi_{(2)} = \left(\varphi_{(2)} + \frac{\partial}{\partial\vartheta}\sum_{j=1}^{2}\sigma_{(2)}^{(j)}\psi_{(2)}^{(j)}\right)d\tau + \psi_{(2)}^{(1)}dW + \psi_{(2)}^{(2)}dB \end{cases} \tag{A3ii}$$

$$\begin{cases} d\vartheta_{(3)} = (b_{(3)} + 2\sigma_{(3),1})d\tau + \sigma_{(3)}^{(1)}dW + \sigma_{(3)}^{(2)}dB \\ d\phi_{(3)} = \left(\varphi_{(3)} + 2\frac{\partial}{\partial\vartheta}\sum_{j=1}^{2}\sigma_{(3)}^{(j)}\psi_{(3)}^{(j)}\right)d\tau + \psi_{(3)}^{(1)}dW + \psi_{(3)}^{(2)}dB \end{cases} \tag{A3iii}$$

and $b_{(i),1} \equiv \partial b(\vartheta_{(i)}, \phi_{(i)})/\partial\vartheta$, $\varphi_{(i)} \equiv \varphi(\vartheta_{(i)}, \phi_{(i)})$, $\psi_{(i),2} \equiv \partial\psi(\vartheta_{(i)}, \phi_{(i)})/\partial\phi$, etc.

Proof. By taking the appropriate partial derivatives in Equation (3), one obtains that w^i are solutions to the following partial differential equations:

$$\begin{cases} 0 = \left(\frac{\partial}{\partial\tau} + L^i - k^i\right)w^i(\vartheta, \phi, \tau, T) + h^i(\vartheta, \phi, \tau, T), & \forall(\vartheta, \phi, \tau) \in \mathbb{R}_{++} \times \mathbb{R} \times [t, T) \\ w^i(\vartheta, \phi, T, T) = 0, & \forall(\vartheta, \phi) \in \mathbb{R}_{++} \times \mathbb{R} \end{cases} \tag{A4}$$

where h^i are as in Equation (A2),

$$\begin{cases} k^1(\vartheta, \phi) = \vartheta - \varphi_2(\vartheta, \phi) \\ k^2(\vartheta, \phi) = \vartheta - b_1(\vartheta, \phi) \\ k^3(\vartheta, \phi) = \vartheta - 2b_1(\vartheta, \phi) - \sigma_{11}(\vartheta, \phi) \end{cases}$$

and

$$\begin{cases} L^1 w^1 = \left(b + \frac{\partial}{\partial \phi}\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_1^1 + (\varphi + \psi_2)w_2^1 + \sigma w_{11}^1 + \psi w_{22}^1 + \left(\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_{12}^1 \\ L^2 w^2 = (b+\sigma_1)w_1^2 + \left(\varphi + \frac{\partial}{\partial \vartheta}\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_2^2 + \sigma w_{11}^2 + \psi w_{22}^2 + \left(\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_{12}^2 \\ L^3 w^3 = (b+2\sigma_1)w_1^3 + \left(\varphi + 2\frac{\partial}{\partial \vartheta}\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_2^3 + \sigma w_{11}^3 + \psi w_{22}^3 + \left(\sum_{j=1}^{2}\sigma^{(j)}\psi^{(j)}\right)w_{12}^3 \end{cases}$$

The result then follows by the Feynman–Kac representation theorem. ■

The following lemma formalizes the idea that at small Δt, the short-term rate $r(t+\Delta\, t)$ is approximately equal to x.

Lemma A2. (Sensitivity of the short-term rate with respect to the initial condition). *Let $(\vartheta,\phi)(\tau)\equiv(\vartheta,\phi)(x,s,\tau;\omega)(\tau\in[t,T])$ denote the flow of the stochastic differential equation [Equation (1)] at τ starting at (x, s) in t at point $\omega\in\Omega$. We have*

(a) $\lim_{\tau\downarrow t}\frac{\partial\vartheta}{\partial x}(\tau)=1$; and,

(b) $\lim_{\tau\downarrow t}\frac{\partial^2\vartheta}{\partial x^2}(\tau)=0$.

Proof. By Theorems 4.7.1 and 4.7.2 in Kunita (1990), p. 177, there exists (up to an explosion time) a unique forward stochastic flow of local C^2-diffeomorphisms (ϑ, ϕ) in (x,s). The processes $\partial\vartheta/\partial x$ and $\partial\phi/\partial x$ then satisfy

$$\begin{cases} \frac{\partial\vartheta}{\partial x}(\tau) = 1 + \int_t^\tau \left(b_1\frac{\partial\vartheta}{\partial x} + b_2\frac{\partial\phi}{\partial x}\right)(u)du + \int_t^\tau \left\{\left(\sigma_1^{(1)}\frac{\partial\vartheta}{\partial x} + \sigma_2^{(1)}\frac{\partial\phi}{\partial x}\right)(u)dW(u).\right. \\ \qquad\qquad \left. + \left(\sigma_1^{(2)}\frac{\partial\vartheta}{\partial x} + \sigma_2^{(2)}\frac{\partial\phi}{\partial x}\right)(u)dB(u)\right\} \\ \frac{\partial\phi}{\partial x}(\tau) = \int_t^\tau (\varphi_1\frac{\partial\vartheta}{\partial x} + \varphi_2\frac{\partial\phi}{\partial x})(u)du + \int_t^\tau \left\{\left(\psi_1^{(1)}\frac{\partial\vartheta}{\partial x} + \psi_2^{(1)}\frac{\partial\phi}{\partial x}\right)(u)dW(u)\right. \\ \qquad\qquad \left. + \left(\psi_1^{(2)}\frac{\partial\vartheta}{\partial x} + \psi_2^{(2)}\frac{\partial\phi}{\partial x}\right)(u)dB(u)\right\} \end{cases}$$

from which (a) and (b) follow. ■

Lemma A3. *(Slope effects dominate convexity effects at short maturity dates). For each $(x,s)\in\mathbb{R}_{++}\times\mathbb{R}$, $\lim_{T\downarrow t}(u_{11}/u_1)(x,s,t,T)=0$.*

Proof. Clearly $\lim_{T\downarrow t} u_{11}(x, s, t, T)$ and $\lim_{T\downarrow t}u_1(x,s,t,T)$ are both zero [use, for instance, the Lebesgue's dominated convergence theorem in Equations (5) and (6) as I do in showing Proposition 1 in Appendix B; alternatively, just notice that $\lim_{T\downarrow t} u_{11}(x,s,t,T)$ and $\lim_{T\downarrow t} u_1(x, s, t, T)$ both equal zero by the boundary conditions in Equation (A4) given in the course of the proof of Lemma A1]. However, $\lim(u_{11}/u_1)$ can be written as

$$\lim_{T\downarrow t}\frac{u_{11}}{-u_1}(x,s,t,T)$$
$$=\lim_{T\downarrow t}\frac{\mathbb{E}\left\{\left[\left(\int_t^T\frac{\partial\vartheta}{\partial \mathrm{x}}(\tau)\mathrm{d}\tau\right)^2-\int_t^T\frac{\partial^2\vartheta}{\partial \mathrm{x}^2}(\tau)\mathrm{d}\tau\right]\exp\left(-\int_t^T\vartheta(\tau)\mathrm{d}\tau\right)\right\}}{\mathbb{E}\left\{\left(\int_t^T\frac{\partial\vartheta}{\partial \mathrm{x}}(\tau)\mathrm{d}\tau\right)\exp\left(-\int_t^T\vartheta(\tau)\mathrm{d}\tau\right)\right\}}$$
$$=\lim_{T\downarrow t}\frac{\mathbb{E}\left\{\left[(T-t)\left(\frac{1}{T-t}\int_t^T\frac{\partial\vartheta}{\partial x}(\tau)d\tau\right)^2-\frac{1}{T-t}\int_t^T\frac{\partial^2\vartheta}{\partial x^2}(\tau)d\tau\right]\exp\left(-\int_{\mathrm{t}}^{\mathrm{T}}\vartheta(\tau)\mathrm{d}\tau\right)\right\}}{\mathbb{E}\left\{\left(\frac{1}{T-t}\int_t^T\frac{\partial\vartheta}{\partial x}(\tau)d\tau\right)\exp\left(-\int_t^T\vartheta(\tau)d\tau\right)\right\}},$$

and given the assumptions of the main text, the result follows from the Lebesgue's dominated convergence theorem, and the fact that $\int\partial\vartheta/\partial x$ and $\int\partial^2\vartheta/\partial x^2$ are Riemann integrals and then $(T-t)^{-1}\int_t^T(\partial\vartheta(\tau)/\partial x)d\tau\to\partial\vartheta(t)/\partial x$ and $(T-t)^{-1}\int_t^T(\partial^2\vartheta(\tau)/\partial x^2)d\tau\to\partial^2\vartheta(t)/\partial x^2$ as $T\downarrow t$, where $\partial\vartheta(t)/\partial x=1$ by Lemma A2 part (a) and $\partial^2\vartheta(t)/\partial x^2=0$ by Lemma A2 part (b). ■

Appendix B: Proofs, Examples, and Comparison Theory for Section 3

Proposition 1 will be proven in the scalar case. The general case is treated by making use of Lemma A1 in Appendix A.

Proof of Proposition 1 (*slope issues*). In the constant volatility case, the stochastic representations of w^2 and w^3 given in Lemma A1 simplify to

$$\begin{cases} w^2(x,t,T)=\mathbb{E}\left\{\int_t^T\kappa^2(t,\tau)h^2(\vartheta_{(2)}(\tau),\tau,T)d\tau\right\}\\ w^3(x,t,T)=\mathbb{E}\left\{\int_t^T\kappa^3(t,\tau)h^3(\vartheta_{(3)}(\tau),\tau,T)d\tau\right\}\end{cases}\tag{B1}$$

where

$$\begin{cases}\kappa^2(t,\tau)=\exp\left\{-\int_t^\tau(\vartheta_{(2)}(u)-b'(\vartheta_{(2)}(u)))du\right\}\\ \kappa^3(t,\tau)=\exp\left\{-\int_t^\tau(\vartheta_{(3)}(u)-2b'(\vartheta_{(3)}(u))-a''(\vartheta_{(3)}(u)))du\right\}\end{cases}$$

and

$$\begin{cases}h^2=-u\\ h^3=-(2-b'')w^2\end{cases}\tag{B2}$$

and $\vartheta_{(2)}$, $\vartheta_{(3)}$ denote the stochastic flows of the following equations:

$$dr(\tau)=(b(r(\tau))+a'(r(\tau)))d\tau+\sqrt{2a(r(\tau))}dW(\tau)$$

and

$$dr(\tau)=(b(r(\tau))+2a'(r(\tau)))d\tau+\sqrt{2a(r(\tau))}dW(\tau).$$

Since the price of a bond cannot be negative by Equation (4), the first line in Equation (B1) and the first line in Equation (B2) ensure that w^2 is strictly negative and by Lemma A1 [see the second relation in Equation (A2)], this is also true in the stochastic volatility case under the restrictions stated in the proposition. ■

Proof of Proposition 1 (*convexity issues*). The second claim of the proposition is confirmed by the second line in Equation (B2), since w^3 can be written as

$$w^3(x,t,T)=\mathbb{E}\left\{\int_t^T \kappa^3(t,\tau)(2-b''(\vartheta_{(3)}(\tau)))(-w^2(\vartheta_{(3)}(\tau),\tau,T))\mathrm{d}\tau\right\}.$$

As regards the final claim of the proposition, use the Fubini–Tonelli theorem (see note 6) and write the previous relation as

$$w^3(x,t,T)=\left\{\int_t^T G_2(x,t,\tau,T)d\tau\right\}\cdot\left\{2-\frac{\int_t^T G_1(x,t,\tau,T)d\tau}{\int_t^T G_2(x,t,\tau,T)d\tau}\right\},$$

where

$$\begin{cases} G_1(x,t,\tau,T)\equiv-\mathbb{E}\{\mathrm{b}''(\vartheta_{(3)}(\tau))\cdot\kappa^3(t,\tau)\cdot w^2(\vartheta_{(3)}(\tau),\tau,T)\} \\ G_2(x,t,\tau,T)\equiv-\mathbb{E}\{\kappa^3(t,\tau)\cdot w^2(\vartheta_{(3)}(\tau),\tau,T)\}. \end{cases}.$$

Writing w^3 as before is justified because there exists a maturity date $T>t$ within which $\int_t^T G_2\neq 0$. We have

$$w^3(x,t,T)>0\ (<0)\quad \text{if } \zeta(x,t,T)\equiv\frac{\int_t^T G_1(x,t,\tau,T)d\tau}{\int_t^T G_2(x,t,\tau,T)d\tau}<2\ (>2), \tag{B3}$$

and

$$\begin{aligned}
&\lim_{T\downarrow t}\zeta(x,t,T)\\
&=\lim_{T\downarrow t}\frac{\int_t^T \mathbb{E}\left\{\kappa^3(t,\tau)b''(\vartheta_{(3)}(\tau))\cdot\mathbb{E}\left\{\left(\int_\tau^T \frac{\partial\vartheta}{\partial \mathrm{x}}(u)du\right)\exp\left(-\int_\tau^T\vartheta(u)du\right)\Big/\mathcal{F}(\tau)\right\}\right\}\mathrm{d}\tau}{\int_t^T \mathbb{E}\left\{\kappa^3(t,\tau)\cdot\mathbb{E}\left\{\left(\int_\tau^T \frac{\partial\vartheta}{\partial \mathrm{x}}(u)du\right)\exp\left(-\int_\tau^T\vartheta(u)du\right)\Big/\mathcal{F}(\tau)\right\}\right\}\mathrm{d}\tau}\\
&=\lim_{T\downarrow t}\frac{\frac{1}{T-t}\int_t^T \mathbb{E}\left\{\kappa^3(t,\tau)b''(\vartheta_{(3)}(\tau))\cdot\frac{\partial}{\partial T}\mathbb{E}\left\{\left(\int_\tau^T \frac{\partial\vartheta}{\partial \mathrm{x}}(u)du\right)\exp\left(-\int_\tau^T\vartheta(u)du\right)\Big/\mathcal{F}(\tau)\right\}\right\}\mathrm{d}\tau}{\frac{1}{T-t}\int_t^T \mathbb{E}\left\{\kappa^3(t,\tau)\cdot\frac{\partial}{\partial T}\mathbb{E}\left\{\left(\int_\tau^T \frac{\partial\vartheta}{\partial x}(u)du\right)\exp\left(-\int_\tau^{\mathrm{T}}\vartheta(u)du\right)\Big/\mathcal{F}(\tau)\right\}\right\}}\\
&=b''(x)\ \text{for each } x\colon |b''(x)|<\infty,
\end{aligned} \tag{B4}$$

by Equation (5); the L'Ĥopital's rule; the Liebnitz's rule (noting also that the integrands in the numerator and denominator of ζ evaluated at T are both zero); the Lebesgue's dominated convergence theorem; continuity and differentiability of $\int^T(\partial\vartheta/\partial x)\ \exp(-\int^T\vartheta)$ w.r.t. T (ensured by the fact that ϑ and $\partial\vartheta/\partial x$ have continuous sample paths); the fact that

$$\lim_{T\downarrow t}\frac{\partial}{\partial T}\left\{\left(\int_t^T\frac{\partial\vartheta}{\partial x}(u)du\right)\cdot\exp\left(-\int_t^T\vartheta(u)du\right)\right\}=\frac{\partial\vartheta}{\partial x}(t)=1,$$

by Lemma A2; and finally because $\lim_{T\downarrow t}\kappa^3(t,T)=1$, by Lemma A1. This shows that there exists a $\hat{T}$ depending on x such that $\forall\tau\in[t,\hat{T}]$,

$$\zeta(x,t,\hat{T}(x))<2\ (>2)\quad\text{for each } x: b''(x)<2\ (>2),$$

and the result follows from Equation (B3).

In the stochastic volatility case, the same arguments can be made whenever there exists a maturity date T_* such that $w^2<0$ for all $T\leq T_*$ (which is ensured by Proposition 2 shown below); by Lemma A1; and finally by a strategy of proof similar to the previous one and revealing that the various integrals involving u_2, u_{22}, u_{122}, u_{12} are dominated by the integral involving u_1 [as in Equation (C2) in Appendix C]. ■

A counter example to Proposition 1: the double square-root model. According to the original formulation of Longstaff (1989), the short-term rate is the solution to

$$dr(\tau) = \left(\frac{\eta^2}{4} - \beta\sqrt{r(\tau)} + \lambda r(\tau)\right)d\tau + \eta\sqrt{r(\tau)}dW(\tau), \beta, \lambda, \eta > 0. \tag{B5}$$

Thus formulated, this model implies that its "scale measure" fails to diverge at both boundaries (zero and infinite) [see Karlin and Taylor (1981, chap. 15) and Mele (2002, assumption A2-(H3)]. The infinite boundary can be attained in finite expected time because the term λr dominates the term $\beta\sqrt{r}$ as $r \to \infty$. To avoid this, one may assume that $\lambda < 0$, but this creates a negative term premium whenever $u_1 < 0$. This is the technical reason for which only the case $\lambda = 0$ was considered in Section 3. The drift function considered in the main text was chosen because the origin is regular (attainable) in Equation (B5) even with $\lambda = 0$ and in this case,

$$k^2 = \vartheta_{(2)} + \beta/\left(2\sqrt{\vartheta_{(2)}}\right)$$

in Equation (B1) for w^2. It can be shown that the origin is regular for the auxiliary process $\vartheta_{(2)}$ too, which makes k^2 explode in finite expected time. Therefore no Feynman–Kac stochastic representation for w^2 is possible if r is generated by Equation (B5). In fact, Longstaff (1989, p. 203) shows that $w^2 > 0$ for small values of $r(t) = x$. However, this is not concluding evidence of the violation of the "no-crossing property" [in the sense of Bergman, Grundy, and Wiener (1996)], since no Feynman–Kac stochastic representation for u necessarily exists. In contrast, if the short-term rate is a double square-root process of the form indicated in the main text, both boundaries cannot be attained in finite expected time, and the result that $w^2 < 0$ is restored.

Comparison theory. In the scalar diffusion case, it is also possible to use powerful *comparison results* [e.g., Karatzas and Shreve (1991, pp. 291–295)] to relate very simply bond prices to the location of the short-term rate drift function. Consider two economies A and B in which the corresponding short-term rates r^A and r^B are solutions to

$$dr^j(\tau) = b^j(r^j(\tau))d\tau + \sqrt{2a(r^j(\tau))}dW(\tau), \quad r^j(t) = x^j, \;\; j = A, B,$$

and suppose that $x^A \leq x^B$. Let the bond prices in the two economies be given by u^A and u^B. Under conditions given, for instance, in Karatzas and Shreve (1991, proposition 2.18, p. 293),

$$\Pr\{r^A(\tau) \leq r^B(\tau), \tau \in [t, \infty)\} = 1,$$

whenever $b^A(r) \leq b^B(r)$, $r \in \mathbb{R}_{++}$.[12] Combining this with Equation (4) reveals that $u^A \geq u^B$. Since the thought experiment of a permanent shift in b can be interpreted as a permanent change of the unit risk premium Λ^1 [see Equation (2)], the previous result also means that if we had to visit two economies differing only in the amount of this risk premium, we would observe a higher level of the yield curve in the more risk-premium demanding economy. While the previous result is intuitive, it must be pointed out that it does not need to hold in more complicated diffusion settings [see Mele (2002, section 7.1)]. Furthermore, it is surprising that no proof of it was available within the same general framework used here. As an example, CIR (1985, p. 393) pointed out that bond prices go up when (minus) the market

[12] This result is still valid when one relaxes a Lipschitz condition on one of the two b^j at the expense of strengthening the condition $b^A \leq b^B$ to $b^A < b^B$ [see Karatzas and Shreve (1991, exercise 2.19, p. 294)].

risk-premium goes down, but their observation concerned the specific case of their celebrated one-factor affine model.[13]

Even in the scalar setting, elegant comparison results cannot be used to implement comparative statics relating bond prices to volatility. Nevertheless, it is not hard to show that two economies A and B for which $a^A(r) > a^B(r)$, all $r \in \mathbb{R}_{++}$, generate a price difference $\nabla u(r, \tau, T) \equiv u^A(r, \tau, T) - u^B(r, \tau, T)$ which is the solution to

$$\begin{aligned} 0 = & \frac{\partial}{\partial \tau} \nabla u + (\hat{b} + \sqrt{2a^A} \Lambda_A^1) \nabla u_1 + a^A \nabla u_{11} - r \nabla u \\ & + \left\{ \left(\sqrt{2a^A} (\Lambda_A^1 - \Lambda_B^1) + (\sqrt{2a^A} - \sqrt{2a^B}) \Lambda_B^1 \right) u_1^B + (a^A - a^B) u_{11}^B \right\}, \end{aligned} \tag{B6}$$

with $\nabla u(r, T, T) = 0\ \forall r \in \mathbb{R}_{++}$ [see Mele (2002, subsection 3.3)]. Suppose that the market risk-premium $\sqrt{2a}\Lambda^1$ is positively valued and increases with volatility. Given Lemma A3, the previous equation reveals that at short maturity dates, $\nabla u < 0$ by a direct application of the maximum principle.[14] In fact, in Mele (2002, Appendix B). I show that in the presence of a sufficiently high level of the risk premium Λ, $\nabla u < 0$ at any arbitrary maturity date. See Mele (2002, section 7) for other comparative statics results in multidimensional settings.

Proof of Proposition 2. By Lemma A1, and the Fubini–Tonelli theorem (see note 6),

$$w^2(x, s, t, T) = \int_t^T \{J_2(x, s, t, \tau, T) + J_3(x, s, t, \tau, T) - J_1(x, s, t, \tau, T)\} d\tau, \tag{B7}$$

where

$$\begin{cases} J_1(x, s, t, \tau, T) \equiv \mathbb{E}\{\kappa^2(t, \tau) \cdot u(\vartheta_{(2)}(\tau), \phi_{(2)}(\tau), \tau, T)\} \\ J_2(x, s, t, \tau, T) \equiv \mathbb{E}\{\kappa^2(t, \tau) \cdot \varphi_1(\vartheta_{(2)}(\tau), \phi_{(2)}(\tau)) \cdot u_2(\vartheta_{(2)}(\tau), \phi_{(2)}(\tau), \tau, T)\} \\ J_3(x, s, t, \tau, T) \equiv \mathbb{E}\{\kappa^2(t, \tau) \cdot \psi_1(\vartheta_{(2)}(\tau), \phi_{(2)}(\tau)) \cdot u_{22}(\vartheta_{(2)}(\tau), \phi_{(2)}(\tau), \tau, T)\}. \end{cases}$$

Therefore, $w^2(x, s, t, T) < 0$ whenever

$$\xi(x, s, t, T) \equiv \frac{\int_t^T J_2(x, s, t, \tau, T) d\tau}{\int_t^T J_1(x, s, t, \tau, T) d\tau} + \frac{\int_t^T J_3(x, s, t, \tau, T) d\tau}{\int_t^T J_1(x, s, t, \tau, T) d\tau} < 1,$$

[13] The referee pointed out that in a thought experiment in which the unit risk-premium changes, it is possible that the *physical* drift function also changes. I may illustrate such a remark with the help of the CIR example reported in note 14. There $\Lambda(\cdot) = \epsilon\sqrt{r(\cdot)}/v$, where ϵ and v are constants, and a change in ϵ (say) makes both the volatility and the *physical* drift functions change. To apply the previous comparison results, one has then to think of a change in ϵ, say, as one that is exactly counterbalanced by changes in other parameters [say h, b or k (see note 14)] that keep volatility and physical drift unchanged. Without this kind of interpretation in mind, comparative statics results such as the previous ones may only have a *partial equilibrium* flavor.

[14] CIR (1985, p. 393–394) state that bond prices are an increasing function of the volatility parameter in their single-factor model because they define (as is customary) a market risk premium $\sqrt{2a}\Lambda \equiv \lambda r$ that is not literally taken to be proportional to the volatility parameter [see their Equation (22)]. In terms of Equation (B6), this implies that volatility affects bond prices only through convexity terms. However, using the framework (and some notation) in Duffie (1996, p. 230–233), one finds that a supporting equilibrium for the CIR model generates: $dr(\tau) = \{bv^2 + (k\epsilon + \kappa)r(\tau)\}d\tau + k \cdot v\sqrt{r(\tau)}dW(\tau)$ and $\Lambda(\tau) = \epsilon\sqrt{r(\tau)}/v$, where $\epsilon > 0$ (to ensure positive term-premia), $v \equiv \sqrt{h - \epsilon^2}$, and b, h, κ, k are constants (a similar analysis can be conducted with the original CIR article). Therefore, here a thought experiment of an increase in the short-term volatility that is *unambiguously interpreted only as a change in volatility* (and not also as a change in the short-term rate drift under the *physical* measure and/or Λ) corresponds to a change in k (in Duffie, k represents the volatility parameter of the primitive state process of the economy, that is, the "shock" process affecting capital productivity).

and since J_i, $i=1, 2, 3$, are all continuous with respect to τ and T, and

$$\lim_{T\downarrow t}\frac{\frac{1}{T-t}\int_t^T J_i(x,s,t,\tau,T)d\tau}{\frac{1}{T-t}\int_t^T J_1(x,s,t,\tau,T)d\tau}=0,\quad i=2,3,$$

then there exists a T_* depending on x, s such that $\forall T\leq T_*(x,s), \xi(x,s,t,T)<1$. ■

Uniform bounds for T_*. Are there situations in which there exist values of T_* in Proposition 2 that are independent on the initial state (x, s)? The answer is definitely positive in the case of affine models, that is, when the bond price functions are of the form $u(x,s,t,T)=\exp\left(B(T-t)+C(T-t)\cdot x+D(T-t)\cdot s\right)$, where B, C, and D are not functions of the initial state (x, s), and $\lim_{T\downarrow t} B=\lim_{T\downarrow t} C=\lim_{T\downarrow t} D=0$ (boundary conditions for u). A rigorous proof is as follows. In affine models, functions $\varphi_1(\cdot,\cdot)$ and $\psi_1(\cdot,\cdot)$ reduce to two constants that with a slight abuse of notation I shall refer to as φ_1 and ψ_1. Equation (B7) can then be written as

$$w^2(x,s,t,T)=\int_t^T J_1(x,s,t,\tau,T)\{\varphi_1\cdot D(T-\tau)+\psi_1\cdot D(T-\tau)^2-1\}d\tau,$$

and since J_1 is always positive, $w^2<0$ for all maturity dates $T_{**}:\varphi_1\cdot D(T_{**}-\tau)+\psi_1\cdot D(T_{**}-\tau)^2<1$, all $\tau\in[t, T_{**}]$, independently of x, s. ■

Conditions and examples for bond prices to be increasing in the short-term rate at medium-long maturity dates. Here the starting point is Equation (B7), which clarifies why cases (b) and (d) in Table 1, for instance, are necessary conditions for $\partial u/\partial r>0$ at medium-long maturity dates when $\partial\psi/\partial r\leq 0$. In the general case, one has that $\partial u/\partial r>0$ at medium-long maturity dates whenever $\int h^2>0$ in Lemma A1.

This kind of condition can be illustrated within the LS model [Equation (10)], which has a known closed-form solution that is "typical" of all models examined in Section 3 (see below). As an example, LS (1992, p. 1267) noticed that in their model, $\partial u/\partial r$ is always negative for small $T-\tau$, but can become positive for bonds with longer maturities. This perfectly illustrates Proposition 2. LS also point out $\partial u/\partial y$ can be of either sign, or can be positive within certain maturity dates, negative at the remaining maturity dates, and vice versa. Let us see how the theory in Section 3 may help to clarify such phenomena.

According to the notation introduced by Longstaff and Schwartz (1992, p. 1264; and Equation (9), p. 1263), Equation (10) describes the dynamics of r, y under the risk-neutral measure with coefficients $b_0=\alpha\gamma+\beta\eta, b_1=(\beta\delta-\alpha\nu)/(\beta-\alpha)$, $b_2=(\delta-\nu)/(\beta-\alpha)$, $\varphi_0=\alpha^2\gamma+\beta^2\eta$, $\varphi_1=\alpha\beta\,(\nu-\delta)/(\beta-\alpha)$, $\varphi_2=(\beta\nu-\alpha\delta)/(\beta-\alpha)$, where δ,ν,γ,η are constants entering the primitive dynamical system of the model. Therefore it is not hard to find that the equilibrium price satisfies

$$\begin{cases} 0=\left(\dfrac{\partial}{\partial\tau}+L-r\right)u, & \forall(r,y,\tau)\in\mathbb{D}_{\alpha,\beta}\times[t,T) \\ u(r,y,T,T)=1, & \forall(r,y)\in\mathbb{D}_{\alpha,\beta}\end{cases}\tag{B8}$$

where, for given constants ψ_j and ρ_j, $j=1, 2$, given below,

$$\begin{aligned}Lu(r,y,\tau,T)=&\ (b_0-b_1r+b_2y)u_1(r,y,\tau,T)+(\varphi_0+\varphi_1r-\varphi_2y)u_2(r,y,\tau,T)\\ &+\frac{y}{2}\cdot u_{11}(r,y,\tau,T)+(\psi_1r+\psi_2y)u_{22}(r,y,\tau,T)+(\rho_3r+\rho_4y)u_{12}(r,y,\tau,T),\end{aligned}$$

and $\mathbb{D}_{\alpha,\beta}=\{(r,y)\in\mathbb{R}_{++}\times\mathbb{R}_{++}:(y/r)\in(\alpha,\beta)\}$. Under mild parameter restrictions such as those given in LS (1992, note 9, p. 1264), the process $(y(\tau)/r(\tau))_{\tau\geq 0}$ cannot attain the boundary $\partial\mathbb{D}_{\alpha,\beta}$ in finite expected time, and so no further transversality and/or boundary condition is needed.

The solution to Equation (B8) reported by the authors [Equation (20), p. 1266] (with coefficients ψ_i and ρ_i given by $\psi_1 = -\alpha\beta(\beta+\alpha)/2, \psi_2 = (\beta^3-\alpha^3)/(2(\beta-\alpha))$, $\rho_3 = -\alpha\beta$, $\rho_4 = \beta+\alpha$) is

$$u(x, s, t, T) = A(T-t)^{2\gamma} \cdot B(T-t)^{2\eta} \cdot \exp(\kappa \cdot (T-t) + C(T-t) \cdot x + D(T-t) \cdot s), \qquad \text{(B9)}$$

where

$$\begin{cases} A(\tau) = 2\phi/((\delta+\phi)(\exp(\phi\tau)-1)+2\phi) \\ B(\tau) = 2\bar{\psi}/((\nu+\bar{\psi})(\exp(\bar{\psi}\tau)-1)+2\bar{\psi}) \\ C(\tau) = (\alpha\phi(\exp(\bar{\psi}\tau)-1)B(\tau) - \beta\bar{\psi}(\exp(\phi\tau)-1)A(\tau))/(\phi\bar{\psi}(\beta-\alpha)) \\ D(\tau) = (\bar{\psi}(\exp(\phi\tau)-1)A(\tau) - \phi(\exp(\bar{\psi}\tau)-1)B(\tau))/(\phi\bar{\psi}(\beta-\alpha)) \end{cases}$$

and $\kappa = \gamma(\delta+\phi) + \eta(\nu+\bar{\psi})$, $\phi = \sqrt{2\alpha+\delta^2}$, $\bar{\psi} = \sqrt{2\beta+\nu^2}$.

Functions C and D are factor loadings of the short-term rate and volatility, respectively. To compute them, I first use the estimates reported in Longstaff and Schwartz (1993, exhibit 3, p. 10), which are $\alpha = 0.001149$, $\beta = 0.1325$, $\delta = 0.05658$, $\nu = 0.335$. In Table A.1, "case I," the previous figures are used to compute four important coefficients of Equation (10); "case II," instead, reports coefficients computed using ad hoc chosen coefficients: $\alpha = 0.10$, $\beta = 0.13$, $\delta = 0.55$, $\nu = 0.33$; the column corresponding to "eigenvalues" reports the eigenvalues of matrix $\begin{pmatrix} -b_1 & b_2 \\ \varphi_1 & -\varphi_2 \end{pmatrix}$.

Figure 2 depicts the two factor loadings in these two cases. Consistently with Proposition 2, $C \leq 0$ at short maturity dates. In case II, $C > 0$ at medium-long maturity dates. To see this with the methods of Section 3, notice that for Equation (10),

$$h^2 = -u + \varphi_1 u_2 + \psi_1 u_{22} = (\varphi_1 D + \psi_1 D^2 - 1)u,$$

and since $\lim_{T \downarrow t} D = 0$, u_1 can never be positive at short maturity dates. Because $\psi_1 < 0$, a necessary condition for $C = w^2/u = (\int h^2)/u > 0$ at longer maturity dates is that $\varphi_1 \cdot u_2 > 0$. In case II, φ_1 and u_2 are both negative, and as Figure 2 shows, u_1 becomes positive at medium-long maturity dates. In case I, φ_1 and u_2 are both positive; given the small value of φ_1, however, $u_1 \leq 0$ always. More on the role of φ_1 in this kind of models below.

To understand the sign of u_2 in the two cases, consider Proposition 3. In case I, $u_1 < 0$, $b_2 < 0$ and, by affinity of the LS model, $u_{11} > 0$: by Equation (12), then, bond prices can never be decreasing in volatility. In case II, $b_2 > 0$ and in addition, slope effects dominate convexity effects at short maturity dates: therefore, bond prices are decreasing in volatility at short maturity dates, which is exactly Proposition 3. Given the parameter values of Table A.1, it also turns out that slope effects dominate convexity effects even at long maturity dates, thus making bond prices react negatively to volatility changes even at long maturity dates.

To examine further the role of the volatility drift function in qualifying how bond prices react to the short-term rate at medium-long maturity dates, consider a toy, *parametrized* model with $\psi_1(r, y) = 0$ and a semilinear volatility drift function:

$$\varphi(r, y) = \varphi_1 r + \hat{\varphi}(y),$$

where φ_1 is a constant and $\hat{\varphi}$ is a well-defined function. By Lemma A1,

Table A.1

	b_1	b_2	φ_1	φ_2	Eigenvalues
Case I	5.4145×10^{-2}	-2.1197	3.227×10^{-4}	0.3374	-5.658×10^{-2}; -0.3349
Case II	1.2833	7.3333	-9.5333×10^{-2}	-0.4033	-0.5497; -0.3302

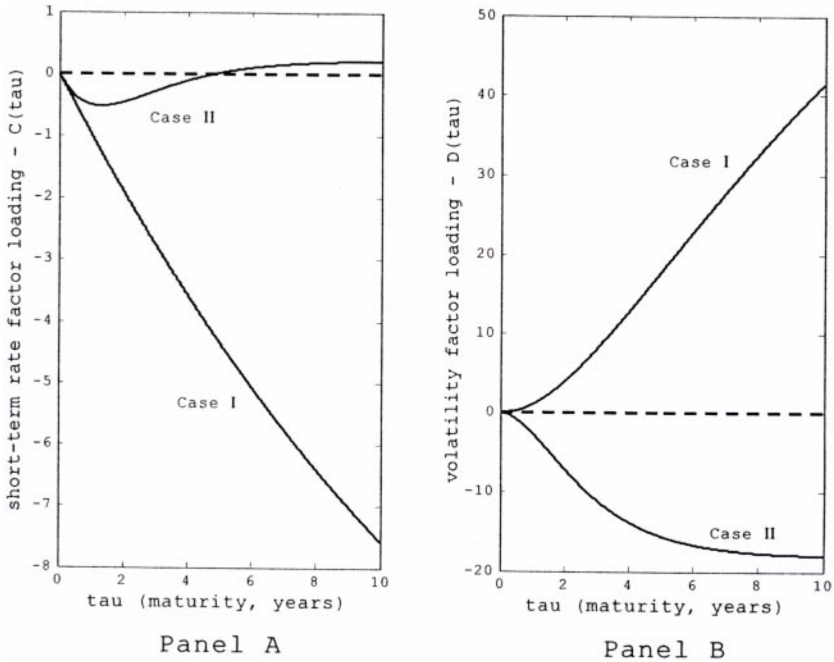

Figure 2
Factor loadings for Equations (10)
The two graphs depict factor loadings for the pricing equation [Equation (B9)]. Panel A depicts the short-term rate factor loading C under case I and case II of Table 3. Panel B shows the volatility factor loading D, also computed under case I and case II of Table 3. When $C<0$ (>0), bond prices are negatively (positively) related to random changes of the short-term rate. When $D>0$ (<0), bond prices are positively (negatively) related to random volatility changes.

$$w^2(x,s,t,T;\varphi_1) = -\mathbb{E}\left\{\int_t^T \kappa^2(t,\tau;\varphi_1)\cdot u(\vartheta_{(2)}(\tau),\phi_{(2)}(\tau),\tau,T;\varphi_1)\cdot d\tau\right\}$$
$$+\varphi_1\cdot\mathbb{E}\left\{\int_t^T \kappa^2(t,\tau;\varphi_1)\cdot w^1(\vartheta_{(2)}(\tau),\phi_{(2)}(\tau),\tau,T;\varphi_1)\cdot d\tau\right\},$$

where I have emphasized the dependence of the various functions on φ_1. A sufficient condition for $w^2>0$ at an arbitrarily small but strictly positive maturity date T is that $|\varphi_1|$ may be increased without bounds and the remaining parameters of the model may at the same time be smoothly changed in such a way that

$$\lim_{|\varphi_1|\to\infty}|\Phi(\varphi_1)|<\infty, \tag{B10}$$

where

$$\Phi(\varphi_1)\equiv\frac{\mathbb{E}\left\{\int_t^T \kappa^2(t,\tau;\varphi_1)\cdot u\left(\vartheta_{(2)}(\tau),\phi_{(2)}(\tau),\tau,T;\varphi_1\right)\cdot d\tau\right\}}{\mathbb{E}\left\{\int_t^T \kappa^2(t,\tau;\varphi_1)\cdot w^1\left(\vartheta_{(2)}(\tau),\phi_{(2)}(\tau),\tau,T;\varphi_1\right)\cdot d\tau\right\}},$$

a continuous function. Indeed, if Equation (B10) holds, Φ has at least one fixed point because

$$\lim_{|\varphi_1|\to 0+}|\Phi(\varphi_1)|<\infty.$$

Assume that a single fixed point exists [multiple fixed points can be coped with using the strategy of proofs in Appendix B of Mele (2002)]. One can then take as a critical value of φ_1 the fixed point

$$\varphi_1^* = \Phi(\varphi_1^*). \tag{B11}$$

If $\varphi_1^* < 0$ (> 0), bond prices are increasing in the short-term rate at any arbitrarily small maturity date $\tilde{T}$ for any $\varphi_1 < \varphi_1^*$ ($> \varphi_1^*$). Of course, all such φ_1 will be functions of x, s, $\tilde{T}$.

Proof of Proposition 3.[15] I only provide the proof of the weak term-structure *augmenting* volatility property, the proof of the weak term structure *decreasing* volatility property being nearly identical. Furthermore, I first present a heuristic proof that helps to develop intuition on the main features of Proposition 3, and that is based on some regularity conditions. I then prove Proposition 3 in full generality.

Suppose first that $w^3 > 0$ for all $T \in (t, T^{**})$, where $T^{**} < \infty$ is given, and that $w^2 < 0$ for all $T \in (t, T^{**})$. Also, assume that $\sigma_2(x, s) > 0$ (the case $\sigma_2(x, s) < 0$ is treated similarly), and that for any $T \in (t, T^{**}), (-u_{11}/u_1)(x, s, t, T)$ is $((x, s)-)$ uniformly bounded. Then there exists a continuous function $K(t, \cdot)$ such that $\forall (r, y) \in \mathbb{R}_{++} \times \mathbb{R}, (-u_{II}/u_I)(r, y, \tau, T) \leq K(\tau, T)$ and that, by Lemma A3,

$$\lim_{T \downarrow t} K(t, T) = 0. \tag{B12}$$

Next, suppose that there exists a maturity date within which

$$\begin{cases} F_1(x, s, t, \tau, T) \equiv -\mathbb{E}\{\kappa^1(t, \tau) \cdot \sigma_2(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau)) \cdot u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T)\} \\ F_2(x, s, t, \tau, T) \equiv \dfrac{\mathbb{E}\{\kappa^1(t, \tau) \cdot u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T) \cdot \mathrm{b}_2(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau))\}}{\mathbb{E}\{\kappa^1(t, \tau) \cdot u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T) \cdot \sigma_2(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau))\}} \end{cases}$$

are well defined. By the Fubini–Tonelli theorem (see note 6),

$$\forall T \in (t, T^{**}), \quad w^1(x, s, t, T) \leq \int_t^T F_1(x, s, t, \tau, T) \cdot \{K(\tau, T) - F_2(x, s, t, \tau, T)\} d\tau, \tag{B13}$$

and by applying the same kind of arguments used to derive Equation (B4),

$$\begin{aligned} \lim_{T \downarrow t} F_2(x, s, t, \tau, T) &= \lim_{T \downarrow t} \frac{\mathbb{E}\{\kappa^1(t, \tau) \cdot u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T) \cdot b_2(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau))\}}{\mathbb{E}\{\kappa^1(t, \tau) \cdot u_1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \tau, T), \sigma_2(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau))\}} \\ &= \frac{b_2(x, s)}{\sigma_2(x, s)} > 0. \end{aligned} \tag{B14}$$

Combining Equation (B12) with Equation (B14) shows that there exists a T^* depending on x, s such that $T^* < T^{**}$ and $\forall \tau \in [t, T^*], K(\tau, T^*) \leq F_2(x, s, t, \tau, T^*)$. The result then follows by Equation (B13). The case $w^3 \leq 0$ is trivial.

The previous case makes clear how Lemma A3 is related to the bond price reaction to random volatility changes at short-maturity dates. When $(u_{11}/u_1)(x, s, t, T)$ is not $((x, s)-)$ uniformly bounded and the regularity conditions underlying Equation (B13) are not easy to check (these conditions are always satisfied by affine models), the proof is similar to the proof

[15] An alternative proof based on the derivative of the yield curve at the origin is available upon request from the author. Unfortunately the method of such an alternative proof cannot be used to show any of the other results appearing in this article. Specifically, such an alternative method of proof does not make it possible to uncover the important role that slope and convexity issues have in explaining the origins of the strong versions of the term structure augmenting (decreasing) volatility property [see Equation (12), Proposition 5, Appendices B and C, and all of the model examples worked out in Sections 3 and 4)]. Even the intuition about the origins of the phenomenon described in Proposition 3 can only be obtained by using the method of proof presented here (see the discussion in Section 2 on the connection of bond prices convexity to the Rothschild–Stiglitz–Jagannathan theory).

of Equation (B4) and so here it will be sketchy. For each $T \in (t, T^{**})$, rewrite w^1 in Lemma A1 as

$$w^1(x,s,t,T) = \mathbb{E}\left\{\int_t^T (\kappa^1(-u_1))(\tau)d\tau\right\}\left\{\frac{\mathbb{E}\{\int_t^T(\kappa^1\sigma_2 u_{11})(\tau)d\tau\}}{\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)\mathrm{d}\tau\}} - \frac{\mathbb{E}\{\int_t^T(\kappa^1 b_2(-u_1))(\tau)d\tau\}}{\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)d\tau\}}\right\},$$

and conclude by showing that $\lim_{T\downarrow t}\mathbb{E}\{\int_t^T(\kappa^1\sigma_2 u_{11})(\tau)d\tau\}/\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)d\tau\} = 0$ and $\lim_{T\downarrow t}\mathbb{E}\{\int_t^T(\kappa^1 b_2(-u_1))(\tau)d\tau\}/\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)d\tau\} = b_2(x,s) > 0$ [use (1) the Fubini–Tonelli theorem (see note 6), and the L'Hôpital's rule; (2) the Liebnitz's rule; (3) the Lebesgue's dominated convergence theorem; (4) $\lim_{T\downarrow t}(\partial u_1/\partial T) = -1$, $\lim_{T\downarrow t}(\partial u_{11}/\partial T) = 0$; and (5) Lemma A2], and noting that there exists a maturity date within which 1: $\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)d\tau\} \geq 0$, with equality as $T\downarrow t$.

Finally, relax the assumption that $w^2 < 0$ for all $T \in (t, T^{**})$. By Proposition 2, there still exists a maturity date $T_{**}(x,s)$ within which $\mathbb{E}\{\int_t^T(\kappa^1(-u_1))(\tau)d\tau\} \geq 0$ (with equality as $T\downarrow t$), and the proof is complete by repeating the same arguments produced before. ■

Proof of Equation (13). Let $\nu \equiv \partial\vartheta / \partial x$ denote the first partial of the short-term rate flow $\vartheta(\tau)$ with respect to the initial condition $\vartheta(t) = x$. A simple computation reveals that $\nu(\tau) = \exp(-\theta(\tau - t))$. Substituting this into Equations (5) and (6) leaves $u_1(x,s,t,T)/u(x,s,t,T) = ((1-\exp(-\theta(T-t)))/\theta$ and $u_{11}(x,s,t,T)/u(x,s,t,T) = ((1-\exp(-\theta(T-t)))/\theta)^2$. Substituting these expressions of u_1 and u_{11} into Equation (7) and using Lemma A1 enable one to conclude that $u_2(x,s,t,T) < 0$ whenever $\lambda_1 \geq \{1-\exp(-\theta(T-t))\}/(2\theta)$. Equation (13) then follows immediately. ■

Appendix C: Proofs for Section 4

Proof of Propositions 4 and 5 (sketchy). I use the same arguments used to show Proposition 3 and Equation (12). Specifically, I find that $w^1 \equiv u_2$ and $w^2 \equiv u_1$ satisfy

$$\begin{cases} 0 = \left(\dfrac{\partial}{\partial\tau} + L^i - k^i\right)w^i(\vartheta,\phi,\chi,\tau,T) + h^i(\vartheta,\phi,\chi,\tau,T), & \forall(\vartheta,\phi,\chi,\tau) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathbb{R}\times[t,T) \\ w^i(\vartheta,\phi,\chi,T,T) = 0, & \forall(\vartheta,\phi,\chi) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathbb{R}, \end{cases} \tag{C1}$$

where L^i ($i = 1,2$) are partial differential operators, and $h^1 = b_2u_1 + \sigma_2u_{11} + \varepsilon_2u_3 + \pi_2u_{33} + [\sum_{j=1}^3 \partial(\sigma^{(j)}\pi^{(j)})/\partial y]\cdot u_{13}$, $k^1(\vartheta,\phi,\chi) = \vartheta - \varphi_2(\vartheta,\phi,\chi)$ $h^2 = \varphi_1u_2 + \psi_1u_{22} + \varepsilon_1u_3 + \pi_1u_{33} + [\sum_{j=1}^3 \partial(\psi^{(j)}\pi^{(j)})/\partial r]\cdot u_{23} - u$, and $k^2(\vartheta,\phi,\chi) = \varepsilon - b_1(\vartheta,\phi,\chi)$ [(see Mele (2002, Appendix D) for additional details]. Proposition 4 (bounded case) is then proved thanks to the following results leading to a generalization of Lemma A3: for each $(x,s,c) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathbb{R}$,

$$\begin{aligned} &\lim_{T\downarrow t}\frac{u_{12}}{u_1}(x,s,c,t,T) \\ &= \lim_{T\downarrow t}\frac{\mathbb{E}\left\{\left[\frac{1}{T-t}\int_t^T\frac{\partial^2\vartheta}{\partial x\partial s}(\tau)d\tau - (T-t)\left(\frac{1}{T-t}\int_t^T\frac{\partial\vartheta}{\partial x}(\tau)d\tau\right)\left(\frac{1}{T-t}\int_t^T\frac{\partial\vartheta}{\partial s}(\tau)d\tau\right)\right]\exp\left(-\int_t^T\vartheta(\tau)d\tau\right)\right\}}{\mathbb{E}\left\{\left(\frac{1}{T-t}\int_t^T\frac{\partial\vartheta}{\partial x}(\tau)d\tau\right)\exp\left(-\int_t^T\vartheta(\tau)d\tau\right)\right\}} \\ &= 0, \end{aligned} \tag{C2}$$

by Lebesgue's dominated convergence theorem and the fact that $\partial\vartheta(t)/\partial x = 1$, $\partial^2\vartheta(t)/\partial x\partial s = 0$, and $\partial\vartheta(t)/\partial s = 0$ (by a straightforward generalization of Lemma A2). The proofs for the other partial derivatives are similar. The unbounded case is treated similarly as in Proposition 3. Finally, Proposition 5 also follows from Equation (C1).

Analysis of Equations (14) and (17). First I show the claim in the main text that Equation (19) is sufficient to guarantee that bond prices are increasing in volatility at any finite maturity date whenever $\lambda_A + \lambda_B\sigma_{rv} > 0$, $\lambda_A\sigma_{\theta r} < 0$ and $\kappa > \nu$ in Equations (14). According to Proposition 5, it is sufficient to show that for any $(r, y, \ell, \tau) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [t, T)$,

$$h^1 = -(\lambda_A + \lambda_B\sigma_{rv})u_1 + \frac{1+\sigma_{rv}^2\eta^2}{2}u_{11} - \lambda_A\sigma_{\theta r}u_3 + \frac{\sigma_{\theta r}^2}{2}u_{33} + \sigma_{\theta r}u_{13} > 0. \tag{C3}$$

Let $m_{\vartheta,\,x} \equiv \partial\vartheta/\partial x$ and $m_{\vartheta,\,c} \equiv \partial\vartheta/\partial c$ denote the sensitivity of the interest rate flow to initial conditions $r(\tau) = x$ and $\chi(\tau) = c$. We have $u_1(\cdot) = -(\int_\tau^T m_{\vartheta,x}(v)dv)u(\cdot)$, $u_{11}(\cdot) = (\int_\tau^T m_{\vartheta,x}(v)dv)^2 u(\cdot)$, $u_3(\cdot) = -(\int_\tau^T m_{\vartheta,c}(v)dv)u(\cdot)$, $u_{33}(\cdot) = (\int_\tau^T m_{\vartheta,c}(v)dv)^2 u(\cdot)$, $u_{13}(\cdot) = (\int_\tau^T m_{\vartheta,x}(v)dv)(\int_\tau^T m_{\vartheta,c}(v)dv)u(\cdot)$, where

$$\begin{cases} \displaystyle\int_\tau^T m_{\vartheta,x}(v)dv = \frac{1-\exp(-\kappa(T-\tau))}{\kappa} \\ \displaystyle\int_\tau^T m_{\vartheta,c}(v)dv = \left(\int_\tau^T m_{\vartheta,x}(u)du\right)\cdot\frac{\kappa}{\kappa-\nu}\cdot\left(\frac{1-\exp(-\nu(T-\tau))}{1-\exp(-\kappa(T-\tau))}\frac{\kappa}{\nu} - 1\right). \end{cases} \tag{C4}$$

Equation (C3) can thus be written as

$$\begin{aligned} \frac{h^1(\tau,T)}{u(r,y,\ell,\tau,T)} &= \tilde{h}^1(\tau,T) \\ &\equiv \left(\int_\tau^T m_{\vartheta,x}(v)dv\right)\cdot\left\{\lambda_A + \lambda_B\sigma_{rv} + \lambda_A\sigma_{\theta r}\frac{\kappa}{\kappa-\nu}\left(\frac{1-\exp(-\nu(T-\tau))}{1-\exp(-\kappa(T-\tau))}\frac{\kappa}{\nu} - 1\right)\right\} \\ &\quad + \frac{\sigma_{rv}^2\eta^2}{2}\left(\int_\tau^T m_{\vartheta,x}(v)dv\right)^2 + \frac{1}{2}\left\{\int_\tau^T m_{\vartheta,x}(v)dv + \sigma_{\theta r}\int_\tau^T m_{\vartheta,c}(v)dv\right\}^2. \end{aligned} \tag{C5}$$

When $\lambda_A + \lambda_B\sigma_{rv} > 0$ and $\lambda_A\sigma_{\theta r} < 0$, a sufficient condition for $h^1 > 0$ for all $\tau \in [t, T)$ is then that

$$\frac{\lambda_A + \lambda_B\sigma_{rv}}{-\lambda_A\sigma_{\theta r}} > \frac{\kappa}{\kappa-\nu}\left(\frac{1-\exp(-\nu(T-\tau))}{1-\exp(-\kappa(T-\tau))}\frac{\kappa}{\nu} - 1\right).$$

Now, function $f(T-\tau) \equiv (1-\exp\ (-\nu\ (T-\tau)))/(1-\exp\ (-\kappa\ (T-\tau)))$, τ varying, is continuous in $[t,\ T)$, with $f(T-\tau) < 1$ for any $\tau \in [t,\ T)$ and $\lim_{\tau\uparrow T} f(T-\tau) = \nu/\kappa < 1$, which shows that the inequality [Equation (19)] is sufficient to guarantee that $h^1 > 0$. Note also that the second relation in Equation (C4) reveals that $u_3 < 0$ for any finite maturity date.

When the inequality [Equation (19)] does not hold, Equation (C5) may be used to develop a condition under which bond prices react negatively to volatility at longer maturity dates. By a straightforward extension of Lemma A1, $u_2(x,s,c,t,T) = \mathbb{E}\{\int_t^T \kappa^1(t,\tau)\cdot h^1(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \chi_{(1)}(u), \tau, T)d\tau\}$, where $\kappa^1(t,\tau) = \exp(-\int_t^\tau(\vartheta_{(1)}(u) - \varphi_2(\vartheta_{(1)}(u), \phi_{(1)}(u), \chi_{(1)}(u)))du) \in (0,1]$ for $\tau \in [t,\ T]$, and $(\vartheta_{(1)},\ \phi_{(1)},\ \chi_{(1)})$ is solution to the system of stochastic differential of Equation (14), except that the drifts are as in operator L^1 in Equation (C1). Furthermore, the parameters estimates reported by Dai and Singleton (2000) are such that $\exists\tau^* : \forall T > \tau^*$, $\tilde{h}^1(\tau,T) < 0$ for all $\tau \in [t,\ \tau^*]$, and $\tilde{h}^1(\tau,T) > 0$ for all $\tau \in [\tau^*,\ T]$. Let $\zeta(x,s,c,t,\tau,T) \equiv \mathbb{E}\{\kappa^1(t,\tau)\cdot u(\vartheta_{(1)}(\tau), \phi_{(1)}(\tau), \chi_{(1)}(u), \tau, T)\} \in (0,1]$. By Fubini,

$$u_2(x,s,c,t,T) = \int_t^{\tau^*}\zeta(x,s,c,t,\tau,T)\cdot\tilde{h}^1(\tau,T)d\tau + \int_{\tau^*}^T\zeta(x,s,c,t,\tau,T)\cdot\tilde{h}^1(\tau,T)d\tau.$$

Therefore, $u_2 < 0$ at longer maturity dates whenever

$$\int_{\tau^*}^T \tilde{h}^1(\tau,T)d\tau < \frac{\min_{\tau\in[t,\tau^*]}\zeta(x,s,c,t,\tau,T)}{\mathbb{E}\{\kappa^1(t,\tau^*)\}}\cdot\int_t^{\tau^*}\{-\tilde{h}^1(\tau,T)\}d\tau. \tag{C6}$$

To illustrate, Equation (C6) for $T=2$ holds when $0.2981 < \min_{\tau\in[0,1.25]} \zeta(x, s, c, 0, \tau, 2)$, where $\zeta \equiv \zeta e^{-(\mu+\lambda_B)}$. As an example, by taking $(x, s, c)=(0.1041, 0.1134, 0.0151)$ [which are the values of r, y, and ℓ under which the drift functions of Equation (14) are zeroed], I find that $\min_{\tau\in[0,1.25]} \zeta(x, s, c, 0, \tau, 2)=0.8132$ after a straightforward numerical analysis in which ζ was computed as $\overline{\zeta}(x,s,c,0,\tau,T)=\mathbb{E}\{\exp(-\int_0^\tau \vartheta_{(1)}(u)du)\mathbb{E}\{\exp(-\int_\tau^T \vartheta(u)du)/(\vartheta,\phi,\chi)(\tau)=(\vartheta_{(1)},\phi_{(1)},\chi_{(1)})(\tau)\}/(\vartheta_{(1)},\phi_{(1)},\chi_{(1)})(0)=x,s,c\}=\mathbb{E}\{\exp(\int_0^\tau(\vartheta(u)-\vartheta_{(1)}(u))du)\cdot\exp(-\int_0^T \vartheta(u)du)/\text{initial state}=x,s,c\}$.

Finally, I show the claim in the main text that the inequality [Equation (18)] is sufficient to guarantee that bond prices are decreasing in volatility at any finite maturity whenever $\kappa_{rv}+\lambda_A+\lambda_B\sigma_{rv}<0$ in Equation (17). First, I show that $u_3<0$. It suffices to use Equation (16) (for factor ℓ) and conclude that for any $(r, y, \ell, \tau)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[t, T)$, $u_3(r, y, \ell, \tau)<0$ whenever $\kappa u_1(r, y, \ell, \tau)<0$. But $\kappa>0$, and $\kappa u_1<0$ because $u_1<0$ by an application of Equation (C1) for $i=2$ (the coefficients of y and ℓ don't depend on r here). To show that $u_2<0$, use Equation (16) (for factor y) and note that it is sufficient to show that for any $(r, y, \ell, \tau)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[t, T)$,

$$-(\kappa_{rv}+\lambda_A+\lambda_B\sigma_{rv})u_1+\frac{1+\eta^2\sigma_{rv}^2}{2}u_{11}-\kappa_{\theta v}u_3<0. \tag{C7}$$

Because $\kappa_{\theta v}<0$ and $u_3<0$, it is sufficient to show that $-(\kappa_{rv}+\lambda_A+\lambda_B\sigma_{rv})u_1+((1+\eta^2\sigma_{rv})/2)u_{11}<0$. But again, $u_1(\cdot)=-(\int_\tau^T m_{\vartheta,x}(v)dv)u(\cdot)$ and $u_{11}(\cdot)=(\int_\tau^T m_{\vartheta,x}(v)dv)^2u(\cdot)$, where $\int_\tau^T m_{\vartheta,x}(v)dv=\{1-\exp(-\kappa(T-\tau))\}/\kappa$, which implies that the inequality [Equation (C7)] holds for any $(r, y, \ell, \tau)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[t, T)$ whenever the inequality [Equation (18)] in the main text is true.

Appendix D: Proofs for Section 5

Proof of Proposition 6. By a standard argument, the following equation will be satisfied by the price function $u(r, \tau, T)$ in the absence of arbitrage opportunities:

$$\begin{cases} 0=\left(\dfrac{\partial}{\partial\tau}+L^J-r\right)u(r,\tau,T), & \forall(r,\tau)\in\mathbb{R}_{++}\times[t,T) \\ u(r,T,T)=1, & \forall r\in\mathbb{R}_{++}, \end{cases} \tag{D1}$$

where $\partial\cdot/\partial\tau+L^J\cdot$ is the jump-diffusion infinitesimal generator of Equation (20), with

$$L^J u(r,\tau,T)=Lu(r,\tau,T)+v(r)\int\{u(r+a_2(r)\,\mathcal{S},\tau,T)-u(r,\tau,T)\}p(d\mathcal{S}),$$

and $\partial\cdot/\partial\tau+L\cdot$ is the usual infinitesimal generator for diffusion processes.

Next, differentiate Equation (D1) twice with respect to r to obtain

$$\begin{cases} 0=\left(\dfrac{\partial}{\partial\tau}+L^{J,i}-k^i\right)w^i(\vartheta,\tau,T)+h^{J,i}(\vartheta,\tau,T), & \forall(\vartheta,\tau)\in\mathbb{R}_{++}\times[t,T) \\ w^i(\vartheta,T,T)=0, & \forall\vartheta\in\mathbb{R}_{++}, \end{cases} \tag{D2}$$

where $w^2\equiv u_1$, $w_3\equiv u_{11}$,

$$h^{J,2}(\vartheta,\tau,T)=-u(\vartheta,\tau,T)+v'(\vartheta)\int\{u(\vartheta+a_2(\vartheta)\,\mathcal{S},\tau,T)-u(\vartheta,\tau,T)\}\cdot p(d\mathcal{S})$$
$$+v(\vartheta)\int w^2(\vartheta+a_2(\vartheta)\,\mathcal{S},\tau,T)\cdot a_2'(\vartheta)\,\mathcal{S}\cdot p(d\mathcal{S}),$$

$$h^{J,3}(\vartheta,\tau,T) = -(2-b^{J\prime\prime})u_1(\vartheta,\tau,T) + v(\vartheta)\int u_1(\vartheta+a_2(\vartheta)\mathcal{S},\tau,T)\cdot a_2''(\vartheta)\mathcal{S}\cdot p(d\mathcal{S})$$
$$+v(\vartheta)\int w^3(\vartheta+a_2(\vartheta)\mathcal{S},\tau,T)\cdot(a_2'(\vartheta)^2\mathcal{S}^2+2a_2'(\vartheta)\mathcal{S})\cdot p(d\mathcal{S})$$
$$+2v'(\vartheta)\int\{u_1(\vartheta+a_2(\vartheta)\mathcal{S},\tau,T)\cdot(1+a_2'(\vartheta)\mathcal{S})-u_1(\vartheta,\tau,T)\}\cdot p(d\mathcal{S})$$
$$+v''(\vartheta)\int\{u(\vartheta+a_2(\vartheta)\mathcal{S},\tau,T)-u(\vartheta,\tau,T)\}\cdot p(d\mathcal{S}),$$

and $k^2=\vartheta-b^{J'}(\vartheta), k^3=\vartheta-2b^{J''}(\vartheta)-a''(\vartheta)$, with operators $L^{J,i}$, $i=2, 3$, satisfying

$$L^{J,i}w^i(\vartheta,\tau,T)=L^iw^i(\vartheta,\tau,T)+v(\vartheta)\int\{w^i(\vartheta+a_2(\vartheta)\mathcal{S},\tau,T)-w^i(\vartheta,\tau,T)\}\cdot p(d\mathcal{S}),$$

where $L^i\cdot$ are defined similarly as in the proof of Lemma A1 in Appendix A.

Since $h^{J,2}(\vartheta,\tau,T)$ tends to minus one as T approach τ and $\mathbb{E}\{e^{-\int_t^T k^2}h^{J,2}(\vartheta(\tau),\tau,T)\}$ is continuous in both τ and T, claim a) of Proposition 6 follows from an argument nearly identical to the one used to show Proposition 2.

Claim (b) follows from rearranging terms in Equation (D2), and claim (c) follows because when $v'(r)=a_2'(r)=0$ for all $r\in\mathbb{R}_{++}$, $h^{J,2}$ and $h^{J,3}$ reduce to functionals h^2 and h^3 encountered during the proof of Lemma A1.

Finally, claim (d) is correct because the price difference $\nabla u\equiv u^A-u^B$ satisfies

$$\begin{cases} 0=\left(\dfrac{\partial}{\partial\tau}+L_\nabla^J-r\right)\nabla u(r,\tau,T) \\ \quad +(v^A(r)-v^B(r))\displaystyle\int\{u^B(r+a_2(r)\mathcal{S},\tau,T)-u^B(r,\tau,T)\}p(d\mathcal{S}), \quad \forall(r,\tau)\in\mathbb{R}_{++}\times[t,T) \\ \nabla u(r,T,T)=0, \quad \forall r\in\mathbb{R}_{++}, \end{cases}$$

where $L_\nabla^J\nabla u=b^J\nabla u_1+a\nabla u_{11}+v^A(r)\int\{\nabla u(r+a_2(r)\mathcal{S},\tau,T)-\nabla u(r,\tau,T))\}p(d\mathcal{S})$. ■

Proof of Equation (21). Let $\hat{\tau}$ be the random default time, and define an auxiliary state process g with the property that

$$g(\tau)=\begin{cases}0, & \text{if } t\leq\tau<\hat{\tau}\\ 1, & \text{otherwise.}\end{cases}$$

In this economy, all relevant information is thus subsumed by the following risk-neutral dynamics:

$$\begin{cases} dr(\tau)=b(r(\tau))d\tau+\sqrt{2a(r(\tau))}dW(\tau)\\ dg(\tau)=\mathcal{S}\cdot dN(\tau), \text{ where } \mathcal{S}\equiv 1 \text{ with probability one.}\end{cases}$$

Denote the rational bond price function as $u(r,g,\tau,T)$, $\tau\in[t,T]$. By a standard argument, the following equation is satisfied by the predefault bond price $u(r,0,\tau,T)=u^{\text{pre}}(r,\tau,T)$ in the absence of arbitrage opportunities:

$$\begin{aligned} 0&=\left(\frac{\partial}{\partial\tau}+L-r\right)u(r,0,\tau,T)+v(r)\cdot(u(r,1,\tau,T)-u(r,0,\tau,T))\\ &=\left(\frac{\partial}{\partial\tau}+L-(r+v(r))\right)u(r,0,\tau,T)+v(r)\bar{u}(\tau), \quad \tau\in[t,T), \end{aligned} \tag{D3}$$

with the usual boundary condition $u(r,0,T,T)=1$. The second line of Equation (D3) follows by the definition of the recovery payment and by rearranging terms. Equation (D3) has exactly the same form as the equations treated in Appendix A (see Lemma A1). Under the usual regularity conditions, the solution for the predefault bond price is

$$u^{\mathrm{pre}}(x,t,T)=\mathbb{E}^*\left\{\exp\left(-\int_t^T (r(\tau)+v(r(\tau)))d\tau\right)\right\} + \mathbb{E}^*\left\{\int_t^T \exp\left(-\int_t^\tau (r(u)+v(r(u)))du\right)\cdot v(r(\tau))\overline{u}(\tau)d\tau\right\}. \quad \text{(D4)}$$

The previous formula is an easy extension of the evaluation formula reported by Duffie, Pan, and Singleton (2000, Equation (1.3), p. 1345) in the case of a constant recovery payment. To show that Equation (D4) coincides with the original derivation of Duffie and Singleton (1999, Equation (10), p. 696) [or with the derivation in Lando (1998, example 3.5, p. 107)], that is with Equation (21), insert $\overline{u}=(1-l)\cdot u(\cdot,0,\cdot,\cdot)$ into Equation (D3) to obtain

$$0=\left(\frac{\partial}{\partial\tau}+L-(r+l(\tau)v(r))\right)u(r,0,\tau,T), \quad \forall(r,\tau)\in\mathbb{R}_{++}\times[t,T),$$

with the usual boundary condition, the solution to which is exactly Equation (21). ∎

References

Aït-Sahalia, Y., 1996a, "Nonparametric Pricing of Interest Rate Derivative Securities," *Econometrica*, 64, 527–560.

Aït-Sahalia, Y., 1996b, "Testing Continuous-Time Models of the Spot Interest Rate," *Review of Financial Studies*, 9, 385–426.

Ahn, C.-M., and H. E. Thompson, 1988, "Jump-Diffusion Processes and the Term Structure of Interest Rates," *Journal of Finance*, 43, 155–174.

Ahn, D.-H., and B. Gao, 1999, "A Parametric Nonlinear Model of Term Structure Dynamics," *Review of Financial Studies*, 12, 721–762.

Andersen, T. G., and J. Lund, 1997, "Stochastic Volatility and Mean Drift in the Short Rate Diffusion: Sources of Steepness, Level, and Curvature in the Yield Curve," unpublished manuscript, Aarhus University.

Bajeux-Besnainou, I., and J.-C. Rochet, 1996, "Dynamic Spanning: Are Options an Appropriate Instrument?" *Mathematical Finance*, 6, 1–16.

Bakshi, G., C. Cao, and Z. Chen, 2000, "Do Call Prices and the Underlying Stock Always Move in the Same Direction?" *Review of Financial Studies*, 13, 549–584.

Balduzzi, P., S. R. Das, S. Foresi, and R. K. Sundaram, 1996, "A Simple Approach to Three Factor Affine Term Structure Models," *Journal of Fixed Income*, 6, 43–53.

Bergman, Y. Z., B. D. Grundy, and Z. Wiener, 1996, "General Properties of Option Prices," *Journal of Finance*, 51, 1573–1610.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–659.

Boudoukh, J., M. Richardson, R. Stanton, and R. F. Whitelaw, 1998, "The Stochastic Behavior of Interest Rates: Implications from a Multifactor, Nonlinear Continuous-Time Model," unpublished manuscript, University of California, Berkeley.

Chan, K. C., G. A. Karolyi, F. A. Longstaff, and A. B. Sanders, 1992, "An Empirical Comparison of Alternative Models of Short-Term Interest Rate," *Journal of Finance*, 47, 1209–1228.

Chapman, D. A., J. B. Long, and N. D. Pearson, 1999, "Using Proxies for the Short Rate: When Are Three Months like an Instant?" *Review of Financial Studies*, 12, 763–806.

Chen, L., 1996, *Interest Rate Dynamics, Derivative Pricing, and Risk Management*, Lecture Notes in Economics and Mathematical Systems no. 435, Springer, Berlin.

Chernov, M., and E. Ghysels, 2000, "A Study Towards a Unified Approach to the Joint Estimation of Objective and Risk-Neutral Measures for the Purpose of Options Valuation," *Journal of Financial Economics*, 56, 407–458.

Conley, T. G., L. P. Hansen, E. G. J. Luttmer, and J. A. Scheinkman, 1997, "Short-Term Interest Rates as Subordinated Diffusions," *Review of Financial Studies*, 10, 525–577.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385–407.

Cox, J. C., and S. A. Ross, 1976, "A Survey of Some New Results in Financial Option Pricing Theory," *Journal of Finance*, 31, 383–402.

Dai, Q., and K. J. Singleton, 2000, "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, 55, 1943–1978.

Das, S. R., 2000, "The Surprise Element: Jumps in Interest Rates," mimeo, Santa Clara University.

Duffie, D., 1996, *Dynamic Asset Pricing Theory*, 2nd ed., Princeton University Press, Princeton, NJ.

Duffie, D., J. Pan, and K. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343–1376.

Duffie, D., and K. Singleton, 1999, "Modeling Term Structures of Defaultable Bonds," *Review of Financial Studies*, 12, 687–720.

Dunn, K. B., and C. S. Spatt, 1999, "Call Options, Points, and Dominance Restrictions on Debt Contracts," *Journal of Finance*, 54, 2317–2337.

Dybvig, P., J. E. Ingersoll, Jr., and S. A. Ross, 1996, "Long Forward and Zero-Coupon Rates Can Never Fall," *Journal of Business*, 69, 1–25.

El Karoui, N., M. Jeanblanc-Picqué, and S. E. Shreve, 1998, "Robustness of the Black and Scholes Formula," *Mathematical Finance*, 8, 93–126.

Fong, H. G., and O. A. Vasicek, 1991, "Fixed Income Volatility Management," *Journal of Portfolio Optimization*, Summer, 41–46.

Fouque, J.-P., G. Papanicolaou and K. R. Sircar, 2000, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge.

Friedman, A., 1975, *Stochastic Differential Equations and Applications*, vol. 1, Academic Press, New York.

Heath, D., R. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60, 77–105.

Jacod, J., and A. N. Shiryaev, 1987, *Limit Theorems for Stochastic Processes*, Springer Verlag, Berlin.

Jagannathan, R., 1984, "Call Options and the Risk of Underlying Securities," *Journal of Financial Economics*, 13, 425–434.

Jiang G. J., 1998, "Nonparametric Modeling of U.S. Interest Rate Term Structure Dynamics and Implications on the Prices of Derivative Decurities," *Journal of Financial and Quantitative Analysis*, 33, 465–497.

Karatzas, I., and S. E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, Springer Verlag, Berlin.

Karlin, S., and H. M. Taylor, 1981, *A Second Course in Stochastic Processes*, Academic Press, San Diego.

Kunita, H., 1990, *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge.

Lando, D., 1998, "On Cox Processes and Credit Risky Securities," *Review of Derivatives Research*, 2, 99–120.

Lewis, A. L., 2000, *Option Valuation Under Stochastic Volatility*, Finance Press, Newport Beach, CA.

Litterman, R., and J. Scheinkman, 1991, "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, 1, 54–61.

Litterman, R., J. Scheinkman, and L. Weiss, 1991, "Volatility and the Yield Curve," *Journal of Fixed Income*, 1, 49–53.

Longstaff, F. A., 1989, "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 23, 195–224.

Longstaff, F. A., and E. S. Schwartz, 1992, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model," *Journal of Finance*, 47, 1259–1282.

Longstaff, F. A., and E. S. Schwartz, 1993, "Implementation of the Longstaff-Schwartz Interest Rate Model," *Journal of Fixed Income*, September, 7–14.

Mele, A., 2002, "Fundamental Properties of Bond Prices in Models of the Short-Term Rate," WP no. 460, Queen Mary, University of London.

Mele, A., and F. Fornari, 2000, *Stochastic Volatility in Financial Markets: Crossing the Bridge to Continuous Time*, Kluwer Academic, Boston.

Mele, A., and F. Fornari, 2001, "Recovering the Probability Density Function of Asset Prices Using ARCH Models as Diffusion Approximations," *Journal of Empirical Finance*, 8, 83–110.

Merton, R. C., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 141–183.

Romano, M., and N. Touzi, 1997, "Contingent Claims and Market Completeness in a Stochastic Volatility Model," *Mathematical Finance*, 7, 399–412.

Rothschild, M., and J. E. Stiglitz, 1970, "Increasing Risk: I. A Definition," *Journal of Economic Theory*, 2, 225–243.

Stanton, R., 1997, "A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk," *Journal of Finance*, 52, 1973–2002.

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.